The Sylvester-Kac matrix space

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Outline

1. Introduction
   - The Sylvester-Kac matrix
   - Constructive reduction in triangular form
2. Band matrices
3. Tridiagonal matrices
   - Reduction of tridiagonal matrices
   - The Sylvester-Kac matrix space

Outline

S_n

$$S_n = \begin{pmatrix} 0 & n-1 & & \\ 1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ & & n-1 & 0 \end{pmatrix}$$

It is surprising that

$$\sigma(S_n) = \{-n+1, -n+3, \dots, n-3, n-1\}$$

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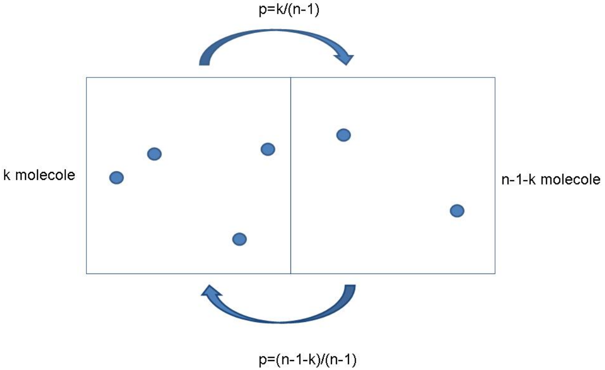
Some papers

- Sylvester, 1854
- Muir, 1882,1923, 1933
- Shrödinger, 1926
- Kac, 1947
- Clement, 1959
- Taussky, Todd, 1991
- Edelman, Kostlan, 1994
- Holtz, 2005
- Boros, Rósza, 2006

Example: Ehrenfest model

$P = \frac{1}{n-1} S_n$ is stochastic



Outline

Schur theorem

If $V = \begin{pmatrix} A & B \end{pmatrix}$ is a $n \times n$ full rank matrix such that $B^T A = O$ then

$$V^{-1} = \begin{pmatrix} A^+ \\ B^+ \end{pmatrix}.$$

If $MA = A\Lambda$ then

$$V^{-1}MV = \begin{pmatrix} A^+ \\ B^+ \end{pmatrix} M \begin{pmatrix} A & B \end{pmatrix} = \begin{pmatrix} \Lambda & A^+MB \\ O & B^+MB \end{pmatrix}.$$

From now on we study the case where $A = a$ is a vector without zero components.

Schur theorem

In order to work with band matrices we choose

$$B = \begin{pmatrix} 1 & 0 & & 0 \\ -a(1)/a(2) & 1 & \ddots & \vdots \\ 0 & -a(2)/a(3) & 1 & 0 \\ 0 & 0 & \ddots & 1 \\ 0 & \ddots & 0 & -a(n-1)/a(n) \end{pmatrix}.$$

Then, it is easy to show that

$$V^{-1}MV = \begin{pmatrix} \lambda & a^{+}MB^{+T} \\ O & B^{T}ML^{T} \end{pmatrix}, \quad \text{where} \quad L = \mathrm{tril}((1./a(1:(n-1)))a^{T})$$

Notation $M = M^{(1)}$ and $B^{T}ML^{T} = M^{(2)}$.

Motivating example

Let $M^{(1)} = S_n$. If $e = (1, \ldots, 1)^T$ then $M^{(1)}e = (n-1)e$. If $a = e$ we find

$$M^{(2)} = S_{n-1} - I.$$

This implies that the reduction step can be repeated, leading to a triangular matrix similar to S_n.

It is interesting to ask if other matrices share with S_n this property.

Reduction of band matrices

Theorem

Let $M^{(1)}$ be a (b_l, b_u) band matrix. Then $M^{(2)}$

- *has lower band of width b_l and its outermost lower diagonal is equal to the outermost lower diagonal of the $(n-1) \times (n-1)$ leading principal submatrix of $M^{(1)}$.*
- *has upper band of width b_u and if*

$$a(k+1)a(k-b_u) = a(k)a(k-b_u+1) \qquad k = b_u+1, \ldots, n-1$$

the outermost upper diagonal of $M^{(2)}$ is equal to the outermost upper diagonal of the $(n-1) \times (n-1)$ trailing principal submatrix of $M^{(1)}$.

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Form of the eigenvector

Lemma

Let a be a vector without zero components. Then

$$a(k+1)a(k-1) = a(k)^2 \iff a(k) = \frac{a(2)^{k-1}}{a(1)^{k-2}}.$$

If $a(1) = 1$, setting $a(2) = \rho$ yields

$$a(k) = \rho^{k-1}.$$

Reduction of tridiagonal matrices

Theorem

Let $M^{(1)}$ be tridiagonal. If

$$M^{(1)}(1, \rho, \ldots, \rho^{n-1})^T = \lambda_1(1, \rho, \ldots, \rho^{n-1})^T$$

then

$$M^{(2)}(1, \rho, \ldots, \rho^{n-2})^T = \lambda_2(1, \rho, \ldots, \rho^{n-2})^T$$

if and only if

$$\begin{aligned} \lambda_2 - \lambda_1 &= \rho(M^{(1)}(k+1, k+2) - M^{(1)}(k, k+1)) \\ &\quad +1/\rho(M^{(1)}(k, k-1) - M^{(1)}(k+1, k)). \end{aligned}$$

for $k = 1, \ldots, n-1$ where $M^{(1)}(1, 0) = M^{(1)}(n, n+1) = 0$.

Outline

Reduction completion

It is possible to complete the reduction process by solving a particular homogeneous linear system that has:

- $(n-1)+(n-2)+\ldots+1=n(n-1)/2$ equations;
- $3n-2$ unknowns (the $2n-2$ off diagonal entries of $M^{(1)}$ and the n eigenvalues λ_i).

The solutions of the system for $\rho \neq 1$ can be obtained from the solutions for $\rho = 1$ by a simple scaling.

Example

In the case where $n = 5$, the system has 10 equations and 13 unknowns.

$$\left(\begin{array}{cccc|cccc|ccccc}
-1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \\
\hline
0 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 \\
\hline
0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
\hline
0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{array}\right)$$

Dimension and parametrization

The system can be solved by using a block elimination.

Theorem

For every n the space of the solution has dimension 4.

It is possible to represent the space of the solutions by using λ_i, $i = 1, 2, 3$ and $\beta = M^{(1)}(2, 1)$ as parameters.

Solution

In the case where $\rho = 1$ we find

$$\lambda_i = \frac{1}{2}(i-2)(i-3)\lambda_1 - (i-1)(i-3)\lambda_2 + \frac{1}{2}(i-1)(i-2)\lambda_3, \quad i = 4, \dots, n,$$

$$M^{(1)}(i+1,i) = i\beta + \frac{1}{2}i(i-1)(\lambda_1 - 2\lambda_2 + \lambda_3) \qquad i = 2, \dots, n-1,$$

$$M^{(1)}(i,i+1) = -(n-i)\beta - \frac{1}{2}(n-1)(n+i-5)\lambda_1 + (n-i)(n+i-4)\lambda_2 - \frac{1}{2}(n-i)(n+i-3)\lambda_3 \quad i = 1, \dots, n-1.$$

$n = 5$

Let $S(\beta, \lambda_1, \lambda_2, \lambda_3)$ the general solution. When $n = 5$ one obtains

$$S_5(1,0,0,0) = \begin{pmatrix} 4 & -4 & 0 & 0 & 0 \\ 1 & 2 & -3 & 0 & 0 \\ 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 3 & -2 & -1 \\ 0 & 0 & 0 & 4 & -4 \end{pmatrix},$$

$$S_5(0,1,0,0) = \begin{pmatrix} 3 & -2 & 0 & 0 & 0 \\ 0 & 4 & -3 & 0 & 0 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & 3 & 0 & -2 \\ 0 & 0 & 0 & 6 & -5 \end{pmatrix},$$

$n = 5$

$$S_5(0,0,1,0) = \begin{pmatrix} -8 & 8 & 0 & 0 & 0 \\ 0 & -9 & 9 & 0 & 0 \\ 0 & -2 & -6 & 8 & 0 \\ 0 & 0 & -6 & 1 & 5 \\ 0 & 0 & 0 & -12 & 12 \end{pmatrix},$$

$$S_5(0,0,0,1) = \begin{pmatrix} 6 & -6 & 0 & 0 & 0 \\ 0 & 6 & -6 & 0 & 0 \\ 0 & 1 & 4 & -5 & 0 \\ 0 & 0 & 3 & 0 & -3 \\ 0 & 0 & 0 & 6 & -6 \end{pmatrix}.$$

Theorem

The Sylvester-Kac matrix space contains a two dimensional subspace made up by symmetric matrices.

Example

$$S_5(2,3,2,0) = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & -2 & 3 & 0 & 0 \\ 0 & 3 & -3 & 3 & 0 \\ 0 & 0 & 3 & -2 & 2 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix}.$$

Problems

- Obtain the entries of V such that VSV^{-1} is triangular.
- What happens if different values of ρ are allowed in the different reduction steps.

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