Due Giorni di Algebra Lineare Numerica

UNSUPERVISED BLIND SEPARATION AND DEBBLURING OF MIXTURES OF SOURCES

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Bologna, March 7, 2008

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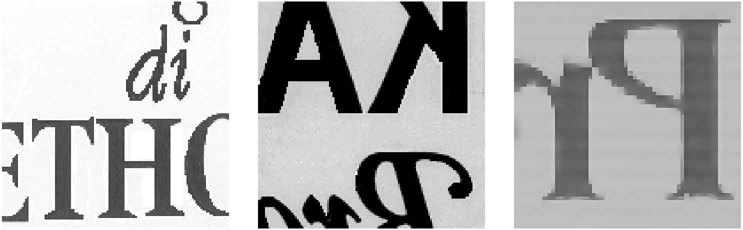
Direct Problem

$$\mathbf{m}_i = H_i \left(\sum_{j=1}^{k} a_{ij} \mathbf{s}_j \right) + \mathbf{n}_i \quad i = 1, \ldots, k$$

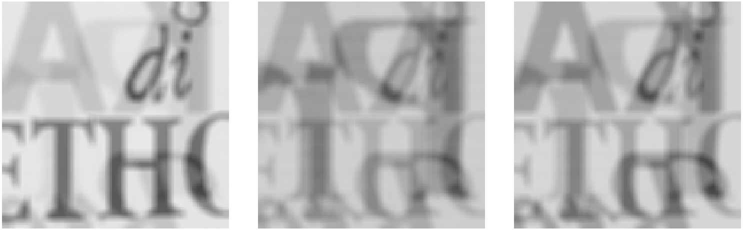
where
$\mathbf{s}_i \in \mathbb{R}^{N^2}$, $i = 1, \ldots, k$, are the map sources,
$A = \{a_{ij}\}_{i,j=1,\ldots,k}$ is the mixing matrix,
$H_i \in \mathbb{R}^{N^2 \times N^2}$, $i = 1, \ldots, k$, are some linear operators,
$\mathbf{n}_i \in \mathbb{R}^{N^2}$, $i = 1, \ldots, k$, are some white, Gaussian and independent noises,
$\mathbf{m}_i \in \mathbb{R}^{N^2}$, $i = 1, \ldots, k$, are the data mixtures.



Map sources

Mixtures

Direct Problem



Blurred mixtures

Direct Problem

Noisy blurred mixtures=data mixtures

The problem of separation and deblurring of mixtures of sources consists of finding an estimation of the original sources $\mathbf{s}_i$, $i = 1, \ldots, k$, given the blur matrices H_i, $i = 1, \ldots, k$, the observed mixtures $\mathbf{m}_i$, $i = 1, \ldots, k$ and the mixing matrix A.

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This problem is ill-posed in the sense of Hadamard.

Associated finite order operator:

$$D_c \boldsymbol{x} = x_s - x_t, \qquad \forall\ c \text{ of kind } (1) \text{ and } (2),$$

$$C = \{c | c \text{ is a first order clique}\}.$$

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$b_c \in B \subset \mathbb{R}_0^+$ indicates the presence of a discontinuities on the clique c.
b is the set of the variable b_c.

Definition

An edge–preserving regularized solution of the problem can be defined as the argument of the minimum of one of the following functions:

- primal energy function

$$E^A(\mathbf{s},\mathbf{b}) = \sum_{i=1}^{k} \left\| \mathbf{m}_i - H_i\left(\sum_{j=1}^{k} a_{ij}\mathbf{s}_j\right)\right\|^2 + \sum_{j=1}^{k}\lambda_j^2 \sum_{c\in C}\left(b_c(D_c(\mathbf{s}_j))^2 + \beta(b_c)\right),$$

- dual energy function

$$E_d^A(\mathbf{s}) = \sum_{i=1}^{k} \left\| \mathbf{m}_i - H_i\left(\sum_{j=1}^{k} a_{ij}\mathbf{s}_j\right)\right\|^2 + \sum_{j=1}^{k}\lambda_j^2 \sum_{c\in C} g(D_c(\mathbf{s}_j)).$$

Primal vs. Dual

A dual energy function can be defined from a primal energy function as follows

$$E_d^A(\mathbf{s}) = \inf_{\mathbf{b} \in B^{|c|}} E^A(\mathbf{s}, \mathbf{b}).$$

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In this case we have that

$$g(t) = \inf_{b \in B} \{t^2 b + \beta(b)\}.$$

DUALITY THEOREM

THEOREM [G.,MARTINELLI AND PUCCI,'08]

Let $g : \mathbb{R} \to \mathbb{R}$ such that

I) $g(0) = 0$, $g \not\equiv 0$, g is a even and continuous function, non decresing in $\mathbb{R}_0^+$;

II) the function $f(t) = \begin{cases} g(\sqrt{t}), & \text{if } t \geq 0 \\ -\infty, & \text{otherwise} \end{cases}$ is concave and

$$\lim_{t \to +\infty} \frac{f(t)}{t} = 0.$$

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Let $\beta : \mathbb{R} \to \mathbb{R}_0^+ \cup +\infty$ such that

III) $g(t) = \inf_{b \in \mathbb{R}} \{bt^2 + \beta(b)\} \quad \forall t \in \mathbb{R}$.

IV) $\beta \not\equiv 0$, $\beta(b) \geq 0 \quad \forall b \in \mathbb{R}$, β is a non incresing and convex function;

V) if $b \neq 0$, $\beta(b) < +\infty$ if and only if $b > 0$;

VI) $\lim_{b \to +\infty} \beta(b) = 0$, $\lim_{b \to 0^+} \beta(b) = \beta(0) > 0$.

THEOREM [G.,MARTINELLI AND PUCCI,'08]

$$\begin{array}{c} \text{I)} \\ \text{II)} \end{array} \iff \begin{array}{c} \text{III)} \\ \text{IV)} \\ \text{V)} \\ \text{VI)} \end{array}$$

Definition [Rockafellar, 1970]

Let f be a function on $\mathbb{R}$, the function

$$f^*(y) = \sup_{x \in \mathbb{R}} \{xy - f(x)\} \ \forall y \in \mathbb{R}$$

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PROPERTIES

Let f be a proper closed convex function on $\mathbb{R}$.
f^* is a proper closed convex function and the conjugate f^{**} di f^* coincides with f.

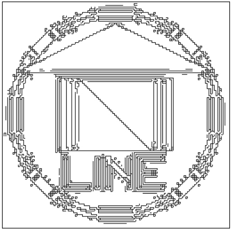
Parallel Lines Inhibition

Observed image

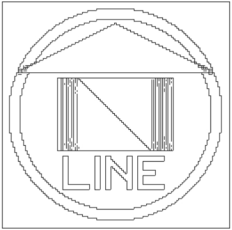
Observed image

Reconstructed image

Reconstructed image



Line elements



Line elements

Blind Problem

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Obviously, this problem is again ill-posed in the sense of Hadamard.

PROPOSED ESTIMATION

DEFINITION

We define the solution $(\tilde{\mathbf{s}}, \tilde{A})$ of the blind problem as

$$\begin{aligned}\tilde{A} &= \arg\min_{A} F(A, \mathbf{s}(A)),\\ \tilde{\mathbf{s}} &= \mathbf{s}(\tilde{A}),\end{aligned}$$

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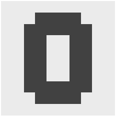
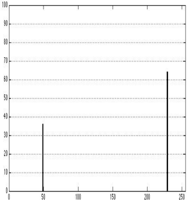
Namely,

$$F(A, \mathbf{s}(A)) = \sum_{i=1}^{k} \left\| \mathbf{m}_i - H_i \left(\sum_{j=1}^{k} a_{ij} \mathbf{s}_j(A) \right) \right\|^2 + K(\mathbf{s}(A)).$$

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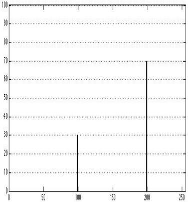
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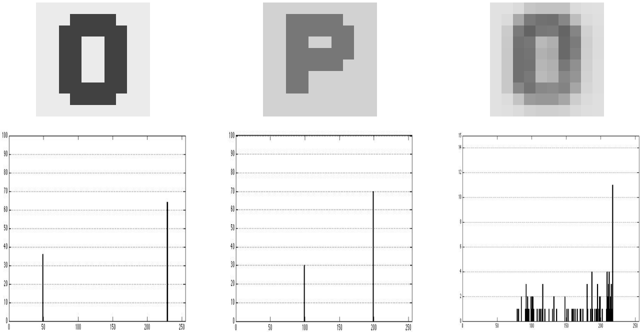

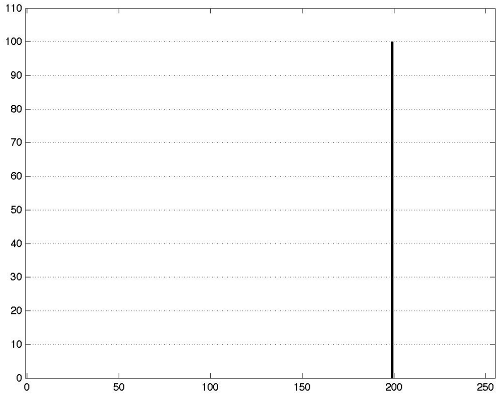
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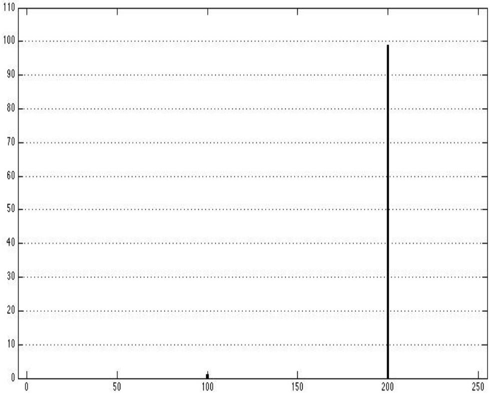
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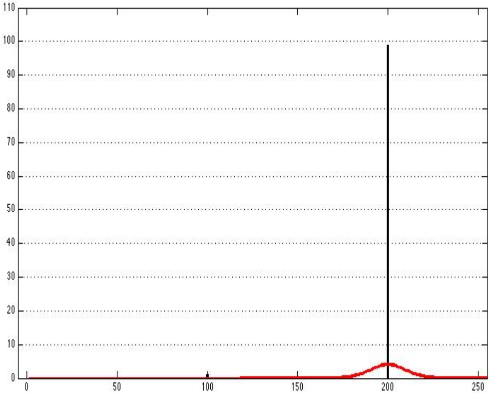
Gaussianity and Non–Gaussianity Constraints



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$$G(\mathbf{s}_j(A)) = \sum_{i=0}^{255} (f_{\mathbf{s}_j(A)}(i) - \varphi_{(\mu,\sigma^2)}(i))^2,$$

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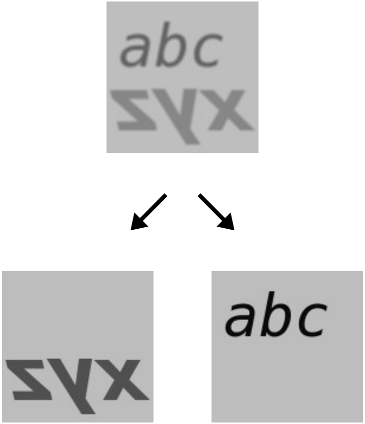
where

$\varphi_{(\mu,\sigma^2)}$ is the best Gaussian that approximates our data,
ν is a decreasing function.


abc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Determination of the background:

$$\gamma_1 = \arg \max_{i \in \{0,\ldots,255\}} \{f_{\mathbf{s}_1(A)}(i)\},$$

$$\gamma_2 = \arg \max_{i \in \{0,\ldots,255\}} \{f_{\mathbf{s}_2(A)}(i)\}.$$

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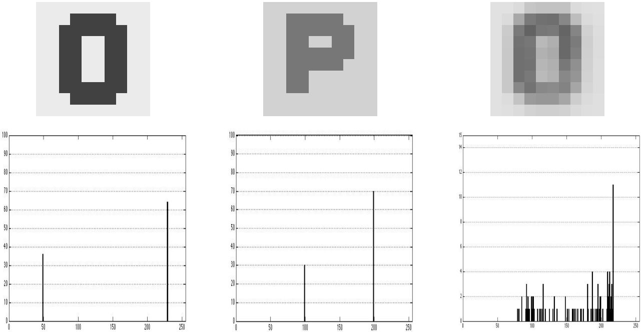
Orthogonality constraint:

$$\Omega(\mathbf{s}_1(A), \mathbf{s}_2(A)) = \sum_{i,j} \left| [s_1(A)]_{(i,j)} - \gamma_1 \right| \left| [s_2(A)]_{(i,j)} - \gamma_2 \right| .$$

Entropy Constraints



Number of states:

$$\tau_j = |\{i \in \{0, \ldots, 255\} : f_{\mathbf{s}_{j(A)}}(i) \neq 0\}|.$$

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Entropy constraint:

$$\Lambda(\mathbf{s}_j(A)) = k_B \log \tau_j,$$

where
k_B is Boltzmann's constant.

- the target function $F(A, \mathbf{s}(A))$ is minimized by a simulated annealing,

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- the energy function $E_d^A(\mathbf{s})$ is minimized by Graduated Non–Convexity Algorithm (GNC).

A family of approximating functions $\{E_d^{(p)}\}_p$ is determinated in such a way that the first one is convex and the last one coincides with the dual energy function E_d^A.

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Then, the following algorithm is executed:

initialize p and $\mathbf{s}^{(prec(p))}$;

while $E_d^{(p)} \neq E_d^A$ do

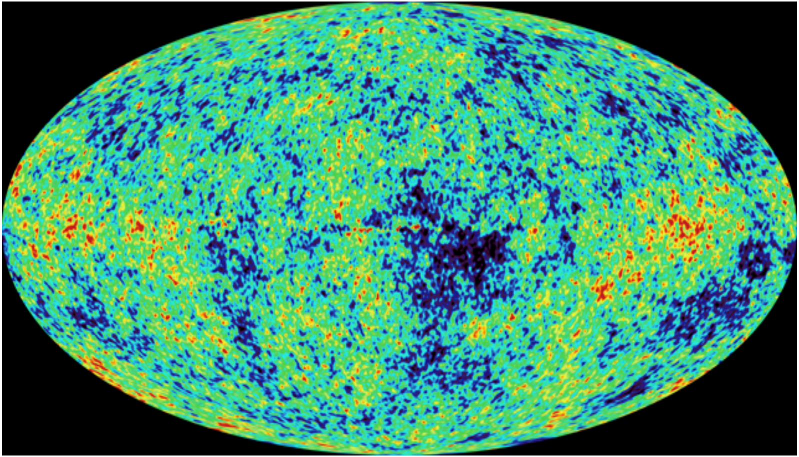
find the minimum of the function $E_d^{(p)}$ starting from the initial point $\mathbf{s}^{(prec(p))}$;

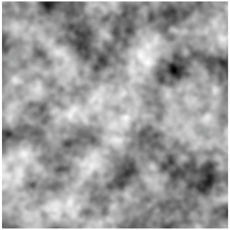
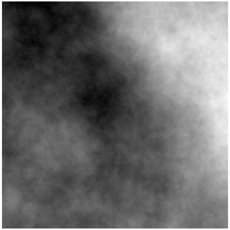
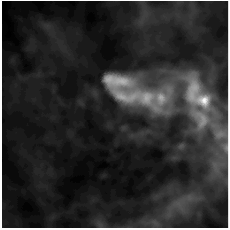
$\mathbf{s}^{(p)} = \arg\min E_d^{(p)}(\mathbf{s})$;

$p = succ(p)$;

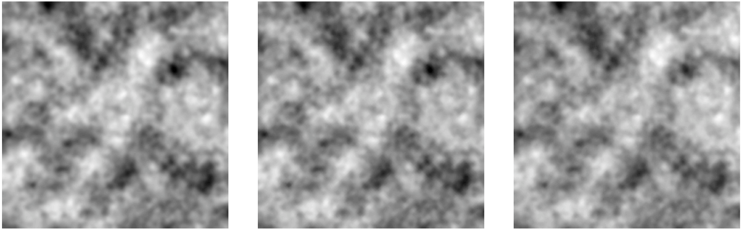
$$\begin{aligned}\Omega^{(p)}(\mathbf{s}_1,\mathbf{s}_2) =& \frac{1}{2}\sum_{i,j}\left|s_1(i,j)-\gamma_1^{(prec(p))}\right|\left|s_2^{(prec(p))}(i,j)-\gamma_2^{(prec(p))}\right| + \\ & \frac{1}{2}\sum_{i,j}\left|s_1^{(prec(p))}(i,j)-\gamma_1^{(prec(p))}\right|\left|s_2(i,j)-\gamma_2^{(prec(p))}\right|\end{aligned}$$

Cosmic Microwave Background

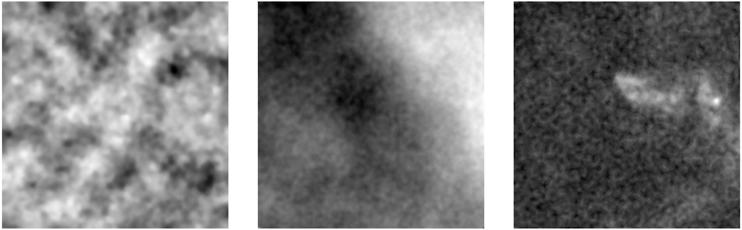


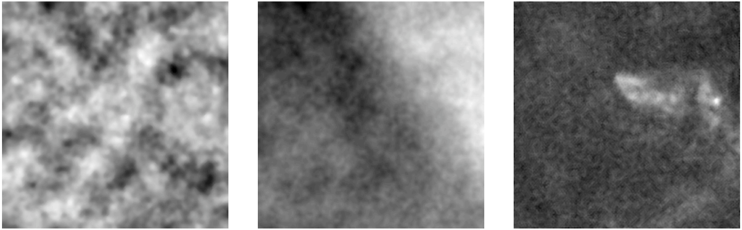
Ideal sources



Data mixtures



Non–blind estimated sources



Blind estimated sources

	MSE		
	medium pixel value	non–blind problem	blind problem
CMB	0.068050	0.0000000001	0.0000322429
Syn	0.018726	0.0000364836	0.0004320682
Dust	0.028039	0.0000000843	0.0000021202

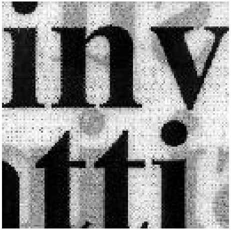
EXPERIMENTAL RESULTS

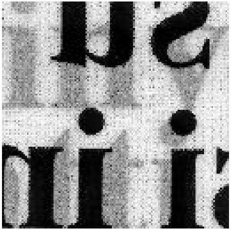
RGB image

RGB components

Blind estimated sources

EXPERIMENTAL RESULTS



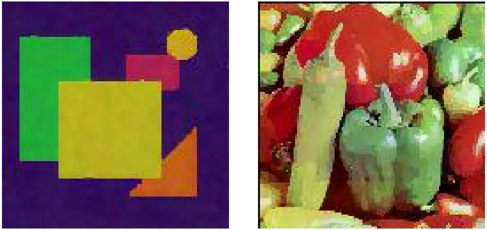


Data mixtures

Blind estimated sources

EXPERIMENTAL RESULTS

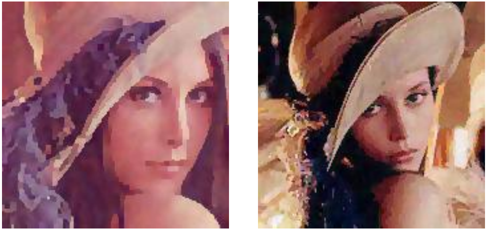
Data mixtures



Blind estimated sources

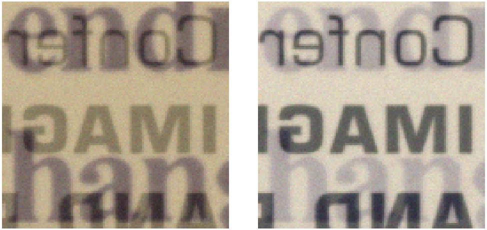
Experimental Results

Data mixtures



Blind estimated sources

Experimental Results



Data mixtures

Blind estimated sources

THE END