Geometric aspects of phase separation

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Several physical phenomena can be described by a certain number of densities (of mass, population, probability, ...) distributed in a domain and subject to laws of diffusion, reaction, and *competitive interaction*. Whenever the competitive interaction is the prevailing phenomenon, the several densities can not coexist and tend to segregate, hence determining a partition of the domain *(Gause's experimental principle of competitive exclusion (1932))*. As a model problem, we consider the system of stationary equations

$$\begin{cases} -\Delta u_i = f_i(u_i) - \beta u_i \sum_{j\neq i} g_{ij}(u_j) \\ u_i > 0\ . \end{cases}$$

The cases $g_{ij}(s) = \beta_{ij} s$ (Lotka-Volterra competitive interactions) and $g_{ij}(s) = \beta_{ij} s^2$ (gradient system for Gross-Pitaevskii energies) are of particular interest in the applications to population dynamics and theoretical physics respectively.

In this series of lectures, we will undertake the analysis of qualitative properties of solutions to systems of semilinear elliptic equations, whenever the parameter β, accounting for the competitive interactions, diverges to infinity. At the limit, when the minimal interspecific competition rate $\beta = \min_{ij} \beta_{ij}$ diverges to infinity, we find a vector $U = (u_1, \cdots, u_h)$ of functions with mutually disjoint supports: *the segregated states*: $u_i \cdot u_j \equiv 0$, for $i \neq j$, satisfying

$$-\Delta u_i = f_i(x, u_i) \qquad \text{whenever } u_i \neq 0\ , \quad i = 1, \dots, h,$$

We will consider the follfowing apsects:

(1) Spectral problems: optimal partitions with respect to eigenvalues in connection with monotonicity formulæ.
(2) Entire solutions of the competitive elliptic system:

$$\begin{cases} -\Delta u_i = -\sum_{j\neq i} u_i u_j^2 & \text{in } \mathbb{R}^N \\ u_i > 0 & \text{in } \mathbb{R}^N \end{cases} \qquad i = 1, \dots, k. \tag{1}$$

(3) Competition-diffusion problems with fractional laplacians.
(4) Competition-diffusion problems with non local interactions.
(5) Spiralling solutions in the non symmetrical case.