Course "Computational methods for linear matrix equations"
Computer exercises
May 28, 2021

Suggestion: Make a script containing all commands used.

1. *Direct methods:*

   (a) Consider the Lyapunov equation $AX+XA=C$ with the *sparse* matrix $A = \text{tridiag}(-1,2,-1)$ of dimension n and the matrix $C = \mathbf{1}\mathbf{1}^T$. Compare the performance of the matlab function `lyap` with that of \ applied to the *sparse* matrix $\mathcal{A} = I \otimes A + A \otimes I$, by reporting on the command window the following information for $n = 100:50:500$:

n	$\frac{\|x_{lyap}-x_{dir}\|}{\|x_{dir}\|}$	time Lyap	time direct
:	:	:	:

   where x_{lyap} is the vectorized version of the solution matrix X (in Matlab use: `x=X(:)`).
   The CPU time should only measure the solver call (`tic-toc` function in Matlab)
   Report in a figure (Figure 1) the graph `mesh(X)` and explain the type of surface.
   Report in a figure (Figure 2) the singular values of `X` as the dimension grows (use `semilogy`).

   (b) Repeat the experiment above with

   $$A = \begin{bmatrix} & & & & \ddots & & & & \\ \ldots & 0 & -1 & -1 & -1 & \underline{4} & -1 & -1 & -1 & 0\ldots \\ & & & & & & \ddots & & \end{bmatrix}$$

   except for the plot in Figure 1.

2. *Iterative methods:* The functions `krylov.m`, `eksm.m` and `rksm.m` generate the basis matrices of Standard Krylov, Extended Krylov and Rational Krylov spaces, respectively. We wish to compare the performance of these spaces in solving $AX + XA^T + C = 0$.

   For each space, first construct a basis of dimension $m_{\max} = 80$ and then determine the approximate solution X_m, $m = 1, \ldots, m_{\max}$ into the corresponding space (with a residual F-norm less than 10^{-8}). Collect the number of iterations (or space dimension) required to meet the residual norm criterion. Take inspiration from the following iteration:

```
normC=norm(C,'fro');
for k=1:m_max
    H = V(:,1:k)'*A*V(:,1:k);
    Cm= V(:,1:k)'*C*V(:,1:k);
    Y = lyap(H,-Cm);
    X = V(:,1:k)*Y*V(:,1:k)';   % this is resource-expensive, should not be done
    norm_res=norm(C-A*X-X*A','fro')/normC;  % this is resource-expensive, should not be done
    if norm_res<tol, break,end
end
```

   For $n = 50$ and $h = 1/(n-1)$, define the matrices $T = -1/h^2 \text{tridiag}(-1,2,-1)$, $B_\alpha = \frac{\alpha}{2h} \cdot \text{tridiag}(1,0,-1)$ and $A = I \otimes (T + B_\alpha) + T \otimes I$, $C = \mathbf{1}\mathbf{1}^T$. For $\alpha \in \{0, 50, 200\}$ display the performance of each method, reporting the number of iterations as in the following table:

α	Krylov	EKSM	RKSM
:	:	:	:

3. Using the data and dimensions in exercise 1(a), propose a cheap stragegy to solve

$$AX + XA^T + uv^T X v u^T = C$$

(*hint: the operator* $\mathcal{L} : X \mapsto AX + XA^T$ *is linear*)