Numerical solution of systems of linear matrix equations and applications

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The problem

Find $\mathbf{X} \in \mathbb{R}^{n_1 \times n_2}$ and $\mathbf{P} \in \mathbb{R}^{m \times n_2}$ such that

$$\begin{aligned} A_1\mathbf{X} + \mathbf{X}A_2 + B^T\mathbf{P} &= F_1 \\ B\mathbf{X} &= F_2 \end{aligned}$$

with $A_i \in \mathbb{R}^{n_i \times n_i}$, $B \in \mathbb{R}^{m \times n_1}$, $F_1 \in \mathbb{R}^{n_1 \times n_2}$, $F_2 \in \mathbb{R}^{m \times n_2}$, $m \leq n_1$

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with $A_i \in \mathbb{R}^{n_i \times n_i}$, $B \in \mathbb{R}^{m \times n_1}$, $F_1 \in \mathbb{R}^{n_1 \times n_2}$, $F_2 \in \mathbb{R}^{m \times n_2}$, $m \leq n_1$

Emerging matrix formulation of different application problems

- Constraint control
- Mixed formulations of stochastic diffusion problems
- Discretized deterministic/stochastic (Navier-)Stokes equations
- ...

A natural generalization

Find $\mathbf{X} \in \mathbb{R}^{n_2 \times n_1}$, $\mathbf{Y} \in \mathbb{R}^{n_1 \times n_2}$ and $\mathbf{P} \in \mathbb{R}^{n_1 \times n_1}$ such that

$$\begin{aligned} A_2\mathbf{X} + \mathbf{X}A_1 + B^T\mathbf{P} &= F_1 \\ A_1\mathbf{Y} + \mathbf{Y}A_2 + \mathbf{P}B &= F_2 \\ B\mathbf{X} + \mathbf{Y}B^T &= F_3 \end{aligned}$$

with corresponding dimensions for the coefficient matrices and rhs

e.g., MAC discretization of Navier-Stokes equation

An example. Regulator equations in contraint Control

Consider the time-invariant dynamical linear system

$$\dot{x} = Ax + Bu, \quad y = Cx \qquad (*)$$

and the reference model $(\dot{\widehat{x}}) = \widehat{A}\widehat{x} + \widehat{B}\widehat{u}$, $\widehat{y} = \widehat{C}\widehat{x}$

Theorem (Duan-Zhang, 2007)

Let K be a stabilizing gain for the dynamical system $(*)$ and assume $\mathbf{X}, \mathbf{P}$ satisfy the equations

$$A\mathbf{X} - \mathbf{X}\widehat{A} + B\mathbf{P} = 0, \quad C\mathbf{X} = \widehat{C}. \qquad (1)$$

When the controller $u = u_s + u_c$ with $u_s = Kx$ and $u_c = (\mathbf{P} - K\mathbf{X})\widehat{x}$ is applied to $(*)$, it holds that $\lim_{t\to\infty}(y(t) - \widehat{y}(t)) = 0$.

Also tracking (see, e.g., Wonham (1979), Duan (2004), Saberietal (1999))

An example. Mixed FE formulation of stochastic Galerkin diffusion pb

$$\begin{aligned} c^{-1}\vec{u} - \nabla p &= 0, \\ -\nabla \cdot \vec{u} &= f, \end{aligned}$$

Assume that $c^{-1} = c_0 + \sum_{r=1}^{\ell} \sqrt{\lambda_r} c_r(\vec{x}) \xi_r(\omega)$ and that an appropriate class of finite elements is used for the discretization of the problem

(see, e.g., the derivation in Elman-Furnival-Powell, 2010)

After discretization the problem reads:

$$\begin{bmatrix} G_0 \otimes K_0 + \sum_{r=1}^{\ell} \sqrt{\lambda} G_r \otimes K_r & G_0^T \otimes B_0^T \\ G_0 \otimes B_0 & \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

An example. Mixed FE formulation of stochastic Galerkin diffusion pb

$$
\begin{aligned}
c^{-1}\vec{u} - \nabla p &= 0, \\
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\end{aligned}
$$

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$$

For $\ell = 1$ we obtain

$$
K_0 \mathbf{X} G_0 + K_1 \mathbf{X} G_1 + B_0^T \mathbf{P} G_0 = 0, \qquad B_0 \mathbf{X} G_0 = F
$$

The two matrix equation case. Computational strategies

$$\begin{aligned} A_1\mathbf{X} + \mathbf{X}A_2 + B^T\mathbf{P} &= F_1 \\ B\mathbf{X} &= F_2 \end{aligned}$$

Kronecker formulation:

$$\begin{bmatrix} \mathcal{A} & \mathcal{B}^T \\ \mathcal{B} & \mathcal{O} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad \mathcal{A} = I \otimes A_1 + A_2^T \otimes I, \quad \mathcal{B} = B \otimes I$$

with $\mathbf{x} = \mathrm{vec}(\mathbf{X})$, $\mathbf{p} = \mathrm{vec}(\mathbf{P})$, $f_1 = \mathrm{vec}(F_1)$ and $f_2 = \mathrm{vec}(F_2)$

Extremely rich literature from saddle point algebraic linear systems

Problem: Coefficient matrix has size $(n_1n_2 + mn_2) \times (n_1n_2 + mn_2)$

The two matrix equation case. Computational strategies. Cont'd

$$\begin{aligned} A_1\mathbf{X} + \mathbf{X}A_2 + B^T\mathbf{P} &= F_1 \\ B\mathbf{X} &= F_2 \end{aligned}$$

⋆ Derive numerical strategies that directly work with the matrix equations:

- Small scale: Nullspace method
- Small and medium scale: Schur complement method (also directly applicable to three matrix equations)
- Large scale: Iterative method for low rank F_i, $i = 1, 2$

"Small and medium scale" actually means "Large scale" for the Kronecker form!

Small problem. Null space method

$$A_1\mathbf{X} + \mathbf{X}A_2 + B^T\mathbf{P} = F_1 \qquad (1)$$
$$B\mathbf{X} = F_2 \qquad (2)$$

U_0 s.t. Range$(U_0) = \mathrm{Null}(B)$, $U_0^T U_0 = I$
U_1 s.t. Range$(U_1) = \mathrm{Range}(B^T)$, $U_1^T U_1 = I$ $\quad \Rightarrow \quad \mathbf{X} = U_0\hat{\mathbf{X}} + U_1\mathbf{X}_\perp$

Small problem. Null space method

$$\begin{aligned} A_1\mathbf{X} + \mathbf{X}A_2 + B^T\mathbf{P} &= F_1 \quad (1) \\ B\mathbf{X} &= F_2 \quad (2) \end{aligned}$$

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* From (2), $\quad \mathbf{X}_\perp = (BU_1)^{-1}F_2$

Small problem. Null space method

$$
\begin{aligned}
A_1\mathbf{X} + \mathbf{X}A_2 + B^T\mathbf{P} &= F_1 \qquad (1)\\
B\mathbf{X} &= F_2 \qquad (2)
\end{aligned}
$$

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* From (2), $\mathbf{X}_\perp = (BU_1)^{-1}F_2$

* From (1), multiplying by U_0^T

$$U_0^T A_1 U_0 \hat{\mathbf{X}} + \hat{\mathbf{X}} A_2 = -U_0^T A_1 U_1 \mathbf{X}_\perp + U_0^T F_1$$

A Sylvester equation, giving $\hat{\mathbf{X}}$

Small problem. Null space method

$$
\begin{aligned}
A_1\mathbf{X} + \mathbf{X}A_2 + B^T\mathbf{P} &= F_1 \qquad (1) \\
B\mathbf{X} &= F_2 \qquad (2)
\end{aligned}
$$

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A Sylvester equation, giving $\hat{\mathbf{X}}$

* From (1), multiplying by U_1^T

$$U_1^T A_1 U_1 \mathbf{X}_\perp + U_1^T A_1 U_0 \hat{\mathbf{X}} + \mathbf{X}_\perp A_2 + U_1^T B^T \mathbf{P} = U_1^T F_1$$

So that $\mathbf{P} = (U_1^T B^T)^{-1}(U_1^T F_1 - U_1^T A_1 U_0 \hat{\mathbf{X}} - \mathbf{X}_\perp A_2 - U_1^T A_1 U_1 \mathbf{X}_\perp)$

Large scale problem. Iterative method. 1/3

$$\begin{aligned} A_1\mathbf{X} + \mathbf{X}A_2 + B^T\mathbf{P} &= F_1 \\ B\mathbf{X} &= F_2 \end{aligned}$$

Rewrite as

$$\begin{bmatrix} A_1 & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{P} \end{bmatrix} + \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{P} \end{bmatrix} A_2 = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad \Leftrightarrow \quad \mathcal{M}\mathbf{Z} + \mathcal{D}_0\mathbf{Z}A_2 = F$$

with

$\mathcal{M}, \mathcal{D}_0 \in \mathbb{R}^{(n_1+m)\times(n_1+m)}$

$A_2 \in \mathbb{R}^{n_2\times n_2}$ nonsingular

$\mathcal{D}_0$ highly singular

If $F \approx$ low rank, exploit projection-type strategies for Sylvester equations

Large scale problem. Iterative method. 2/3

$$\begin{bmatrix} A_1 & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{P} \end{bmatrix} + \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{P} \end{bmatrix} A_2 = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \quad \Leftrightarrow \quad \mathcal{M}\mathbf{Z} + \mathcal{D}_0 \mathbf{Z} A_2 = F$$

with F low rank. We rewrite the matrix equation as

$$\mathbf{Z} A_2^{-1} + \mathcal{M}^{-1} \mathcal{D}_0 \mathbf{Z} = \widehat{F}, \qquad \widehat{F} = \mathcal{M}^{-1} F A_2^{-1}$$

which is a Sylvester equation with one singular coefficient matrix

$$\Rightarrow \qquad \mathbf{Z} \approx \widetilde{\mathbf{Z}}_k = V_k \mathcal{Z}_k W_k^T, \quad \text{for some } \mathcal{Z}_k$$

with Range(V_k), Range(W_k) appropriate approximation spaces of small dimensions

Large scale problem. Iterative method. 3/3

$$\mathbf{Z}A_2^{-1} + \mathcal{M}^{-1}\mathcal{D}_0\mathbf{Z} = \widehat{F}_l \widehat{F}_r^T \quad \Rightarrow \quad \mathbf{Z} \approx \widetilde{\mathbf{Z}}_k = V_k \mathcal{Z}_k W_k^T$$

Choosing V_k, W_k. Let $K_k(T, F) = \mathrm{Range}([F, TF, T^2F, \ldots, T^{k-1}F])$.

Large scale problem. Iterative method. 3/3

$$\mathbf{Z}A_2^{-1} + \mathcal{M}^{-1}\mathcal{D}_0\mathbf{Z} = \widehat{F}_l \widehat{F}_r^T \quad \Rightarrow \quad \mathbf{Z} \approx \widetilde{\mathbf{Z}}_k = V_k \mathcal{Z}_k W_k^T$$

Choosing V_k, W_k. Let $K_k(T, F) = \mathrm{Range}([F, TF, T^2F, \ldots, T^{k-1}F])$.

A possible strategy:

- $W_k = \mathbb{E}\mathbb{K}_k(A_2^{-T}, \widehat{F}_r) := K_k(A_2^{-T}, \widehat{F}_r) + K_k(A_2^T, A_2^T\widehat{F}_r)$
  Extended Krylov subspace
- $V_k = K_k(\mathcal{M}^{-1}\mathcal{D}_0, \widehat{F}_l) + K_k((\mathcal{M}^{-1}\mathcal{D}_0 + \sigma I)^{-1}, \widehat{F}_l)$
  Augmented Krylov subspace, $\sigma \in \mathbb{R}$
  (see, e.g., Shank-Simoncini (2013))

Note: $\mathcal{M}$ has size $(n_1 + m) \times (n_1 + m)$

(Compare with $(n_1 n_2 + m n_2) \times (n_1 n_2 + m n_2)$ of the Kronecker form)

Numerical experiments

$A_1\mathbf{X} - \mathbf{X}A_2 + B^T\mathbf{P} = 0, \qquad B\mathbf{X} = F_2$ vs $\mathcal{A}\mathbf{z} = f$

$A_1 \to \mathcal{L}_1 = -u_{xx} - u_{yy}$

$A_2 \to \mathcal{L}_2 = -(e^{-10xy}u_x)_x - (e^{10xy}u_y)_y + 10(x+y)u_x$

$[F_1; F_2]$ rank-1 matrix

$B = \text{bidiag}(-1, \underline{1}), \ (n_2 - n_1) \times n_2,$ iterative: tol$=10^{-6}$, $\sigma = 10^{-2}$

Elapsed time

n_1	n_2	size($\mathcal{A}$)	Monolithic	Direct Nullspace	Matrix eqns
400	100	79,000	6.9769e-02	9.4012e-02	3.1523e-02 (4)
900	225	401,625	3.4808e-01	6.3597e-01	5.0447e-02 (4)
1600	400	1272,000	1.1319e+00	4.7888e+00	7.8018e-02 (4)
2500	625	3109,375	3.1212e+00	1.5063e+01	1.5282e-01 (5)
3600	900	6453,000	1.0210e+01	3.9419e+01	2.8053e-01 (5)
4900	1225	11,962,125	3.7699e+01	1.0721e+02	1.4754e+00 (5)

Numerical experiments. 1D stochastic Stokes problem. 1/3

$$\begin{bmatrix} \mathcal{H} & \mathcal{B}^T \\ \mathcal{B} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \qquad \mathcal{H} = (\nu_0 G_0 + \nu_1 G_1) \otimes A_x, \quad \mathcal{B} = G_0 \otimes B_x$$

where $\nu = \nu_0 + \nu_1 \xi(\omega)$ uncertain viscosity, ξ random variable

Then

$$A_x \mathbf{X} G_0 \nu_0 + A_x \mathbf{X} G_1 \nu_1 + B_x^T \mathbf{P} G_0 = F_1, \quad B_x \mathbf{X} = F_2$$

with $G_0 = I$. This corresponds to

$$\begin{bmatrix} \nu_0 A_x & B_x^T \\ B_x & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{P} \end{bmatrix} + \begin{bmatrix} \nu_1 A_x & \\ & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{P} \end{bmatrix} G_1 = \begin{bmatrix} F1 \\ F2 \end{bmatrix}$$

that is

$$\mathcal{M} \mathbf{Z} + \mathcal{D}_0 \mathbf{Z} G_1 = F$$

Numerical experiments. 1D stochastic Stokes problem. 2/3

$$\mathcal{M}\mathbf{Z} + \mathcal{D}_0\mathbf{Z}G_1 = F \qquad \text{vs} \qquad \mathcal{A}\mathbf{z} = f$$

n_1	n_2	n_B	size($\mathcal{A}$)	Monolithic direct	Monolithic MINRES	Iterative $\mathbb{EK}(\sigma)$
1256	4	389	6,580	0.1852	0.146 (11)	0.19 (2)
3526	4	990	18,064	0.9063	0.275 (11)	0.52 (2)
9812	4	2615	49,708	4.6418	0.981 (10)	2.09 (2)

$\nu_0 = 1/10, \nu_1 = 3\nu_0/10$ Powell-Silvester (2012)

Numerical experiments. 1D stochastic Stokes problem. 3/3

$$\mathcal{M}\mathbf{Z} + \mathcal{D}_0\mathbf{Z}G_1 = F \qquad \text{vs} \qquad \mathcal{A}\mathbf{z} = f$$

n_1	n_2	n_B	size($\mathcal{A}$)	Monolithic direct	Monolithic MINRES	Iterative $\mathbb{EK}(\sigma)$
1256	165	389	271,425	2.91	1.53 (11)	0.20 (2)
3526	165	990	745,140	12.16	7.43 (11)	0.45 (2)
9812	165	2615	2050,455	-	-	1.87 (2)

$\nu_0 = 1/10, \nu_1 = 3\nu_0/10$ Powell-Silvester (2012)

- n_2 could be much larger, $n_2 = O(10^3)$
- Memory requirements are very limited, $\widetilde{\mathbf{Z}} = Z_1 Z_2^T$ of very low rank

Numerical experiments. 2D stochastic Stokes problem

$$\begin{bmatrix} \mathcal{H} & \mathcal{B}^T \\ \mathcal{B} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \qquad \begin{aligned} \mathcal{H} &= \mathrm{blkdiag}((\nu_0 \mathrm{G}_0 + \nu_1 \mathrm{G}_1) \otimes \mathrm{A_x}, (\nu_0 \mathrm{G}_0 + \nu_1 \mathrm{G}_1) \otimes \mathrm{A_y}) \\ \mathcal{B} &= [G_0 \otimes B_x, \, G_0 \otimes B_y] \end{aligned}$$

$$\mathcal{M}\mathbf{Z} + \mathcal{D}_0 \mathbf{Z} G_1 = F \qquad \text{vs} \qquad \mathcal{A}\mathbf{z} = f$$

n_1	n_2	size($\mathcal{A}$)	Monolithic direct	Monolithic MINRES	Iterative $\mathbb{EK}(\sigma)$
2512	4	11,604	0.55	0.12 (12)	0.28 (2)
7052	4	32,168	3.73	0.36 (12)	1.22 (2)
19624	4	88,956	11.93	1.51 (12)	4.37 (2)

n_1	n_2	size($\mathcal{A}$)	Monolithic direct	Monolithic MINRES	Iterative $\mathbb{EK}(\sigma)$
2512	165	478 665	7.60	3.16 (17)	0.33 (2)
7052	165	1 326 930	34.08	15.52 (18)	1.32 (2)
19624	165	3 669 435	–	–	5.69 (3)

$\nu_0 = 1/10, \nu_1 = 3\nu_0/10$ Powell-Silvester (2012)

Conclusions and further comments

... A new matrix-oriented methodology

- Matrix formulation is computationally effective and memory saving
- Very preliminary numerical results are encouraging
- Three matrix equation solver effective via matrix-oriented Schur complement
- Operator $\mathcal{L} : X \to A_1 X + X A_2$ could be replaced with $\mathcal{L} : X \to \sum_k A_k X G_k$

  (work in progress with C. Powell)

For further information

http://www.dm.unibo.it/~simoncin