Structured Preconditioners for Saddle Point Problems

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Collaborators on this project

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Application problems

- Computational Fluid Dynamics
- Elasticity problems
- Mixed (FE) formulations of II and IV order elliptic PDEs
- Linearly Constrained Programs
- Linear Regression in Statistics
- Weighted Least Squares (Image restoration)
- ...

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$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

Motivational Application

Constrained Quadratic minimization problem

$$\text{minimize} \quad J(u) = \frac{1}{2}\langle Au, u\rangle - \langle f, u\rangle$$

subject to $Bu = g$

$A \quad n \times n$ symmetric, $B\ m \times n$, $m \leq n$ full rank

$$\Downarrow$$

Lagrange multipliers approach

Karush-Kuhn-Tucker (KKT) system

The algebraic Saddle Point Problem

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

- A sym. pos.semidef., B full rank

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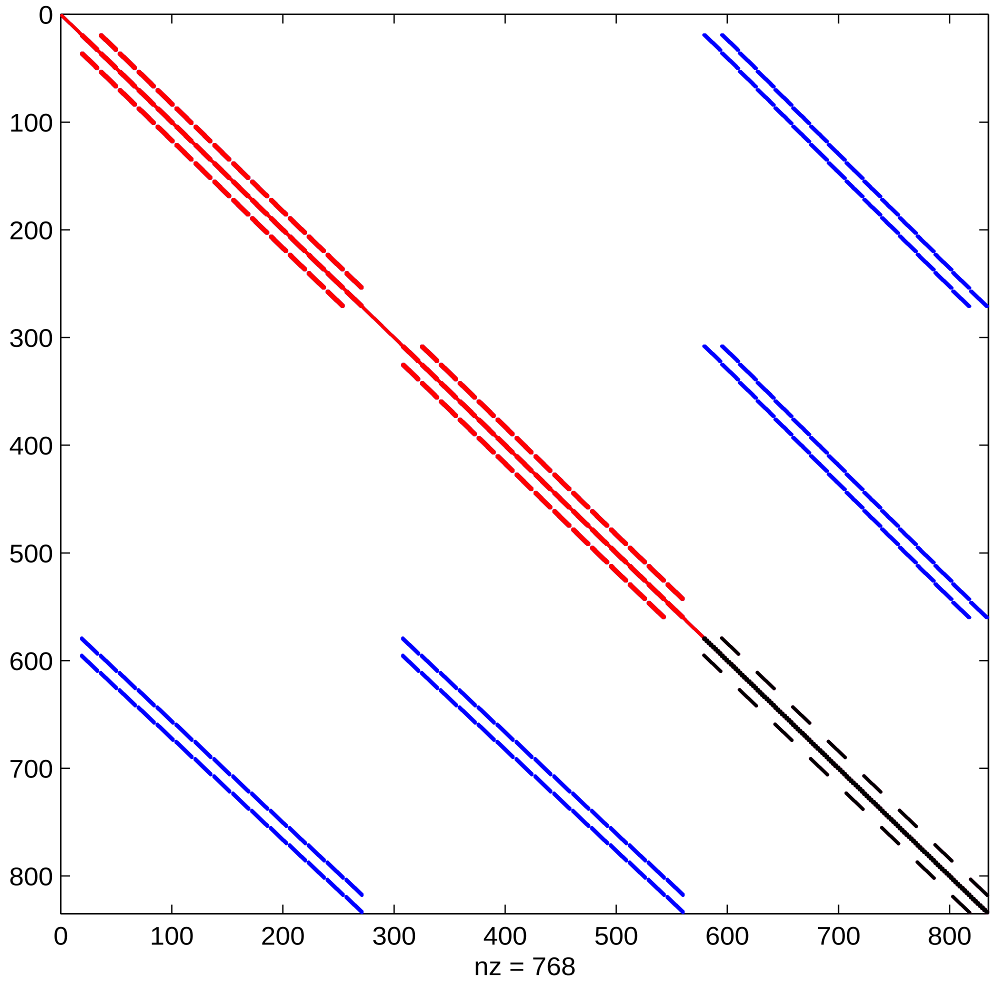
A sym,

$$\mathcal{M}x = b \qquad \mathcal{M} \quad \text{sym. indef.}$$

With n positive and m negative real eigenvalues

Typical Sparsity pattern (3D problem)



Spectral properties

- $\mathcal{M} = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}$ $\quad$ $0 < \lambda_n \leq \cdots \leq \lambda_1$ eigs of A
  $0 < \sigma_m \leq \cdots \leq \sigma_1$ sing. vals of B

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- (Rusten & Winther 1992) $\Lambda(\mathcal{M})$ subset of

$$\left[\frac{1}{2}(\lambda_n - \sqrt{\lambda_n^2 + 4\sigma_1^2}), , \frac{1}{2}(\lambda_1 - \sqrt{\lambda_1^2 + 4\sigma_m^2})\right] \quad \cup \quad \left[\lambda_n, \frac{1}{2}(\lambda_1 + \sqrt{\lambda_1^2 + 4\sigma_m^2})\right]$$

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- (Silvester & Wathen 1994), $0 \le \sigma_m \le \cdots \le \sigma_1$

$$\lambda_n - \lambda_{\max}(C) - \sqrt{(\lambda_n + \lambda_{\max}(C))^2 + 4\sigma_1^2}$$

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More results for special cases (e.g. Perugia & S. 2000)

General preconditioning strategy

- Find $\mathcal{P}$ such that

$$\mathcal{M}\mathcal{P}^{-1}\hat{u} = b \qquad \hat{u} = \mathcal{P}u$$

is easier (faster) to solve than $\mathcal{M}u = b$

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- A look at efficiency:
  - Dealing with $\mathcal{P}$ should be cheap
  - Storage for $\mathcal{P}$ should be low
  - Properties (algebraic/functional) exploited

Idealized case:

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⋆ A nonsing., $C = 0$:

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MINRES converges in at most 3 iterations

(Murphy, Golub & Wathen, 2002)

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⋆ A nonsing., $C \neq 0$:

$$\mathcal{P} = \begin{bmatrix} A & B \\ 0 & B^T A^{-1} B + C \end{bmatrix} \quad \Rightarrow \quad \mathcal{M}\mathcal{P}^{-1} = \begin{bmatrix} I & 0 \\ B^T A^{-1} & I \end{bmatrix}$$

GMRES converges in at most 2 iterations

Block diagonal Preconditioner

$$\mathcal{P} = \begin{bmatrix} \widetilde{A} & 0 \\ 0 & \widetilde{C} \end{bmatrix} \qquad \text{sym. pos. def.}$$

$$\widetilde{A} \approx A \qquad \widetilde{C} \approx BA^{-1}B^T + C$$

Rusten Winther (1992), Silvester Wathen (1993-1994), Klawonn (1998)
Fischer Ramage Silvester Wathen (1998...), . . .

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$\lambda \neq 0$ eigs of $\mathcal{P}^{-\frac{1}{2}}\mathcal{M}\mathcal{P}^{-\frac{1}{2}}$,

$$\lambda \in [-a, -b] \cup [c, d]$$

Constraint Preconditioner

$$\mathcal{Q} = \begin{bmatrix} \widetilde{A} & B \\ B^T & -C \end{bmatrix}$$

Axelsson (1979), Ewing Lazarov Lu Vassilevski (1990), Braess Sarazin (1997) Golub Wathen (1998) Vassilevski Lazarov (1996), Lukšan Vlček (1998-1999), Perugia S. Arioli (1999), Keller Gould Wathen (2000) Perugia S. (2000), Gould Hribar Nocedal (2001), Rozloznik S. (2002), Durazzi Ruggiero (2003), Axelsson Neytcheva (2003), ...

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$$\mathcal{Q}^{-1} = \begin{bmatrix} I & -B^T \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -(\mathbf{B}\mathbf{B}^\mathbf{T} + \mathbf{C})^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -B & I \end{bmatrix}$$

3D Magnetostatic problem

Size	QMR	QMR($\mathcal{Q}$)
1119	2368	15
2208	2825	13
4371	5191	17
8622	$>$10000	16
22675	$>$10000	25

Computational Considerations. II

3D Magnetostatic problem

$H \approx BB^T + C$ with H: Incomplete Cholesky fact.

(ICT package, Saad & Chow)

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Elapsed Time

Size	MA27	QMR $\mathcal{Q}$	QMR $\widehat{\mathcal{Q}}(2)(it)$	QMR ILDLT(10)	
1119	**0.6**	**3.0**	**1.7**(18)	**0.7**	
2208	**2.3**	**11.7**	**3.1**(18)	**1.5**	
4371	**10.2**	**64.6**	**8.4**(20)	**5.2**	
8622	**83.4**	**466.0**	**18.3**(29)	**31.0**	
22675	**753.5**	**3745.5**	**63.2**(45)	**246.0**	

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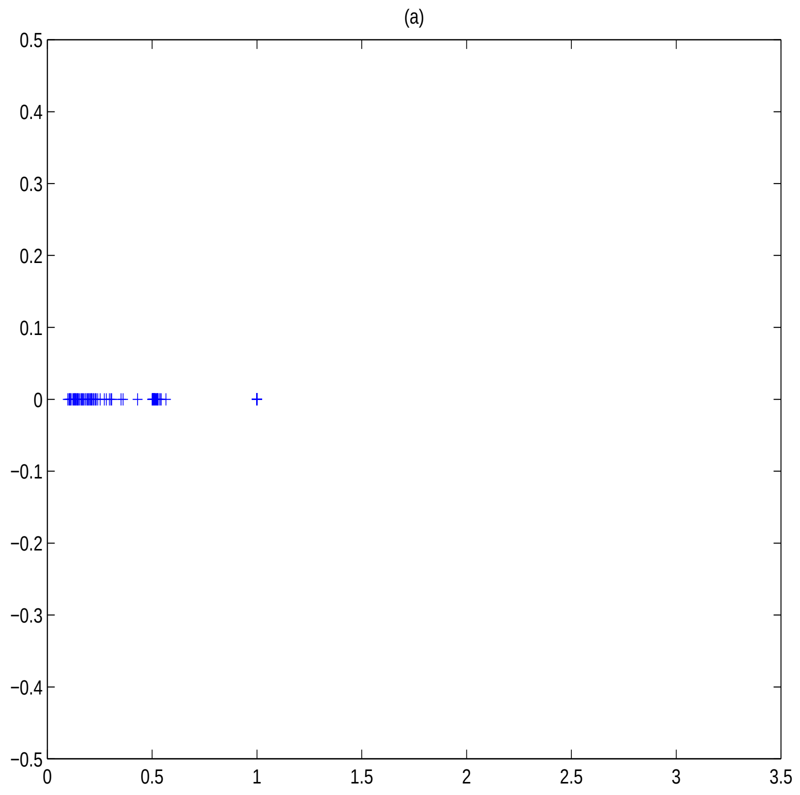
2208	**2.3**	**11.7**	**3.1**(18)	**1.5**	**15.2**	**4.9**
4371	**10.2**	**64.6**	**8.4**(20)	**5.2**	**73.9**	**11.0**
8622	**83.4**	**466.0**	**18.3**(29)	**31.0**	**510.1**	**24.3**
22675	**753.5**	**3745.5**	**63.2**(45)	**246.0**	**4161.4**	**128.2**

Spectrum of perturbed problem

3D Magnetostatic problem



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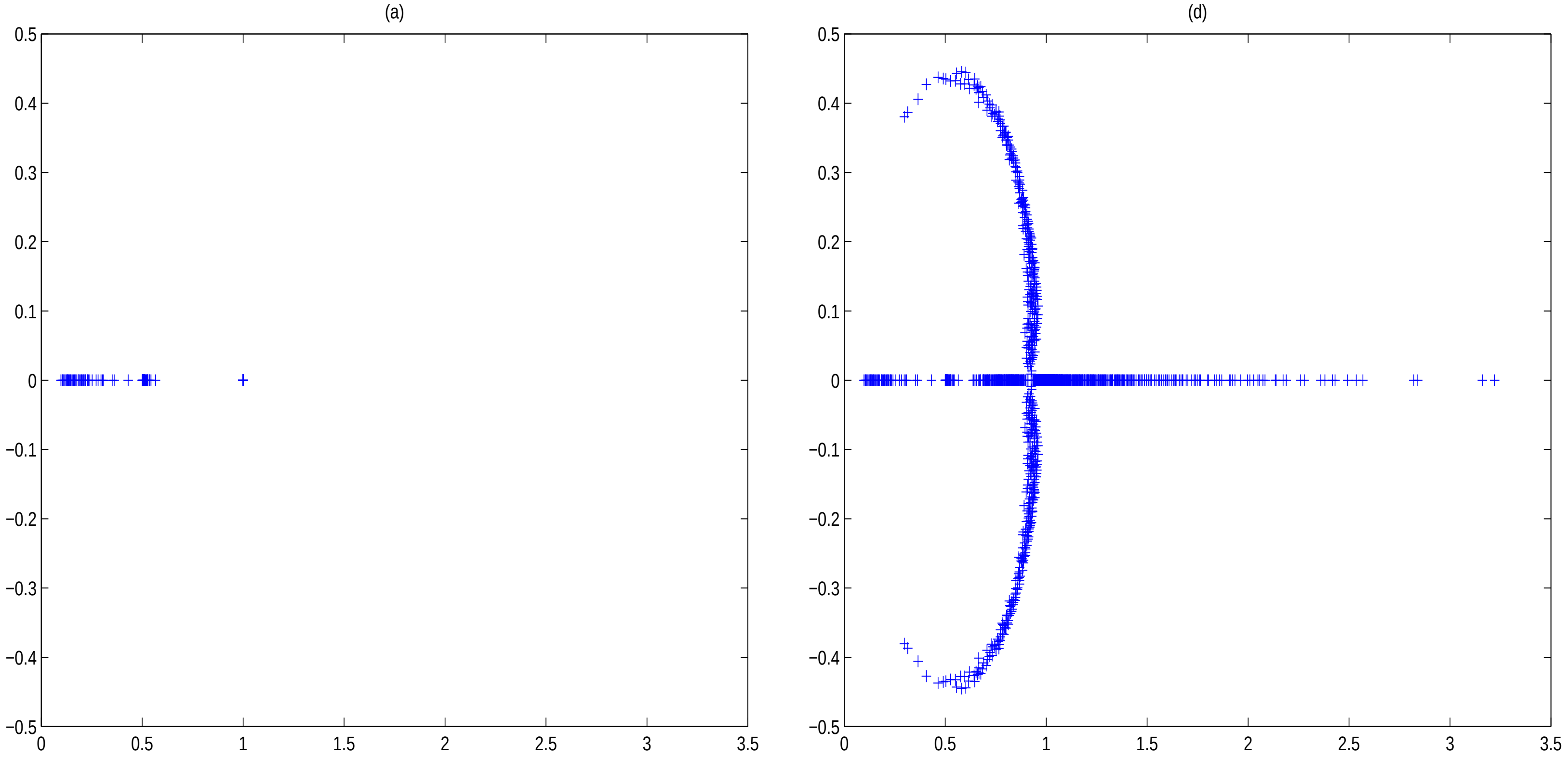

$$\|(BB^T + C) - H\|_\infty \approx 2 \cdot 10^{-1} \|BB^T + C\|_\infty$$

Triangular preconditioner

$$A \text{ spd}, \quad \mathcal{P} = \begin{bmatrix} \widetilde{A} & B^T \\ 0 & -\widetilde{C} \end{bmatrix}$$

Bramble & Pasciak, Elman, Klawonn, Axelsson & Neytcheva, S. 2004

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Spectrum of $\mathcal{M}\mathcal{P}^{-1}$:

small complex cluster around 1 + real interval

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More precisely: $\theta \in \Lambda(\mathcal{M}\mathcal{P}^{-1})$

$$\Im(\theta) \quad \neq 0 \quad \Rightarrow |\theta - 1| \le \sqrt{1 - \lambda_{\min}(A\widetilde{A}^{-1})}$$ (if $1 - \lambda_{\min}(A\widetilde{A}^{-1}) \ge 0$)

$$\Im(\theta) \quad = 0 \quad \Rightarrow \theta \in [\chi_1, \chi_2] \ni 1$$

A “different” perspective

$$\begin{bmatrix} A & B^T \\ -B & C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ -g \end{bmatrix} \qquad \mathcal{M}_- x = d$$

Polyak 1970, ..., Fischer & Ramage & Silvester & Wathen 1997, Bai & Golub & Ng 2003, Sidi 2003, Benzi & Gander & Golub 2003, Benzi & Golub 2004, S. & Benzi 2004, ...

$\mathcal{M}$ positive real $\quad \Rightarrow \quad \Lambda(\mathcal{M}_-)$ in $\mathbb{C}^+$

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- More refined spectral information possible

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- More refined spectral information possible
- New classes of preconditioners

A "different" perspective

$$\begin{bmatrix} A & B^T \\ -B & C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ -g \end{bmatrix} \qquad \mathcal{M}_{-} x = d$$

Polyak 1970, ..., Fischer & Ramage & Silvester & Wathen 1997, Bai & Golub & Ng 2003, Sidi 2003, Benzi & Gander & Golub 2003, Benzi & Golub 2004, S. & Benzi 2004, ...

$\mathcal{M}$ positive real $\quad \Rightarrow \quad \Lambda(\mathcal{M}_{-})$ in $\mathbb{C}^{+}$

- More refined spectral information possible
- New classes of preconditioners
- General framework for spectral analysis of some indefinite preconditioners

Spectral properties of $\mathcal{M}_-$

$$\mathcal{M}_- = \begin{bmatrix} A & B^T \\ -B & C \end{bmatrix}$$

A $n \times n$ sym. semidef. matrix, B $m \times n$, $m \leq n$

$\mathcal{M}_-$ has at least $n - m$ real eigenvalues

Reality condition

Let $C = \beta I$. If

$$\lambda_{\min}(A + \beta I) \geq 4\,\lambda_{\max}(B^T A^{-1} B + \beta I),$$

then all eigenvalues of $\mathcal{M}_-$ are real.

e.g. Stokes problem $(C = 0)$ Benzi & S. (in prep.)

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Moreover, if X is s.t. $\mathcal{M}_- = X \Lambda X^{-1}$, then

$$\kappa(X) \leq \frac{\|X\|^2}{\sqrt{1-\chi}}, \qquad \chi = \frac{4\,\lambda_{\max}(B^T A^{-1} B + \beta I)}{\lambda_{\min}(A + \beta I)}$$

(columns of X have unit norm)

Location of spectrum

Let $\lambda \in \Lambda(\mathcal{M}_-)$, A spd (*cf. Sidi 2003 for* $C = 0$)

$\star$ If $\Im(\lambda) \neq 0$ then

$$\begin{aligned} \frac{1}{2}(\lambda_{\min}(A) + \lambda_{\min}(C)) \leq \quad \Re(\lambda) \quad &\leq \frac{1}{2}(\lambda_{\max}(A) + \lambda_{\max}(C)) \\ |\Im(\lambda)| \quad &\leq \sigma_{\max}(B). \end{aligned}$$

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$\star$ If $\Im(\lambda) = 0$ then

$$\min\{\lambda_{\min}(A), \lambda_{\min}(C)\} \leq \lambda \leq (\lambda_{\max}(A) + \lambda_{\max}(C)).$$

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$$\mathcal{M}_- = \left(\begin{array}{cc|c} 2 & 0 & 1 \\ 0 & 1 & 0 \\ \hline -1 & 0 & 1 \end{array}\right) \qquad \lambda_1 = 1, \quad \lambda_{2,3} = \frac{3}{2} \pm i\frac{\sqrt{3}}{2}.$$

Hermitian Skew-Hermitian Preconditioning

$$\mathcal{M}_- = \begin{bmatrix} A & B^T \\ -B & C \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} + \begin{bmatrix} 0 & B^T \\ -B & 0 \end{bmatrix}$$

$$= \quad \mathcal{H} \quad + \quad \mathcal{S}$$

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$$= \quad \mathcal{H} \quad + \quad \mathcal{S}$$

Use the preconditioner

$$\mathcal{R}_\alpha = \frac{1}{2\alpha}(\mathcal{H} + \alpha I)(\mathcal{S} + \alpha I) \qquad \alpha \in \mathbb{R},\ \alpha > 0$$

Bai & Golub & Ng 2003, Benzi & Gander & Golub 2003,

Benzi & Golub 2004, S. & Benzi 2004, Benzi & Ng, 2004

Reality condition

Assume A is sym. positive definite, $C = 0$. If

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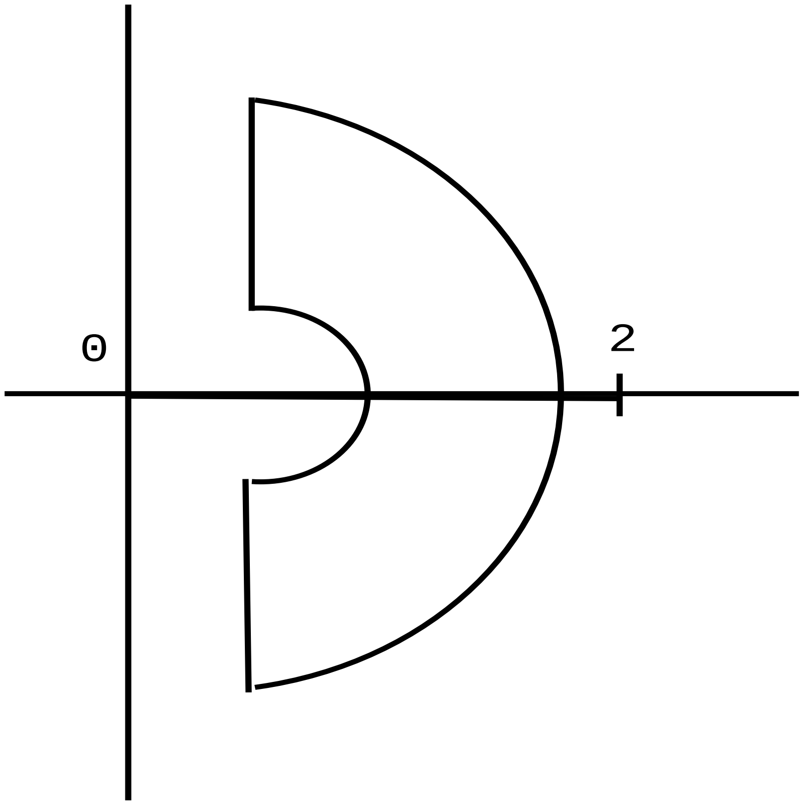
We provide bounds for real and imaginary part of eigenvalues

Stokes Problem

α	Lower bound	$\eta_{\min}$	$\eta_{\max}$	Upper bound
0.001	0.00048902	0.00050629	1.9999	1.9999
0.01	0.00111635	0.00169724	1.9999	1.9999
0.1	0.00014289	0.00022355	1.9929	1.9929
0.5	0.00002866	0.00004485	1.8150	1.8154
0.8	0.00001791	0.00002803	1.6871	1.6880
1.0	0.00001433	0.00002243	1.6137	1.6147

Spectrum of $\mathcal{R}_\alpha^{-1}\mathcal{M}_-$ (case $C = 0$)



(S. & Benzi 2004)

A General framework. I

- Eigenvalue problem

$$\begin{bmatrix} A & B^T \\ -B & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} I & 0^T \\ 0 & -I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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- Generalization

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} K & 0^T \\ 0 & -H \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad K, H \quad \text{spd}$$

A General Framework. II

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \mathbf{z} = \lambda \mathcal{P} \mathbf{z} \quad \longrightarrow \quad \begin{bmatrix} \widehat{A} & \widehat{B}^T \\ \widehat{B} & -\widehat{C} \end{bmatrix} \mathbf{w} = \lambda \begin{bmatrix} I & 0^T \\ 0 & -I \end{bmatrix} \mathbf{w}$$

$$\widehat{A} \geq 0, \ \widehat{C} \geq 0$$

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$$\widehat{A} \geq 0, \ \widehat{C} \geq 0$$

- Indefi nite Block diagonal Preconditioner

$$\mathcal{P} = \begin{bmatrix} \widehat{A} & 0 \\ 0 & -\widehat{C} \end{bmatrix}$$

cf. Fischer et al. 1997

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$$\widehat{A} \geq 0, \;\; \widehat{C} \geq 0$$

- Indefi nite Block diagonal Preconditioner

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cf. Fischer et al. 1997

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- Hermitian Skew-Hermitian Preconditioner ($C = \beta I$)
- Inexact Constraint Preconditioner

$$\mathcal{Q} = \begin{bmatrix} I & 0 \\ B & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -\mathbf{H} \end{bmatrix} \begin{bmatrix} I & B^T \\ 0 & I \end{bmatrix} \qquad H \approx BB^T + C$$

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