



The Sylvester matrix equation: computation and sparsity properties

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Some matrix equations

- Sylvester matrix equation

$$AX + XB + D = 0$$

Eigenvalue pbs and tracking, Control, MOR, Assignment pbs, Riccati eqn

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- Lyapunov matrix equation

$$A\mathbf{X} + \mathbf{X}A^{\top} + D = 0, \quad D = D^{\top}$$

Stability analysis in Control and Dynamical systems, Signal processing, eigenvalue computations

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Bini-Iannazzo-Meini '12

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Focus: All or some of the matrices are large (and possibly sparse)

The Lyapunov operator

$$\mathcal{L} : X \mapsto AX + XA^\top \quad \text{or} \quad \ell : x \mapsto (I \otimes A + A \otimes I)x$$

- In linear matrix equations: Computational aspects
- A mathematical tool

Solving the Lyapunov equation. The problem

Approximate X in:

$$AX + XA^T + BB^T = 0$$

$$A \in \mathbb{R}^{n \times n} \text{ neg.real} \quad B \in \mathbb{R}^{n \times p}, \quad 1 \leq p \ll n$$

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Time-invariant linear system:

$$\mathbf{x}'(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \quad \mathbf{x}(0) = x_0$$

Closed form solution:

$$X = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\omega I + A)^{-1} BB^\top (i\omega I + A)^{-H} d\omega$$

$\Rightarrow X$ symmetric semidef.

see, e.g., Antoulas '05, Benner '06

Linear systems vs linear matrix equations

Large linear systems:

$$Ax = b, \quad A \in \mathbb{R}^{n \times n}$$

- Krylov subspace methods (CG, MINRES, GMRES, BiCGSTAB, etc.)
- Preconditioners: find P such that

$$AP^{-1}\tilde{x} = b \quad x = P^{-1}\tilde{x}$$

is **easier** and **fast** to solve

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Large linear matrix equations:

$$AX + XA^\top + BB^\top = 0$$

- No preconditioning - to preserve symmetry
- X is a large, dense matrix \Rightarrow low rank approximation

$$X \approx \tilde{X} = ZZ^\top, \quad Z \text{ tall}$$

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Large linear matrix equations:

$$AX + XA^\top + BB^\top = 0$$

Kronecker formulation:

$$(A \otimes I + I \otimes A)x = b \quad x = \text{vec}(X)$$

Projection-type methods

Given an approximation space \mathcal{K} ,

$$X \approx X_m \quad \text{col}(X_m) \in \mathcal{K}$$

Galerkin condition: $R := AX_m + X_m A^\top + BB^\top \perp \mathcal{K}$

$$V_m^\top R V_m = 0 \quad \mathcal{K} = \text{Range}(V_m)$$

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Assume $V_m^\top V_m = I_m$ and let $X_m := V_m Y_m V_m^\top$.

Projected Lyapunov equation:

$$V_m^\top (AV_m Y_m V_m^\top + V_m Y_m V_m^\top A^\top + BB^\top) V_m = 0$$

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$$\begin{aligned} V_m^\top (AV_m Y_m V_m^\top + V_m Y_m V_m^\top A^\top + BB^\top) V_m &= 0 \\ (V_m^\top AV_m) Y_m + Y_m (V_m^\top A^\top V_m) + V_m^\top BB^\top V_m &= 0 \end{aligned}$$

Early contributions: Saad '90, Jaimoukha & Kasenally '94, for

$$\mathcal{K} = \mathcal{K}_m(A, B) = \text{Range}([B, AB, \dots, A^{m-1}B])$$

More recent options as approximation space

Enrich space to decrease space dimension

- Extended Krylov subspace

$$\mathcal{K} = \mathcal{K}_m(A, B) + \mathcal{K}_m(A^{-1}, A^{-1}B),$$

that is, $\mathcal{K} = \text{Range}([B, A^{-1}B, AB, A^{-2}B, A^2, A^{-3}B, \dots,])$

(Druskin & Knizhnerman '98, Simoncini '07)

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- Rational Krylov subspace

$$\mathcal{K} = \text{Range}([B, (A - s_1 I)^{-1}B, \dots, (A - s_m I)^{-1}B])$$

usually, $\{s_1, \dots, s_m\} \subset \mathbb{C}^+$ chosen a-priori

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In both cases, for $\text{Range}(V_m) = \mathcal{K}$, **projected Lyapunov equation:**

$$(V_m^\top A V_m) Y_m + Y_m (V_m^\top A^\top V_m) + V_m^\top B B^\top V_m = 0$$

$$X_m = V_m Y_m V_m^\top$$

Rational Krylov Subspaces. A long tradition...

In general,

$$K_m(A, B, \mathbf{s}) = \text{Range}([(A-s_1I)^{-1}B, (A-s_2I)^{-1}B, \dots, (A-s_mI)^{-1}B])$$

- Eigenvalue problems (Ruhe, 1984)
- Model Order Reduction (transfer function evaluation)
- In Alternating Direction Implicit iteration (ADI) for linear matrix equations

The Lyapunov operator

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- In linear matrix equations: Computational aspects
- **A mathematical tool**

The Poisson equation - revisited

$$-u_{xx} - u_{yy} = f, \quad \text{in } \Omega = (0, 1)^2$$

+ Dirichlet b.c. (zero b.c. for simplicity)

$$\text{Usual discretization} \Rightarrow Au = f$$

Discretization: $U_{i,j} \approx u_{x_i, y_j}$, with (x_i, y_j) interior nodes, so that

$$u_{xx}(x_i, y_j) \approx \frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h^2} = \frac{1}{h^2} [1, -2, 1] \begin{bmatrix} U_{i-1,j} \\ U_{i,j} \\ U_{i+1,j} \end{bmatrix}$$

$$u_{yy}(x_i, y_j) \approx \frac{U_{i,j-1} - 2U_{i,j} + U_{i,j+1}}{h^2} = \frac{1}{h^2} [U_{i,j-1}, U_{i,j}, U_{i,j+1}] \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

h : meshsize

The Poisson equation - matrix formulation

Let $T = \frac{1}{h^2} \text{tridiag}(-1, 2, -1)$

$$u_{xx}(x_i, y_j) \approx \frac{1}{h^2} [1, -2, 1] \begin{bmatrix} U_{i-1,j} \\ U_{i,j} \\ U_{i+1,j} \end{bmatrix} \quad u_{yy}(x_i, y_j) \approx \frac{1}{h^2} [U_{i,j-1}, U_{i,j}, U_{i,j+1}] \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Collecting all nodes together,

$$-u_{xx} \approx TU, \quad -u_{yy} \approx UT$$

Therefore,

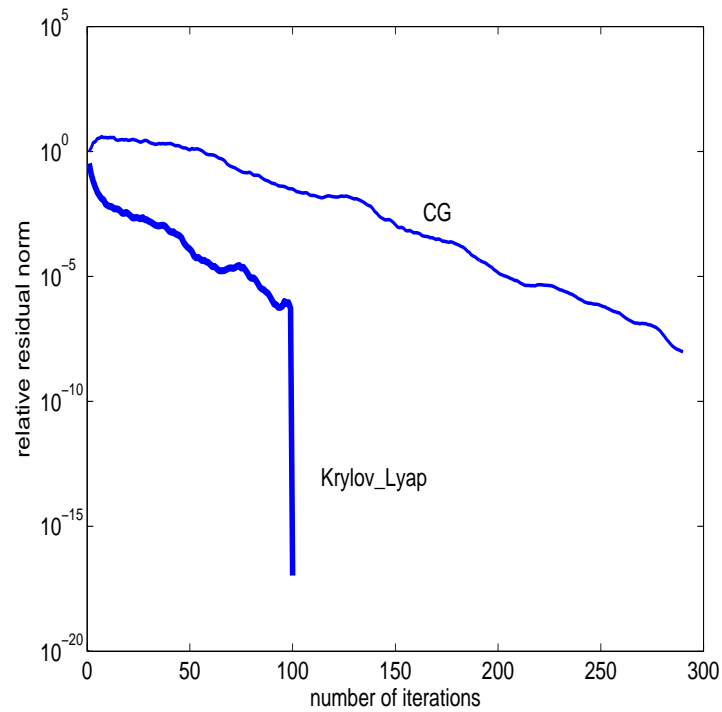
$$-u_{xx} - u_{yy} = f \quad \Rightarrow \quad TU + UT = F$$

(here $F_{ij} = f(x_i, y_j)$)

To be compared with $Au = b$ (here $A = T \otimes I + I \otimes T$, and $b = \text{vec}(F)$)

CG for $Ax = b$ vs Krylov solver for $TU + UT = F$

$$T \in \mathbb{R}^{n \times n}, A \in \mathbb{R}^{n^2 \times n^2}, \quad n = 100$$



For $\text{tol} = 10^{-6}$, Elapsed time: $\text{CG} \approx 0.8$, $\text{Krylov} \approx 0.4$

```
>> tic;lyap(T,F);toc
Elapsed time is 0.019335 seconds.
>> tic;A\b;toc
Elapsed time is 0.030765 seconds.
```

... A classical approach

Matrix formulation is not new...

- Bickley & McNamee, 1960: Early literature on difference equations
- Wachspress, 1963: Model problem for ADI algorithm
- Ellner & Wachspress (1980's): interplay between the matrix and vector formulations (via preconditioning)

The stiffness matrix

$$S := M \otimes I_n + I_n \otimes M,$$

with M symmetric and positive definite, banded with bandwidth b

- Finite differences: M is second order operator in one space dimension ($b = 1$)
 \Rightarrow for instance, S : 2D Laplace operator in $[a, b]^2$
- Legendre Spectral methods: M spd, nonconstant ($b = 1$)
- ...

More generally,

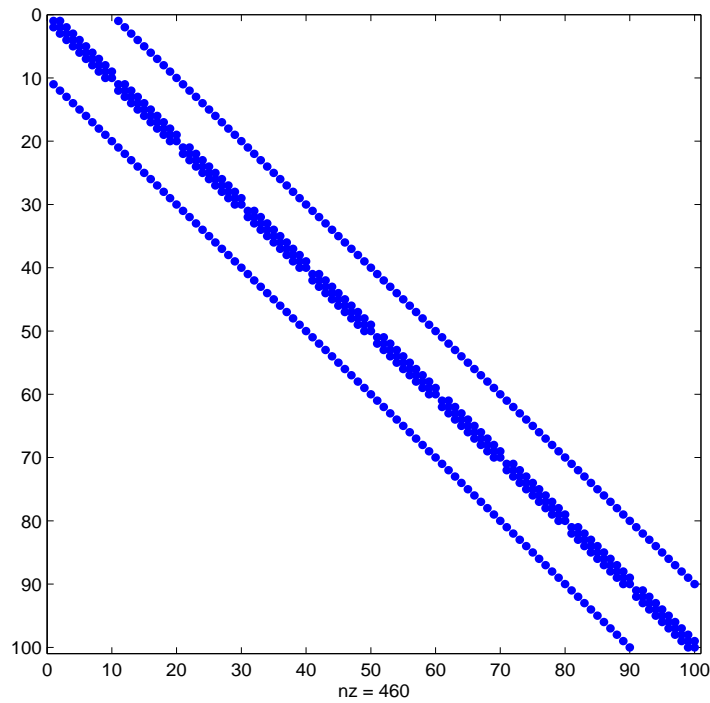
$$S_g := M_1 \otimes I_n + I_n \otimes M_2,$$

with $M_1 \neq M_2$, banded, with not necessarily the same dimensions

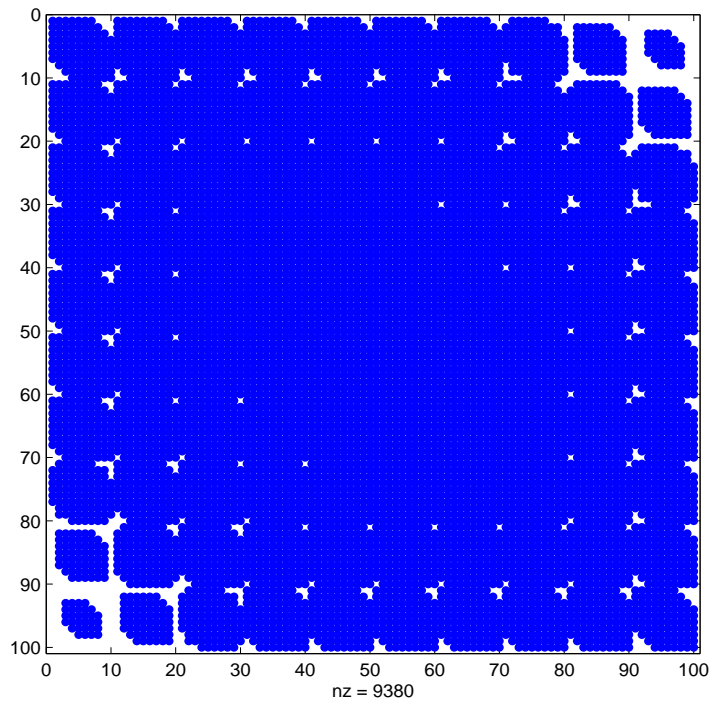
The inverse of the 2D Laplace matrix on the unit square

$$S := M \otimes I_n + I_n \otimes M, \quad M = \text{tridiag}(-1, 2, -1)$$

Sparsity pattern:



Matrix S

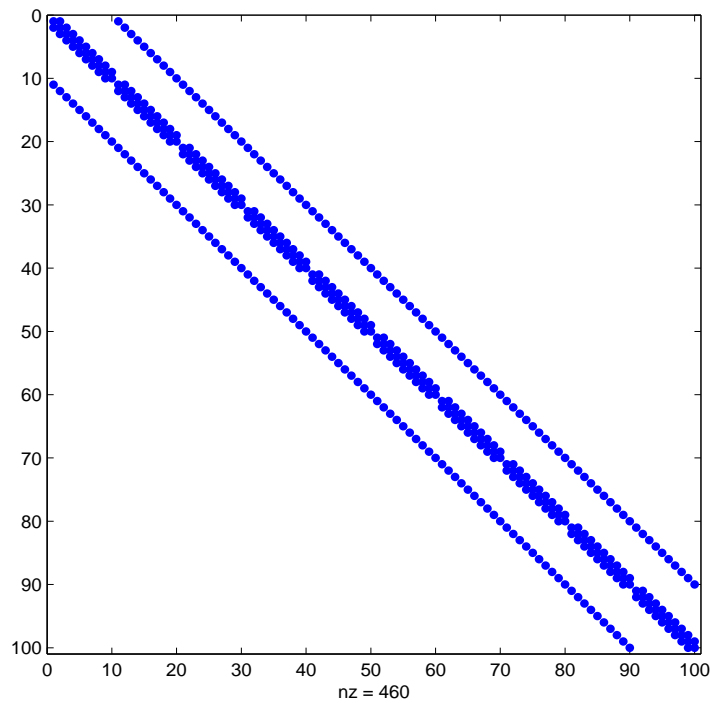


S^{-1}

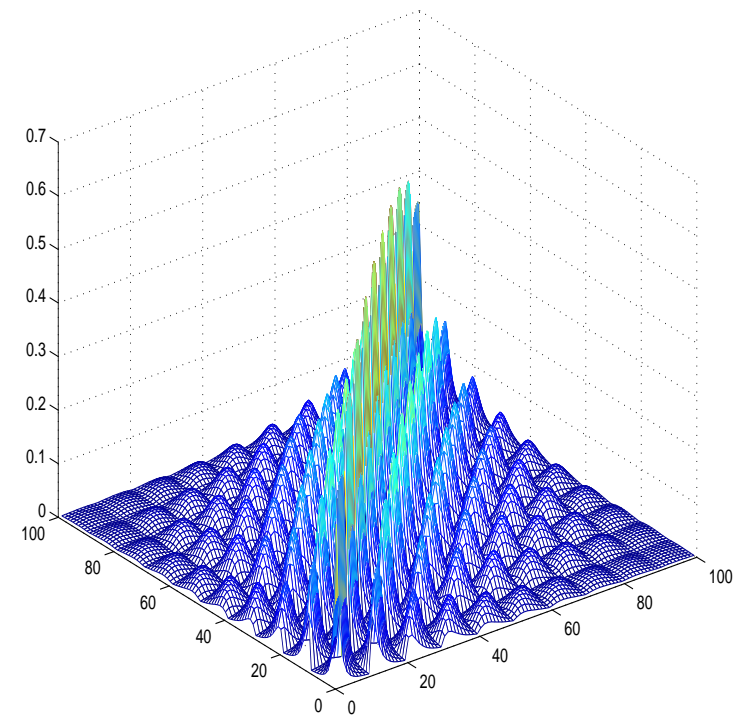
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Sparsity pattern:



S



$|((S^{-1})_{ij})|$

The exponential decay of the entries of S^{-1}

The classical bound (Demko, Moss & Smith):

If S spd is banded with bandwidth b , then

$$|(S^{-1})_{ij}| \leq \gamma q^{\frac{|i-j|}{b}}$$

where

κ : condition number of S

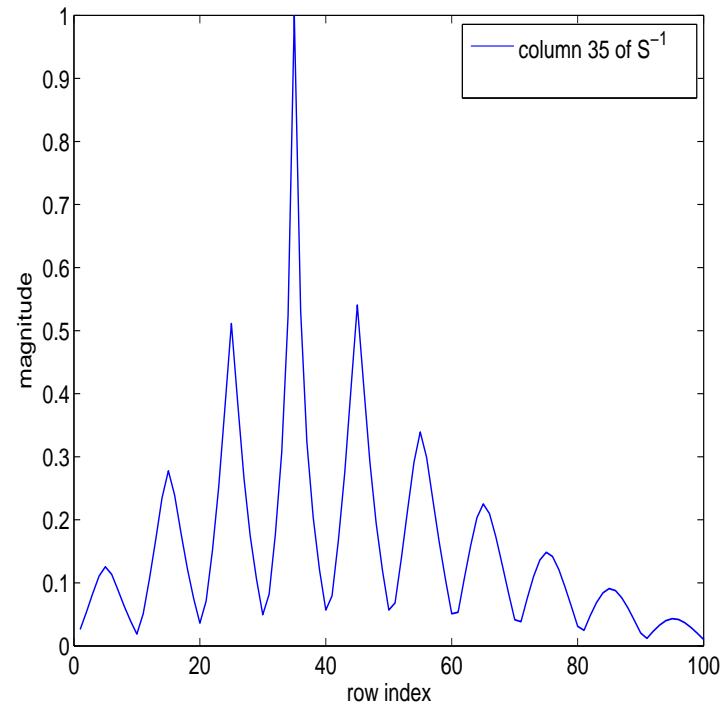
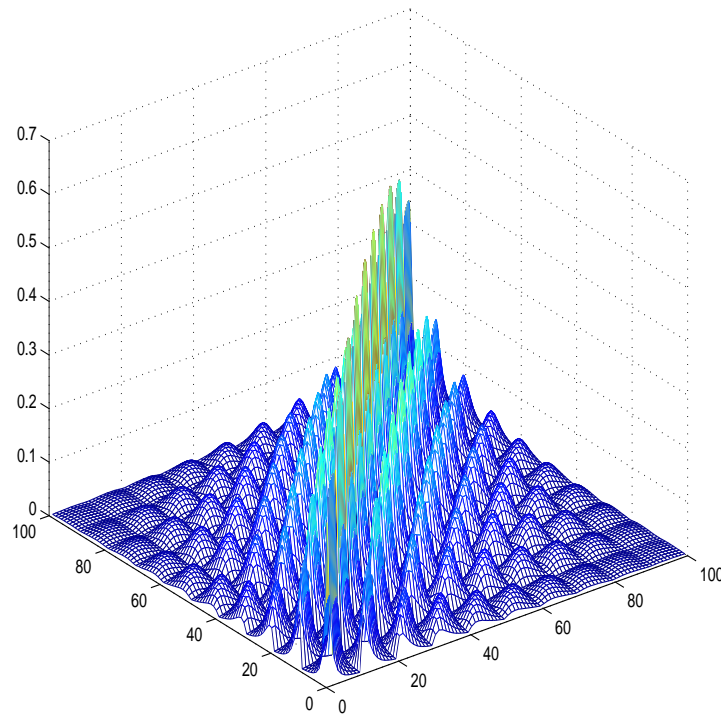
$$q := \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} < 1$$

$$\gamma := \max\{\lambda_{\min}(S)^{-1}, \hat{\gamma}\}, \text{ and } \hat{\gamma} = \frac{(1 + \sqrt{\kappa})^2}{2\lambda_{\max}(S)}$$

($\lambda_{\min}(\cdot)$, $\lambda_{\max}(\cdot)$ smallest and largest eigenvalues of the given symmetric matrix)

Many contributions: Bebendorf, Hackbusch, Benzi, Boito, Razouk, Golub, Tuma, Concus, Meurant, Mastronardi, Ng, Tyrtshnikov, Nabben, ...

The true decay



... a very peculiar pattern

⇒ much higher sparsity

Where do the repeated peaks come from?

For $S = M \otimes I_n + I_n \otimes M \in \mathbb{R}^{n^2 \times n^2}$:

$$x_t := (S^{-1})_{:,t} = S^{-1}e_t \quad \Leftrightarrow \quad \text{Solve : } Sx_t = e_t$$

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Let

$X_t \in \mathbb{R}^{n \times n}$ be such that $x_t = \text{vec}(X_t)$

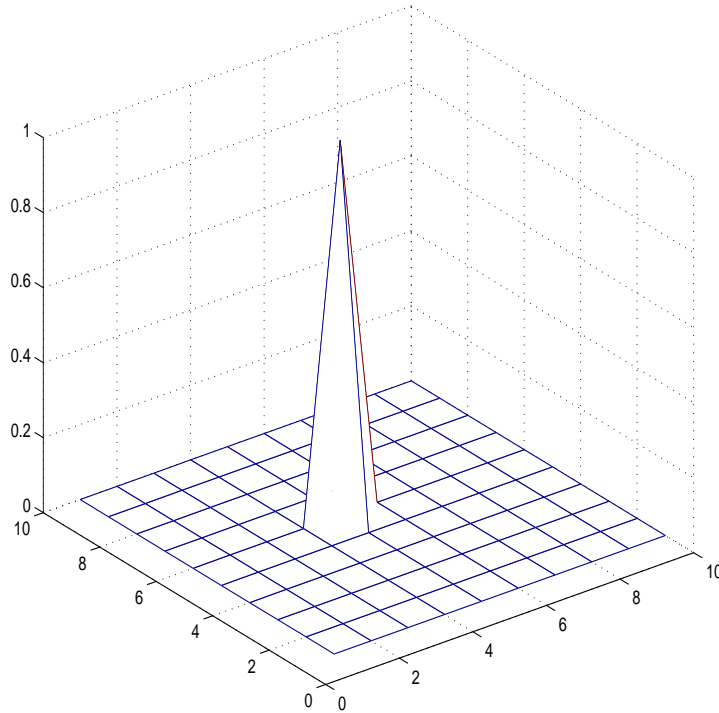
$E_t \in \mathbb{R}^{n \times n}$ be such that $e_t = \text{vec}(E_t)$

Then

$$Sx_t = e_t \quad \Leftrightarrow \quad MX_t + X_tM = E_t$$

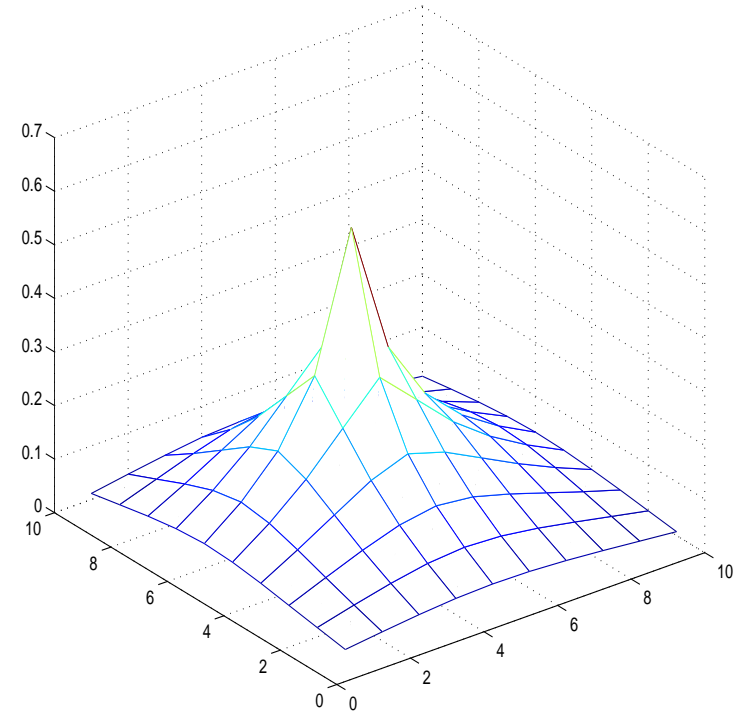
For S the 2D Laplace operator, $t = 1, \dots, n^2$

$$t = 35, \quad Sx_t = e_t \quad \Leftrightarrow \quad MX_t + X_tM = E_t$$



matrix E_t

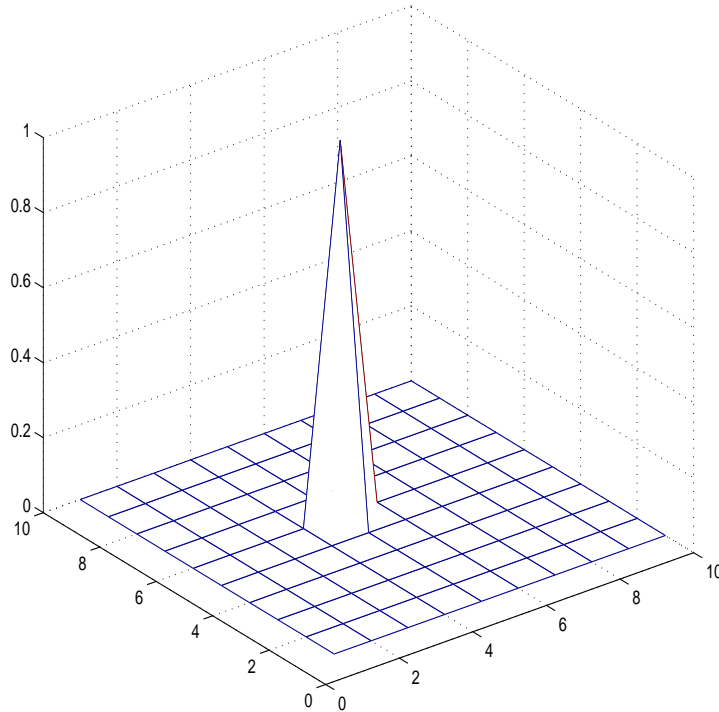
and



matrix X_t

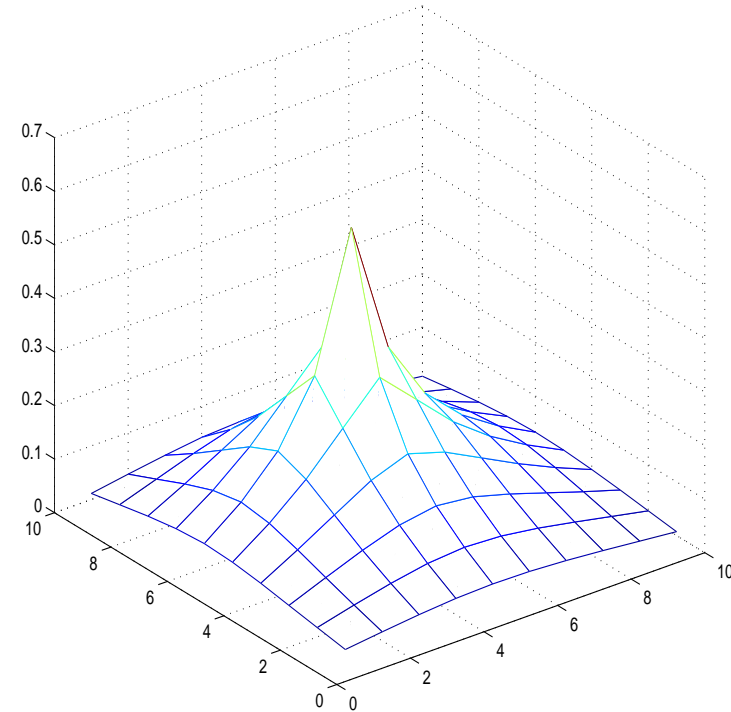
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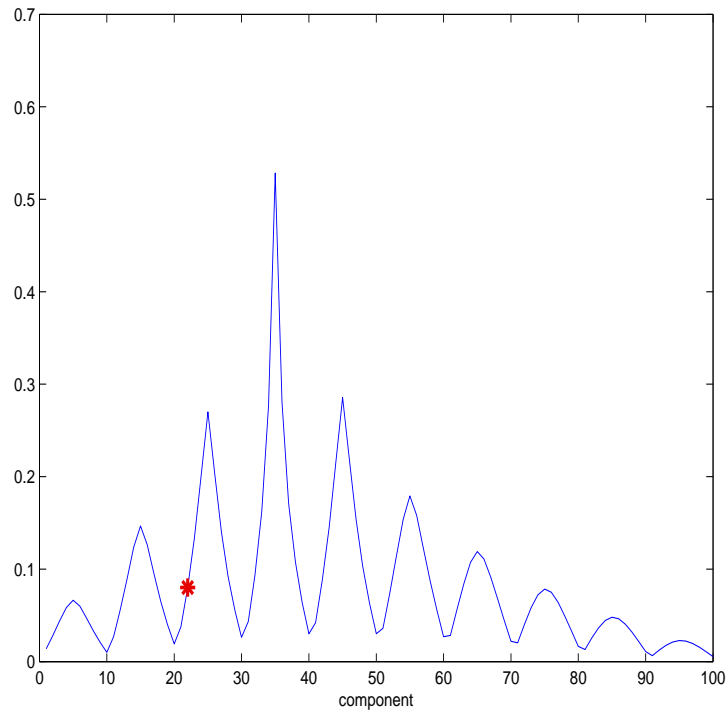
and



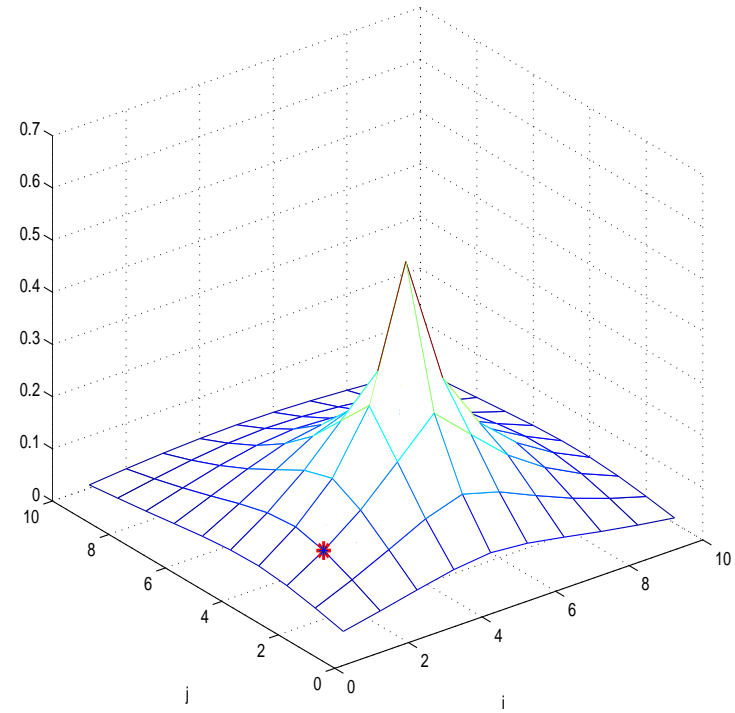
matrix X_t

E_t has only one nonzero element

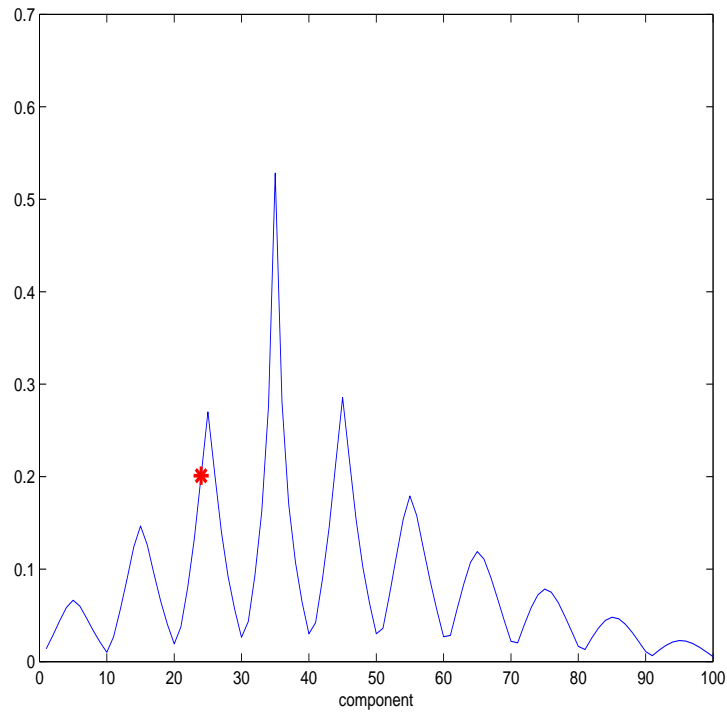
Lexicographic order: $(E_t)_{ij}$, $j = \lfloor (t-1)/n \rfloor + 1$, $i = tn \lfloor (t-1)/n \rfloor$



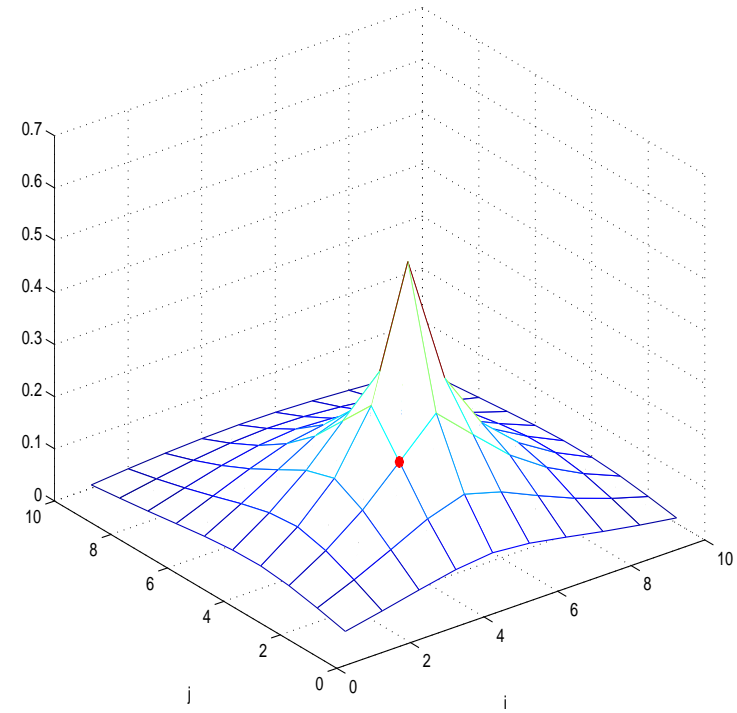
Left: Row of S^{-1}



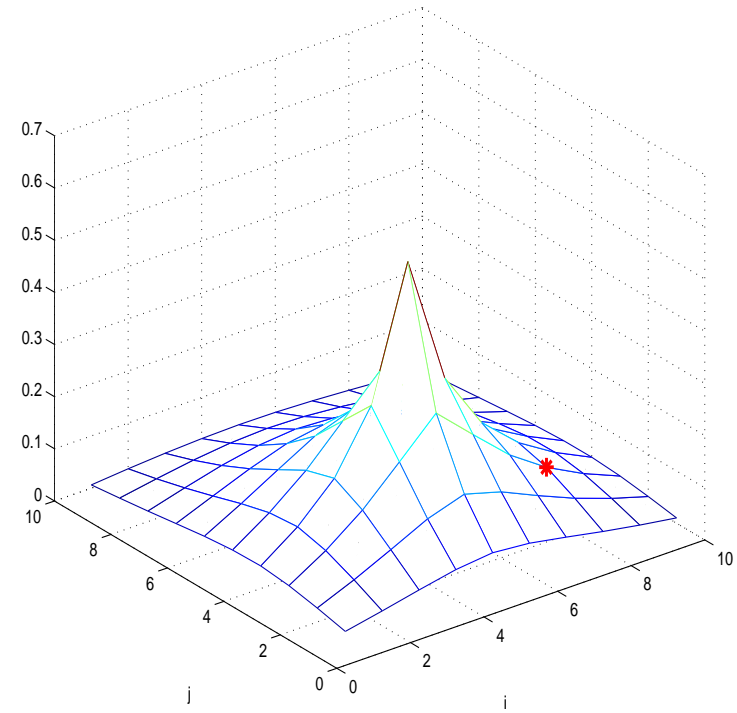
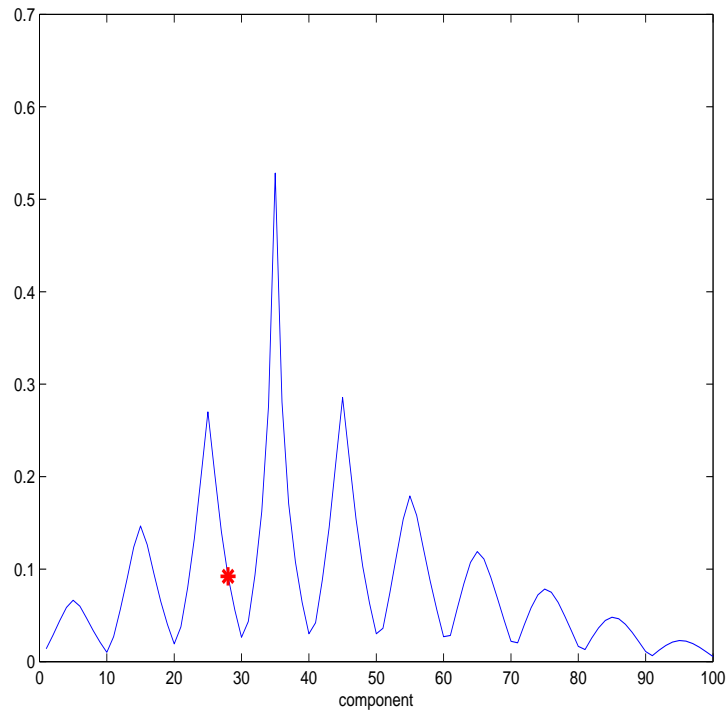
Right: same row on the grid



Left: Row of S^{-1}

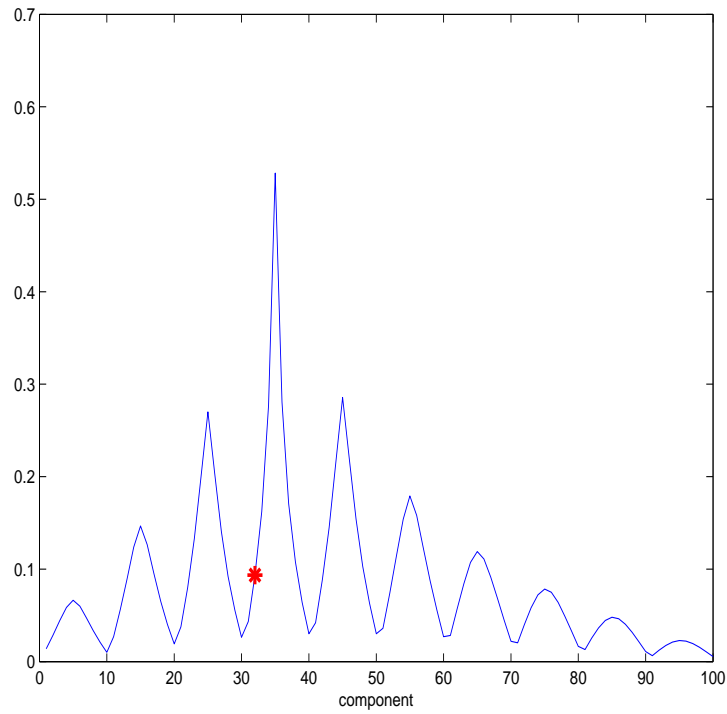


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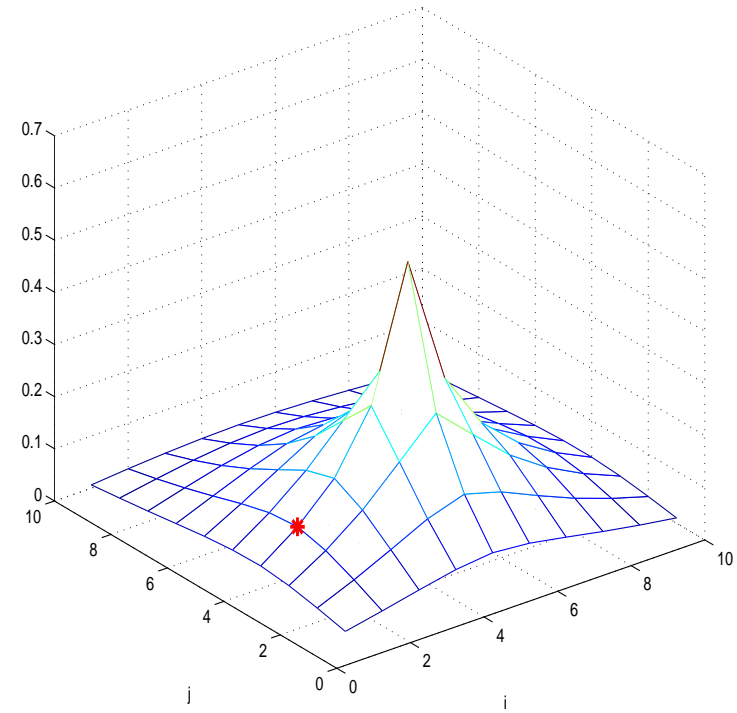


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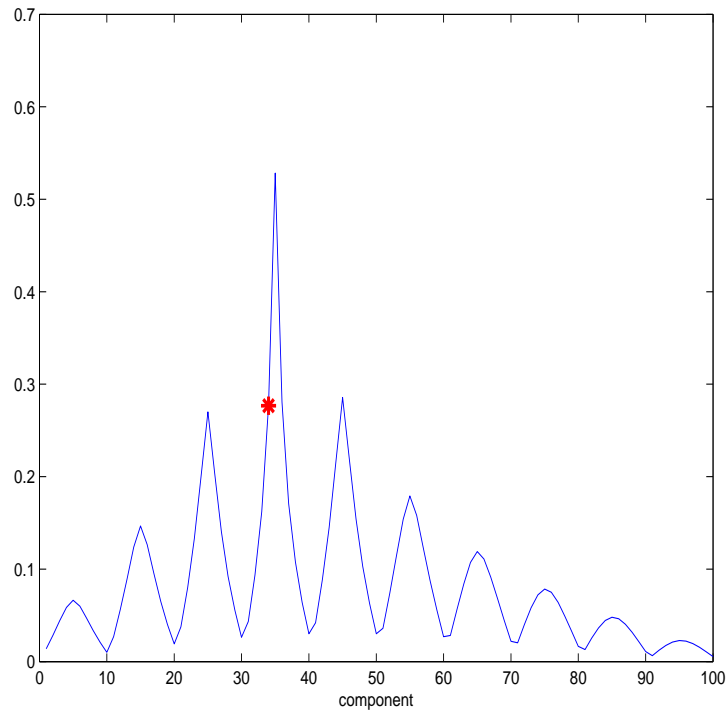
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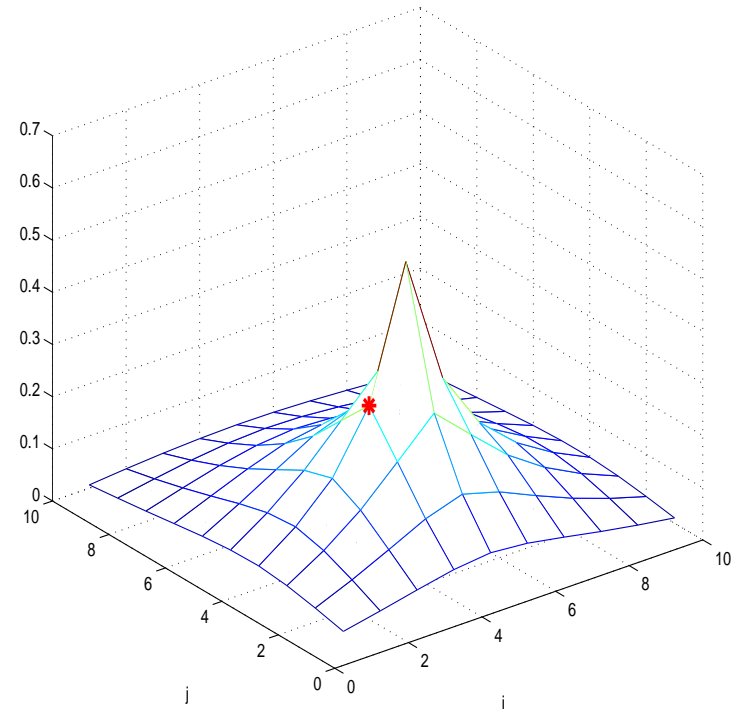
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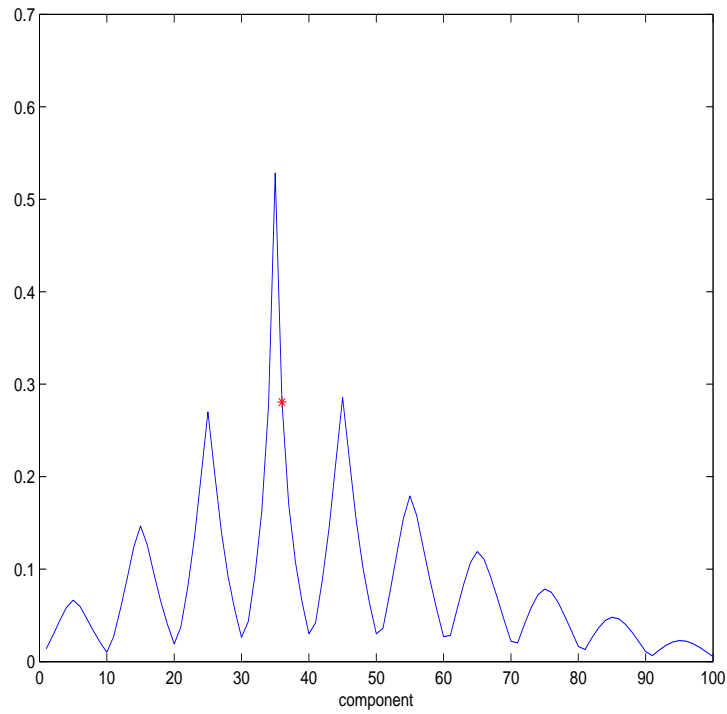
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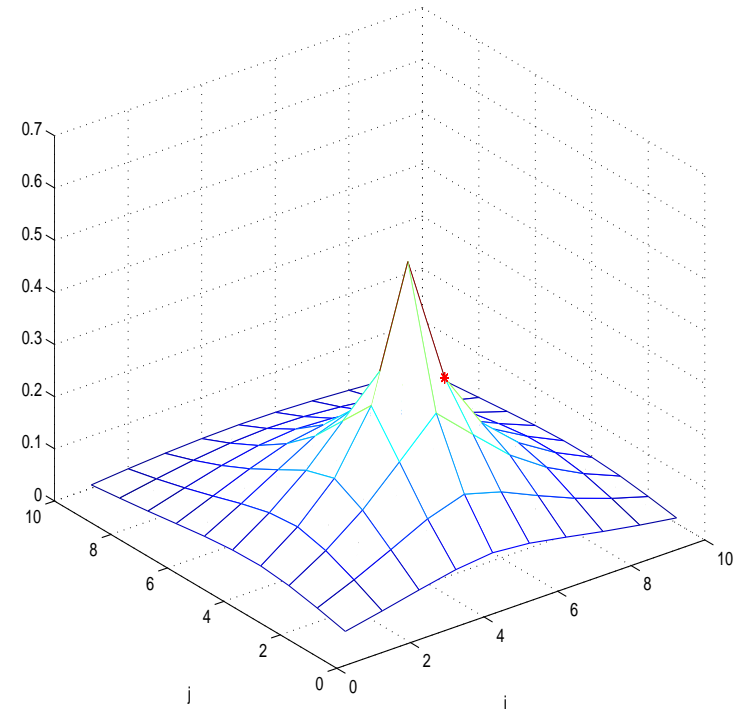
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Resolving the entry indexing using $MX_t + X_tM = E_t$

$$(S^{-1})_{k,t} = (S^{-1})_{\ell+n(m-1),t} = e_\ell^\top X_t e_m, \quad \ell, m \in \{1, \dots, n\}$$

\Rightarrow All the elements of the t -th column, $(S^{-1})_{:,t}$, are obtained by varying $m, \ell \in \{1, \dots, n\}$

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From the Lyapunov equation theory,

$$X_t = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\omega I + M)^{-1} E_t (i\omega I + M)^{-*} d\omega$$

with $E_t = e_i e_j^\top$, $j = \lfloor (t-1)/n \rfloor + 1$, $i = t - n \lfloor (t-1)/n \rfloor$

Therefore,

$$e_\ell^\top X_t e_m = \frac{1}{2\pi} \int_{-\infty}^{\infty} e_\ell^\top (i\omega I + M)^{-1} e_i e_j^\top (i\omega I + M)^{-*} e_m d\omega$$

Qualitative bounds

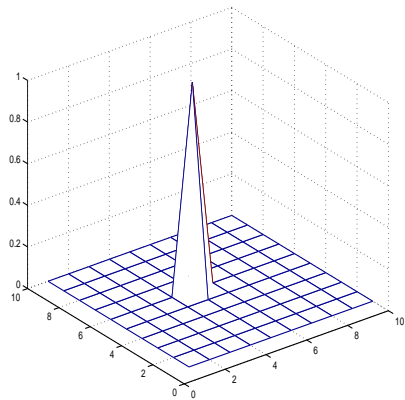
Let $\kappa = \lambda_{\max}/\lambda_{\min} = \text{cond}(M)$

i) Assume $\ell, i, m, j : \ell \neq i, m \neq j$. $\mathbf{n}_2 := |\ell - i| + |m - j| - 2 > 0$

$$|(S^{-1})_{k,t}| \leq \frac{\sqrt{\kappa^2 + 1}}{2\lambda_{\min}} \frac{1}{\sqrt{\mathbf{n}_2}}.$$

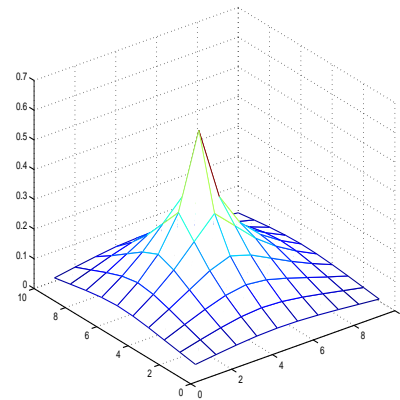
ii) Assume $\ell, i, m, j : \ell = i$ or $m = j$. $\mathbf{n}_1 := |\ell - i| + |m - j| - 1 > 0$

$$|(S^{-1})_{k,t}| \leq \frac{\kappa\sqrt{\kappa^2 + 1}}{2} \frac{1}{\sqrt{\mathbf{n}_1}}.$$



(i, j)

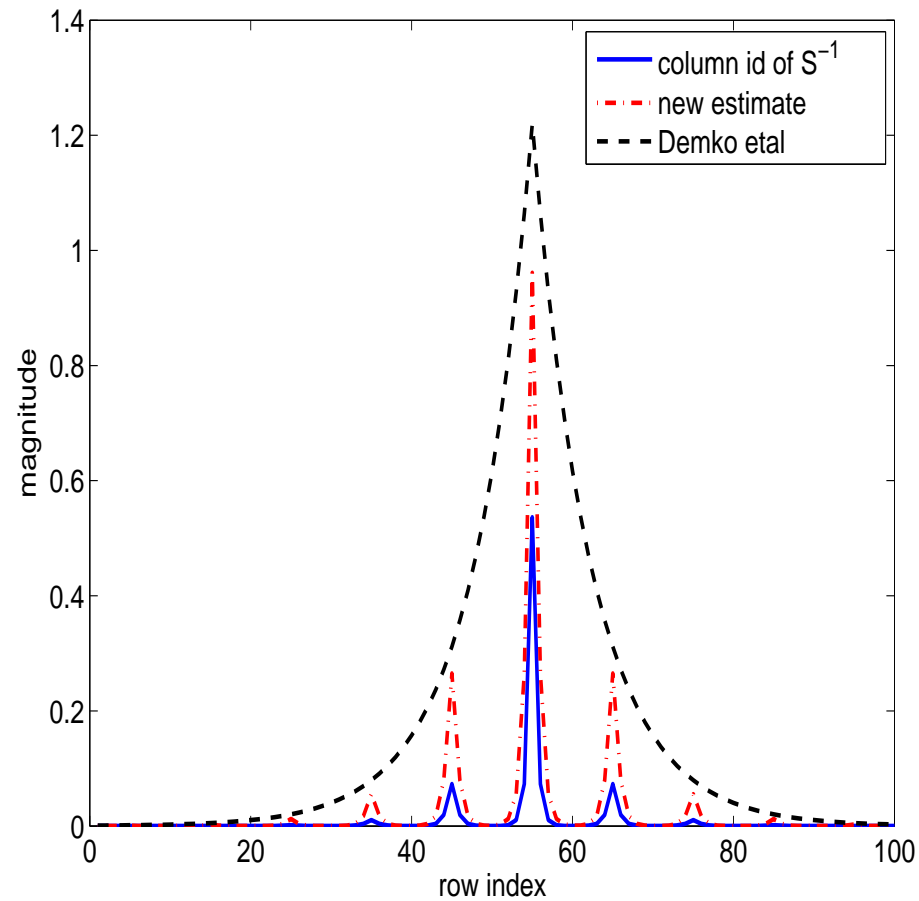
and



(ℓ, m)

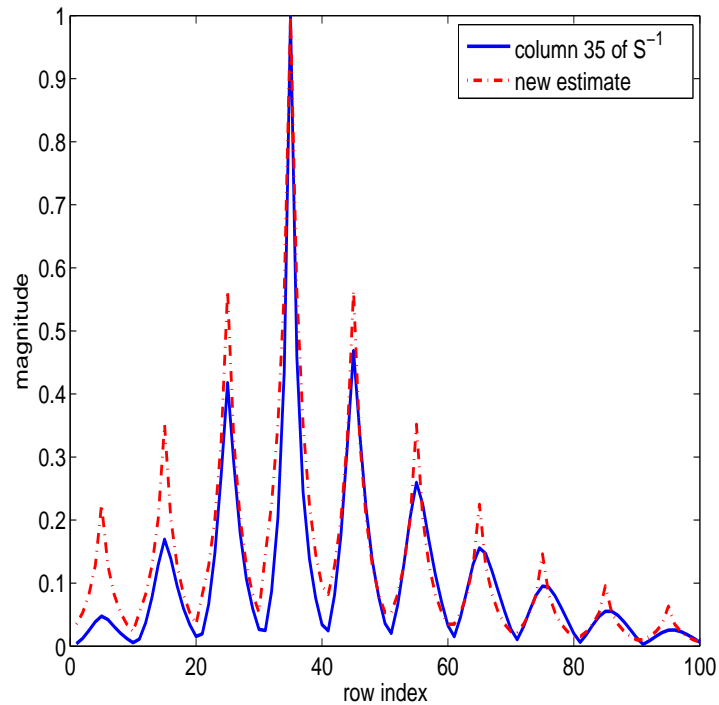
Examples. Symmetric positive definite matrix

$$M = \text{tridiag}(-0.5, \underline{2}, -0.5) \in \mathbb{R}^{10 \times 10}$$



Examples. Legendre stiffness matrix (scaled to have peak equal to 1)

$$M = \text{tridiag}(\delta_k, \underline{\gamma_k}, \delta_k)$$



$$\gamma_k = \frac{2}{(4k - 3)(4k + 1)}$$

$$k = 1, \dots, n, \quad \text{and}$$

$$\delta_k = \frac{-1}{(4k + 1)\sqrt{(4k - 1)(4k + 3)}}$$

$$k = 1, \dots, n - 1$$

Connections to point-wise estimates for discrete Laplacian

For the discrete Green function G_h on the discrete d -dimensional grid R_h , there exist constants h_0 and C such that for $h \leq h_0$, $x, y \in R_h$,

$$G_h(x, y) \leq \begin{cases} C \log \frac{C}{|x-y|+h} & \text{if } d = 2 \\ \frac{C}{(|x-y|+h)^{d-2}} & \text{if } d \geq 3 \end{cases}$$

(Bramble & Thomee, '69)

Our estimate: entries depend on inverse square root of the distance!

Conclusions and further work ahead

- The Lyapunov operator has a very rich structure
- Appropriate computational devices
- Powerful mathematical tool
- ...this structure is recurrent in many application problems...

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