## Stopping criterion: Problem dependence

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- Direct method accurate up to machine precision (likely)
- Iterative method accurate up to what is wanted (hopefully)

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Algebraic problem: Discretization of PDEs

$$\text{error} \quad \to O(h)$$

$h$ discretization parameter...

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For instance, CG optimal: ($\|x\|_A^2 = x^* A x$)

$$\min_{x_k \in x_0 + K_k(A, r_0)} \|b - Ax_k\|_{A^{-1}} = \min_{x_k \in x_0 + K_k(A, r_0)} \|x - x_k\|_A$$

Available: Cheap, reliable estimates of $\|x - x_k\|_A$

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For instance, matrix $G$ associated with FE error measure:

$$\min_{x_k} \|b - Ax_k\|_G$$

# Matrix dependence

$A$ may be very ill-conditioned

$\Rightarrow$ small residual does not necessarily imply small error

$$\frac{1}{\kappa(A)}\frac{\|b - Ax_k\|}{\|b\|} \leq \frac{\|x^\star - x_k\|}{\|x^\star\|} \leq \kappa(A)\frac{\|b - Ax_k\|}{\|b\|}$$

Well-known fact, but often not used

$$\frac{\|b - Ax_k\|}{\|b\|} \quad \text{vs} \quad \frac{\|b - Ax_k\|}{\|b\| + \|A\|_\star \|x_k\|}$$

(here $x_0 = 0$)

## Matrix dependence

Inner-outer methods. e.g. Solve

$$BM^{-1}B^\top x = b$$

Each multiplication with $A = BM^{-1}B^\top$ requires solving a system with $M$

$$u = Av \quad \Leftrightarrow \quad \begin{aligned} &\tilde{u} = B^\top v \\ &\tilde{\tilde{u}} \text{ solves } M\tilde{\tilde{u}} = \tilde{u} \\ &u = B\tilde{\tilde{u}} \end{aligned}$$

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Note: True residual $r_k = b - BM^{-1}B^\top x_k$ not available!

## How accurately should one solve with $M$?

Typically: Inner tolerance $<$ Outer tolerance

But: if optimal Krylov method is used to solve $BM^{-1}B^{\top}x = b$ then:

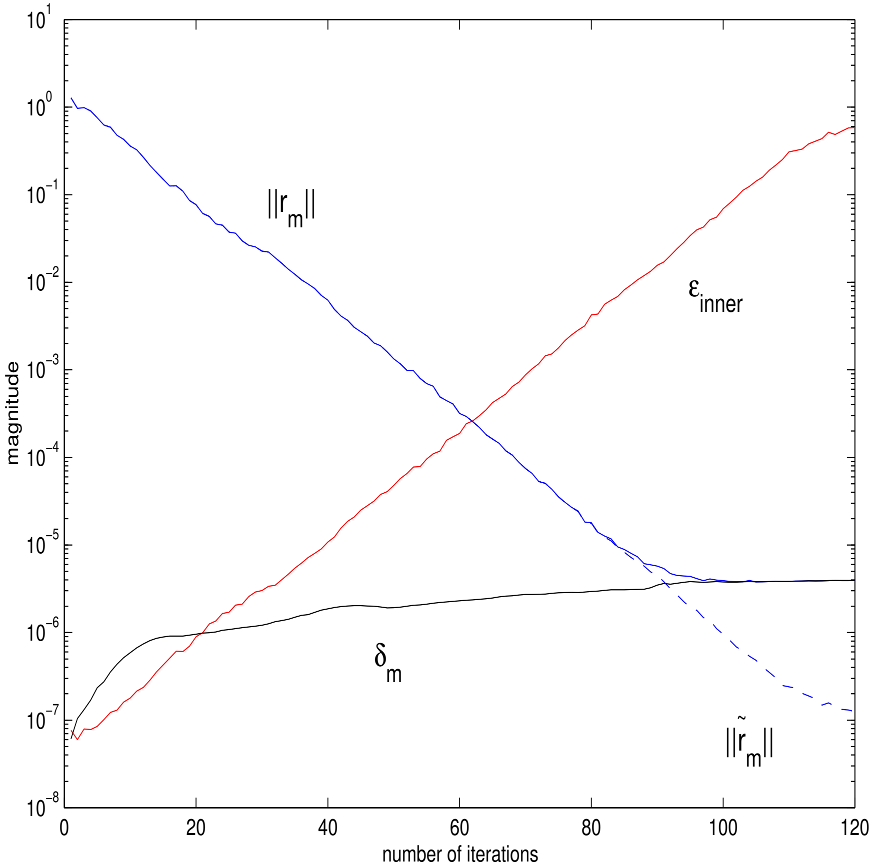
$$\text{Inner tolerance} = \text{c} \cdot \frac{\text{Outer tolerance}}{\text{current outer residual}}$$

## Numerical experiment: Schur complement

$\underbrace{B^T S^{-1} B}_{A} x = b$ at each it. $i$ solve $Sw_i = Bv_i$

Inexact FOM

$\delta_m = \|r_m - (b - V_{m+1} H_m y_m)\|$



## Conclusions

- Computational issues for Krylov solvers well understood
- Other tricks can be used (but not usually in black-box routines)
- Many ideas have wider applicability
- Theory is still under development

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