Advances in projection-type methods for the numerical solution of the Lyapunov equation

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Collaborations with

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The Problem

Given the continuous-time system

$$\mathbf{\Sigma} = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array} \right), \quad A \in \mathbb{C}^{n\times n}$$

Analyse the construction of a reduced system

$$\hat{\mathbf{\Sigma}} = \left(\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \end{array} \right)$$

with $\tilde{A}$ of size $m \ll n$, and issues associated with its accuracy.

The Problem

$$\hat{\mathbf{\Sigma}} = \left(\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \end{array}\right)$$

In particular,

$\star$ Approximate P, Q, the system Gramians. They solve:

$$AP + PA^\top + BB^\top = 0, \quad QA + A^\top Q + C^\top C = 0.$$

$\star$ Approximate the associated Hankel singular values:

$$\sigma_j(\mathbf{\Sigma}) = \sqrt{\lambda_j(PQ)}, \quad j = 1, 2, \ldots, n$$

Projection-type approaches

Given matrices V and W such that $W^\top V = I$, a reduced system onto Range(V) may be obtained as

$$\hat{\mathbf{\Sigma}} = \left(\begin{array}{c|c} W^\top A V & W^\top B \\ \hline C V & \end{array}\right)$$

The associated Gramians solve the systems

$$\begin{aligned} (W^\top A V)X + X(W^\top A V)^\top + W^\top B B^\top W &= 0 \\ Y(W^\top A V) + (W^\top A V)^\top Y + V^\top C^\top C V &= 0 \end{aligned}$$

So that $\quad VXV^\top \approx P \qquad WYW^\top \approx Q$

Approximate Hankel singular values:

$$\sigma_j(\hat{\mathbf{\Sigma}}) = \sqrt{\lambda_j(XY)}$$

Outline

- Brief review of solvers for Lyapunov equations
- Convergence of standard projection methods
- The Extended Krylov subspace: Lyapunov eqn solver
- The Extended Krylov subspace in Model order reduction
- Approximation of Hankel singular values by balanced truncation

Standard Krylov subspace projection for the Lyapunov equation

Hypothesis: $A < 0$

$$P \approx P_m \qquad P_m \in \mathcal{K}$$

Galerkin condition: $\quad R := AP_m + P_m A^\top + BB^\top \quad \perp \quad \mathcal{K}$

$$V_m^\top R V_m = 0 \qquad\qquad \mathcal{K} = \mathrm{range}(V_m)$$

Assume $V_m^\top V_m = I_m$ and let $P_m := V_m X_m V_m^\top$.

Projected Lyapunov equation:

$$(V_m^\top A V_m) X_m + X_m (V_m^\top A^\top V_m) \quad + \quad V_m^\top B B^\top V_m = 0$$

$$\Updownarrow$$

$$T_m X_m + X_m T_m^\top \quad + \quad E_1 E_1 = 0$$

with $B = V_m E_1$ (Saad, '90, for $\mathcal{K} = \mathcal{K}_m(A, B)$; Jaimoukha & Kasenally, '94)

Other related approaches

- Enhanced projection: Different selection of $\mathcal{K}$, e.g.,

$$\mathbf{EK}_m(A,B) = \mathcal{K}_m(A,B) + \mathcal{K}_m(A^{-1}, A^{-1}B) \qquad \text{(Simoncini, 2007)}$$

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- "Global" projection: (Jbilou, Messaoudi, Riquet, Sadok, 1999, 2006)

$$\mathrm{range}(\mathcal{V}) = K_m(A,B), \qquad \mathcal{V} = [V_1, \ldots, V_m]$$
$$\mathrm{trace}(V_i^\top V_j) = 0, i \neq j, \mathrm{trace}(V_i^\top V_i) = 1$$

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- Kronecker formulation: (Preconditioning: Hochbruck & Starke, 1995)

$$AP + PA^\top + BB^\top = 0 \;\Leftrightarrow\; (A \otimes I + I \otimes A)\mathrm{vec}(P) + \mathrm{vec}(BB^\top) = 0$$

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- **Cyclic low rank Smith method:** (see, e.g., Li 2000, Penzl 2000)

$$\begin{aligned} P_0 = 0,\; P_j &= -2p_j(A+p_jI)^{-1}BB^\top(A+p_jI)^{-\top} \quad j = 1, \ldots, \ell \\ &\quad +(A+p_jI)^{-1}(A-p_jI)P_{j-1}(A-p_jI)^\top(A+p_jI)^{-\top} \end{aligned}$$

with $r_\ell(t) = \Pi_{j=1}^{\ell}(t-p_j)$, $\{p_1, \ldots, p_\ell\} = \mathrm{argmin} \max_{t \in \Lambda(A)} |r_\ell(t)/r_\ell(-t)|$

Convergence results and a-priori bounds

- Kronecker formulation: all available results for

$$\mathcal{A}x = f, \qquad \mathcal{A} \in \mathbb{R}^{n^2 \times n^2}$$

- Global projection methods: only a-posteriori estimates (?)

- Cyclic low rank Smith method: results based on

$$r_\ell(t) = \prod_{j=1}^{\ell}(t - p_j), \quad \{p_1, \ldots, p_\ell\} = \operatorname{argmin} \max_{t \in \Lambda(A)} |r_\ell(t)/r_\ell(-t)|$$

- Standard Krylov projection: (Robbè & Sadkane, 2002)

$$\|AP_m^g + P_m^g A^\top + BB^\top\|_F \le \left(1 - \frac{d^2}{\|\mathcal{S}\|^2}\right)^{m/2} \|BB^\top\|_F$$

$$d = \operatorname{dist}(\mathcal{F}(A), \mathcal{F}(-A)) > 0, \quad \mathcal{S} : P \mapsto AP + PA^\top$$

(P_m^g Petrov-Galerkin, originally for the Sylvester equation)

Convergence of the Standard Krylov method

(with V. Druskin, SINUM '09)

$$AP + PA^\top + BB^\top = 0, \qquad P \approx P_m \in K_m(A, B)$$

A symmetric with eigs: $0 < \lambda_{\min} \leq \ldots \leq \lambda_{\max}$

Let

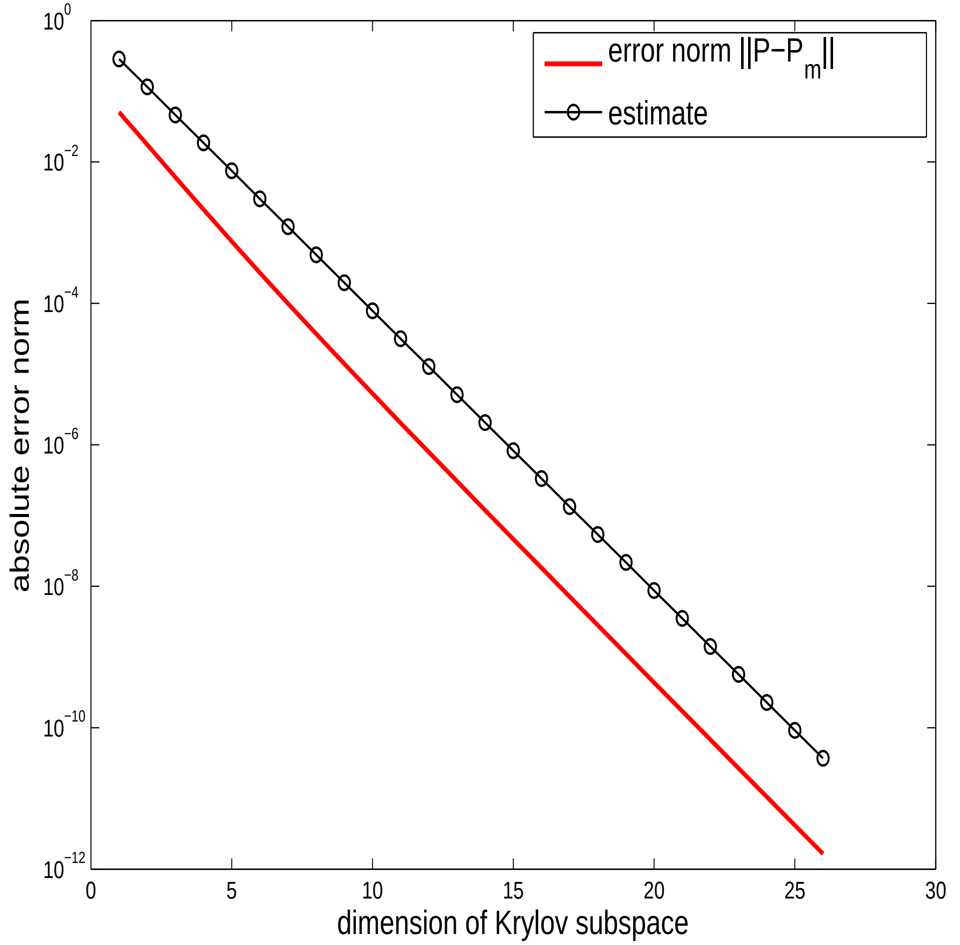
$$\hat{\kappa} := \frac{\lambda_{\max} + \lambda_{\min}}{\lambda_{\min} + \lambda_{\min}}$$

Then

$$\|P - P_m\| \quad \leq \quad \frac{\sqrt{\hat{\kappa}} + 1}{2\lambda_{\min}\sqrt{\hat{\kappa}}} \left(\frac{\sqrt{\hat{\kappa}} - 1}{\sqrt{\hat{\kappa}} + 1} \right)^m$$

Note: same rate as CG for $(A + \lambda_{\min} I)z = b$

The case of A symmetric. An example



A: 400×400 diagonal with uniformly distributed eigenvalues in $[1, 10]$

The case of $\mathcal{F}(A)$ in an ellipse

Assume $\mathcal{F}(A) \subseteq E \subset \mathbb{C}^+$

$\big($$E$ ellipse of center $(c, 0)$, foci $(c \pm d, 0)$ and major semi-axis $a$$\big)$

Let $\alpha_{\min} = \lambda_{\min}((A + A^\top)/2) > 0$. Then

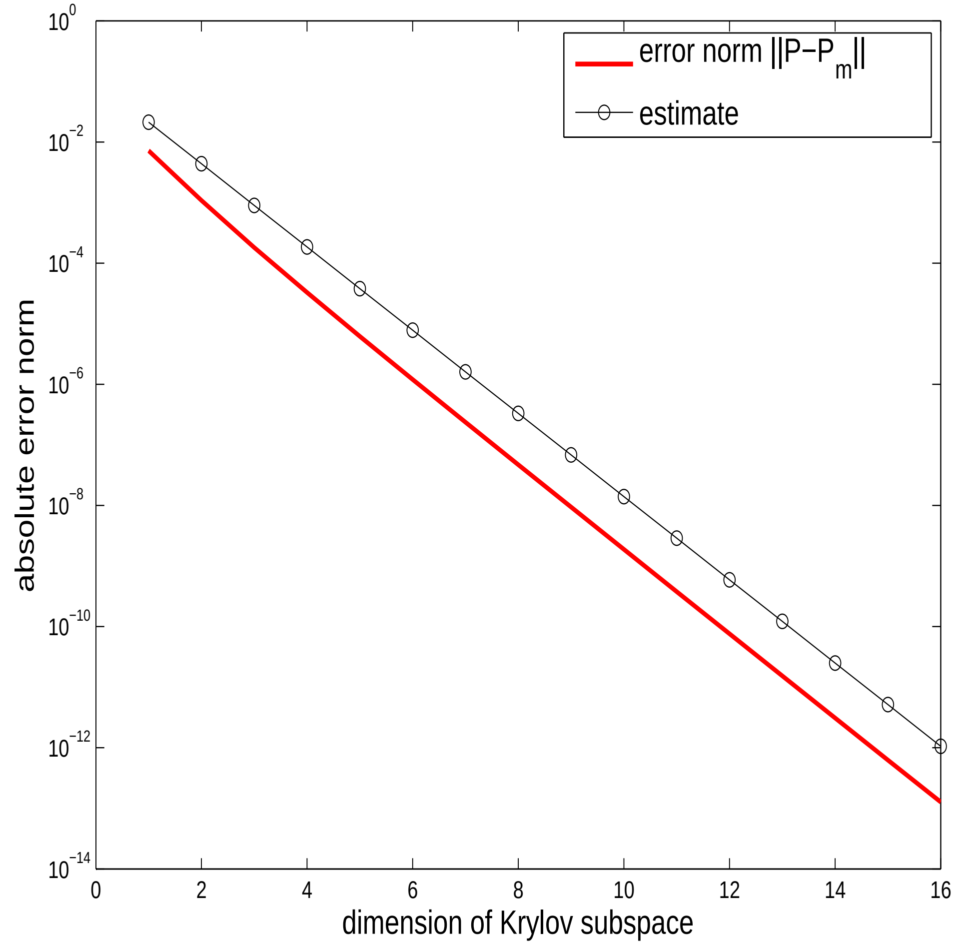
$$\|P - P_m\| \leq \frac{4}{\alpha_{\min}} \frac{r_2}{r_2 - r} \left(\frac{r}{r_2}\right)^m$$

where

$$r = \frac{a}{d} + \sqrt{\left(\frac{a}{d}\right)^2 - 1}, \quad r_2 = \frac{c + \alpha_{\min}}{d} + \sqrt{\left(\frac{c + \alpha_{\min}}{d}\right)^2 - 1}$$

Note: same rate as FOM for $(A + \alpha_{\min} I)z = b$

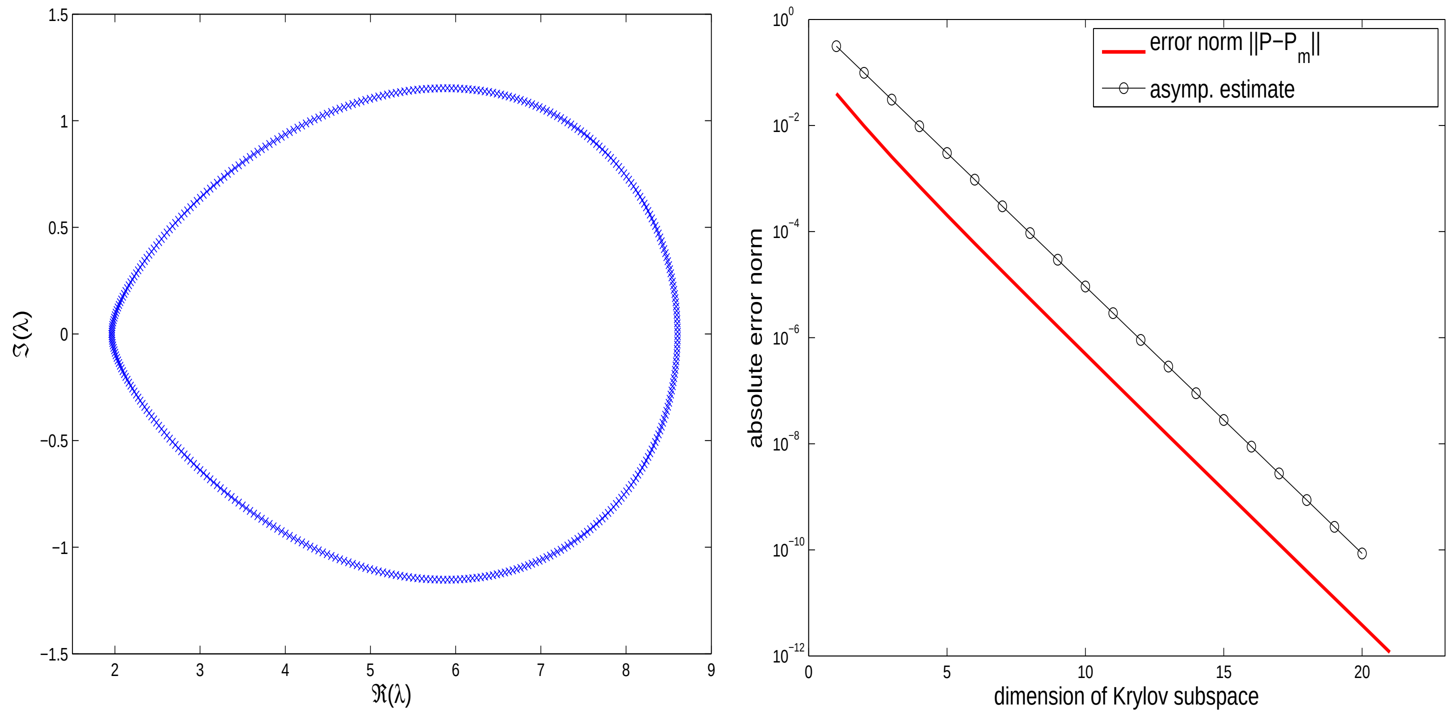
The case of $\mathcal{F}(A)$ in an ellipse. An example



A normal with eigenvalues on an elliptic curve

The case of $\mathcal{F}(A)$ in a wedge-shaped set. An example

Generalization to a wedge-shaped convex set of $\mathbb{C}^+$.

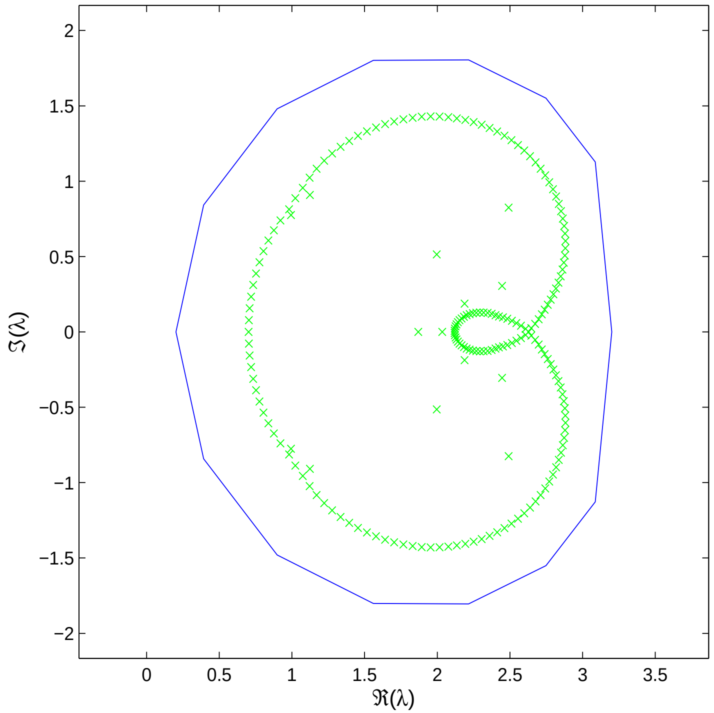


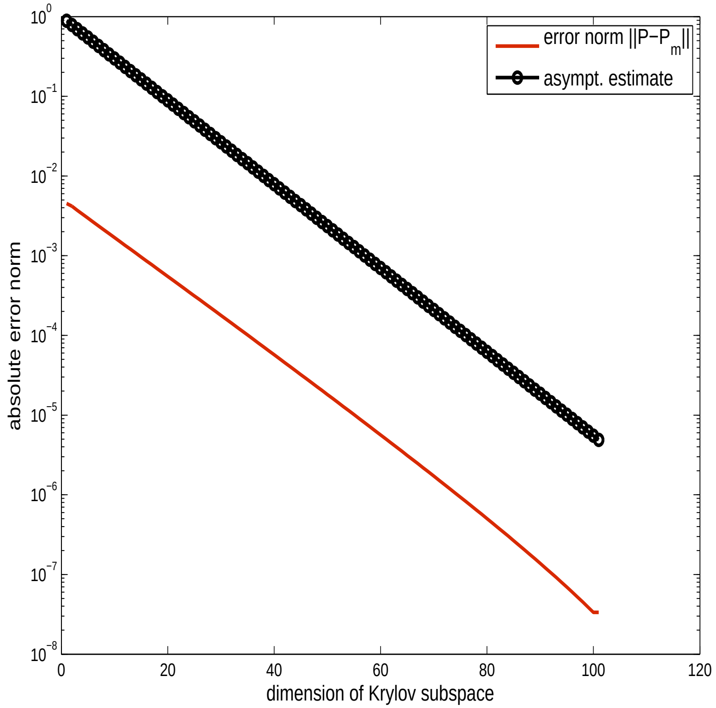
A: diagonal (normal) matrix on the wedge-shaped curve.

(Inclusion set from Hochbruck & Lubich, 1997)

The case of $\mathcal{F}(A)$ in a numerically determined set

Generalization to a Schwarz-Christoffel mapping (Driscoll, Trefethen 2002)





A : Toeplitz$(-1, -1, \underline{2}, 0.1)$

(SC Matlab Toolbox, T. Driscoll 1996)

Extended Krylov subspace method

Galerkin condition:

$$R := AP_m + P_m A^\top + BB^\top \quad \perp \quad \mathbf{EK}_m(A, B)$$

$$P_m \in \mathbf{EK}_m(A, B) = \mathcal{K}_m(A, B) + \mathcal{K}_m(A^{-1}, A^{-1}B) = \text{range}(\mathcal{V}_m)$$

(Druskin-Knizhnerman 1998, S., 2007)

Projected Lyapunov equation:

$$(\mathcal{V}_m^\top A \mathcal{V}_m) X_m + X_m (\mathcal{V}_m^\top A^\top \mathcal{V}_m) \quad + \quad \mathcal{V}_m^\top BB^\top \mathcal{V}_m = 0$$

$$\Updownarrow$$

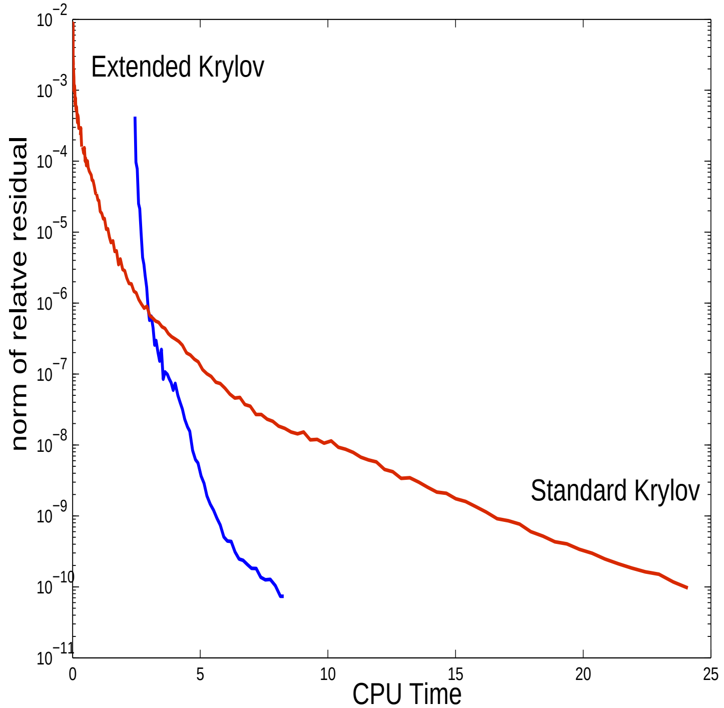
$$\mathcal{T}_m X_m + X_m \mathcal{T}_m^\top \quad + \quad E_1 E_1 = 0$$

Note: Possibility of deflation now included.

Performance evaluation. I

$$\mathbf{x}' = \mathbf{x}_{xx} + \mathbf{x}_{yy} + \mathbf{x}_{zz} - 10x\mathbf{x}_x - 1000y\mathbf{x}_y - 10z\mathbf{x}_z + \mathbf{b}(x,y)\mathbf{u}(t)$$

A matrix $\quad 18^3 \times 18^3$



approximation space dim.: 146 (Standard Krylov) 112 (Extended Krylov)

Performance evaluation. II

Stopping criterion: norm of difference in solution

	s	EKSM		CF-ADI	
		time(#its)	dim.space	time (#its)	dim.space
Example	1	5.95 (12)	24	31.66 (6)	120
`rail_5177`	2	8.08 (10)	40	30.83 (5)	200
`tol`$=10^{-5}$	4	11.11 (7)	56	40.20 (5)	400
	7	18.12 (6)	84	54.22 (5)	700
Example (*)	1	38.95 (34)	68	588.68 (5)	150
`tol`$=10^{-8}$	2	50.50 (33)	132	633.41 (5)	300
	4	90.69 (33)	264	722.92 (5)	600
	7	204.91 (32)	448	857.57 (5)	1050

$$\mathbf{x}' = \mathbf{x}_{xx} + \mathbf{x}_{yy} + \mathbf{x}_{zz} - 10x\mathbf{x}_x - 1000y\mathbf{x}_y - 10\mathbf{x}_z + \mathbf{b}(x,y)\mathbf{u}(t) \qquad (*)$$

Extended Krylov subspace method: Convergence Analysis

For $A \in \mathbb{R}^{n\times n}$ still an open problem (Knizhnerman & S., work in progress)

General considerations (cf. Kressner & Tobler, tr '09) :

$$AP + PA^\top + BB^\top = 0$$

$$A^{-1}P + PA^{-\top} + A^{-1}BB^\top A^{-\top} = 0$$

Summing up for any $\gamma \in \mathbb{R}$, we obtain yet a Lyapunov equation:

$$(A + \gamma A^{-1})P + P(A + \gamma A^{-\top}) + [B,\ \sqrt{\gamma}A^{-1}B][B^\top; \sqrt{\gamma}B^\top A^{-\top}] = 0$$

and $\quad \mathcal{K}_m(A + \gamma A^{-1}, [B,\ \sqrt{\gamma}A^{-1}B]) \subsetneq \mathbf{EK}_m(A, B)$

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and $\quad \mathcal{K}_m(A+\gamma A^{-1}, [B, \sqrt{\gamma}A^{-1}B]) \subsetneq \mathbf{EK}_m(A, B)$

For A symmetric and γ appropriately chosen, convergence rate in $\mathcal{K}_m(A+\gamma A^{-1}, [B, \sqrt{\gamma}A^{-1}B])$ "close" to that in $\mathbf{EK}_m(A, B)$

Transfer function approximation

$$H(\sigma) = C(A - i\sigma I)^{-1}B, \quad \sigma \in [\alpha, \beta]$$

Given space $\mathcal{K}$ and V s.t. $\mathcal{K}=\text{range}(V)$,

$$H(\sigma) \approx CV(V^\top AV - i\sigma I)^{-1}(V^\top B)$$

$\star$ $\mathcal{K} = K_m(A, B)$ (standard Krylov): $h_m(\sigma) = CV_m(H_m - i\sigma I)^{-1}E_1\beta$

$\star$ $\mathcal{K} = K_m((A + s_0I)^{-1}, B)$ (Shift-Invert Krylov):

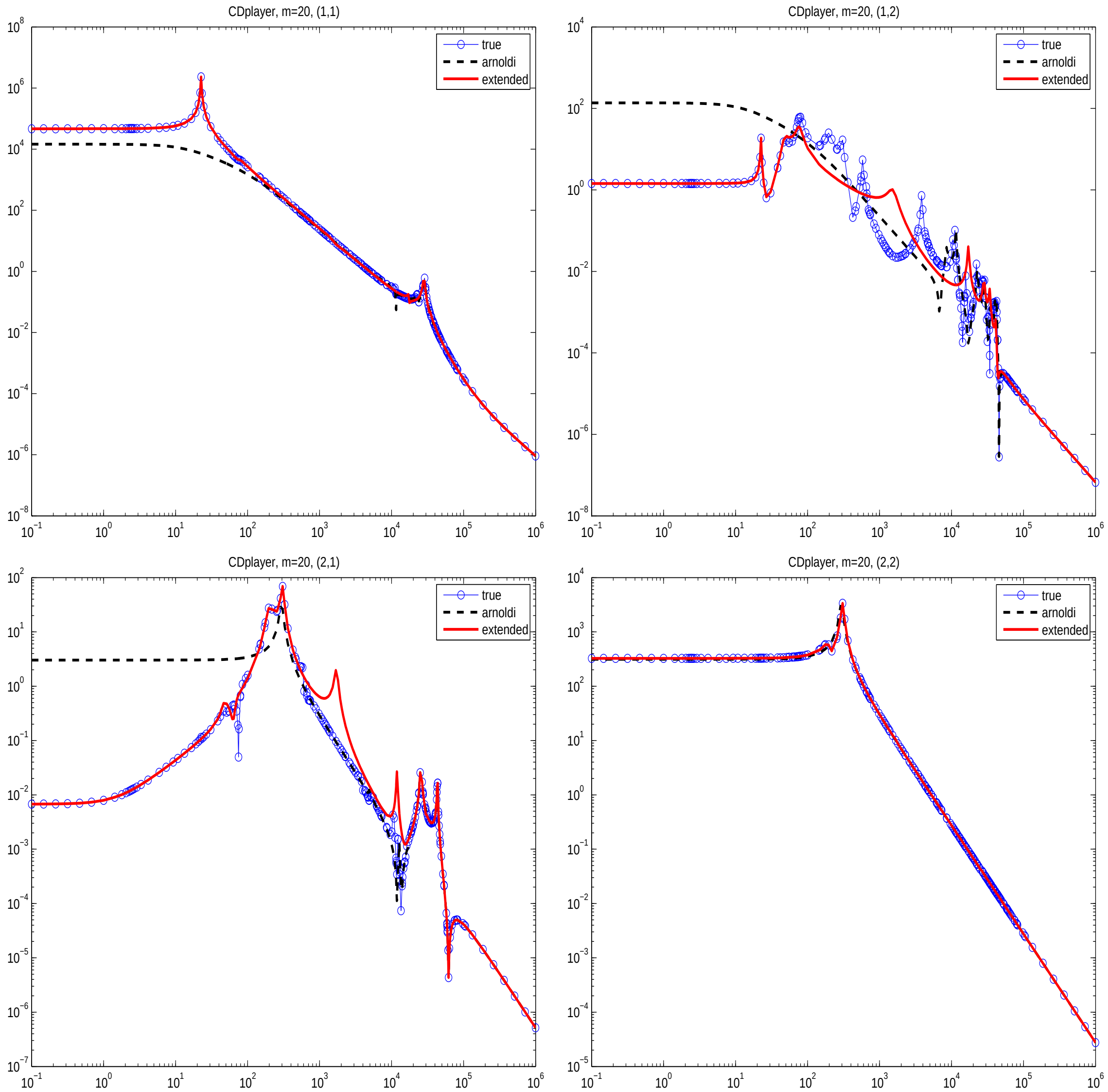
$H_m(\sigma) = CV_m((H_m^{-1} - s_0I) - i\sigma I)^{-1}E_1\beta$

$\star$ $\mathcal{K} = \mathbf{EK}_m(A, B)$: $H_m(\sigma) = C\mathcal{V}_m(\mathcal{T}_m - i\sigma I)^{-1}E_1\beta$

Alternative: Rational Krylov (Grimme-Gallivan-VanDooren etc.)

pole selection is a crucial issue

An example: CD Player, $|H(\sigma)| = |C_{i,:}(A - i\sigma I)^{-1}B_{:,j}|$



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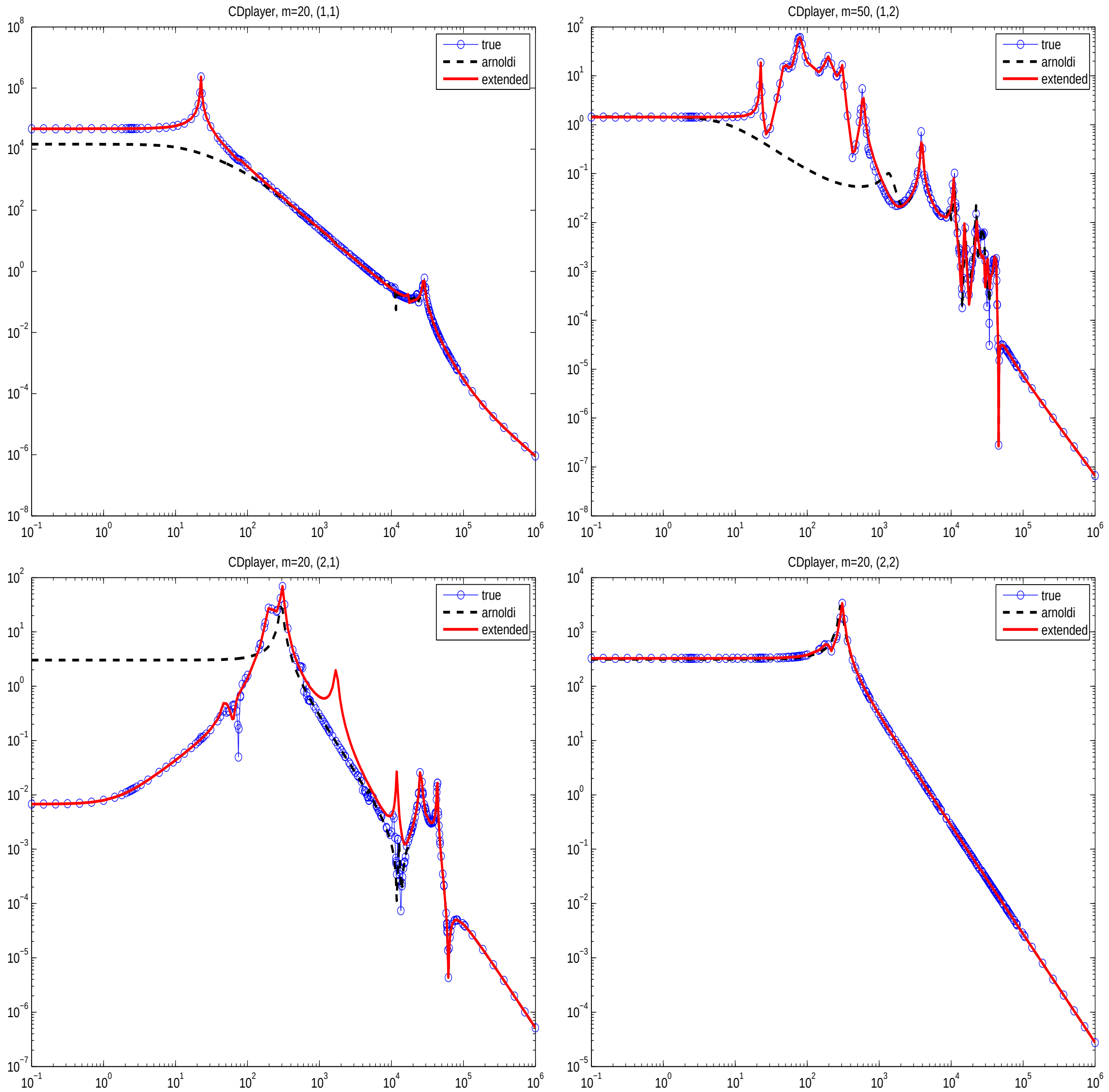


Balanced reduction

Balancing matrix transformation. Given

$$AP + PA^\top + BB^\top = 0, \quad QA + A^\top Q + C^\top C = 0.$$

Find T_r, T_ℓ such that $T_\ell^\top P T_\ell = \Sigma = T_r^\top Q T_r$

The matrix

$$\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \sigma_3, \ldots)$$

contains the Hankel singular values of the system

Large body of literature, and various possibilities, even in the small-scale case (cf., e.g., Antoulas '05)

Error estimate for the reduced system:

$$\|\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}\|_{\mathcal{H}_\infty} \leq 2(\sigma_{k+1} + \cdots + \sigma_{\tilde{n}}),$$

An iterative procedure. Joint work in progress with T. Stykel

Given $\mathcal{K}_0$, $\mathcal{L}_0$.

For $k = 1, 2, \ldots$

1. Update approx. spaces $\mathcal{K}_{k-1} \to \mathcal{K}_k$=range($V_k$), $\mathcal{L}_{k-1} \to \mathcal{L}_k$=range($W_k$)
2. Compute approximate Gramians X_k, Y_k s.t.

$$P \approx P_k = V_k X_k V_k^\top, \quad Q \approx Q_k = W_k Y_k W_k^\top$$

   with $W_k^\top V_k = I$

3. Approximate Hankel singular values:

$$\sqrt{\lambda_j(PQ)} \approx \sigma_j(L_X^\top L_Y), \quad X_k = L_X L_X^\top, \quad Y_k = L_Y L_Y^\top$$

   $U\Sigma Z^\top =$ svd$(L_X^\top L_Y)$

4. If satisfied, compute truncated balancing transformation matrices:

$$T_r = V_k L_X U \Sigma^{-1/2}, \quad T_\ell = W_k L_Y Z \Sigma^{-1/2} \quad \text{and stop}$$

What spaces $\mathcal{K}_k$, $\mathcal{L}_k$ to obtain accurate and small size T_r, T_ℓ ?

Truncated balancing

What spaces $\mathcal{K}_k$, $\mathcal{L}_k$ to obtain accurate and small size T_r, T_ℓ ?

Two possible choices we are exploring:

$\star$ $\mathcal{K}_k = \mathcal{L}_k = \mathbf{EK}_k(A, [B, C^\top])$

(Related to cross-Gramians for A symmetric)

$\star$ $\mathcal{K}_k = \mathbf{EK}_k(A, B) \qquad \mathcal{L}_k = \mathbf{EK}_k(A^\top, C^\top)$

bi-orthogonal bases (à la Lanczos)

Example

Penzl's example (408×408): $A =$ blkdiag(A_1, A_2, A_3, A_4, D)

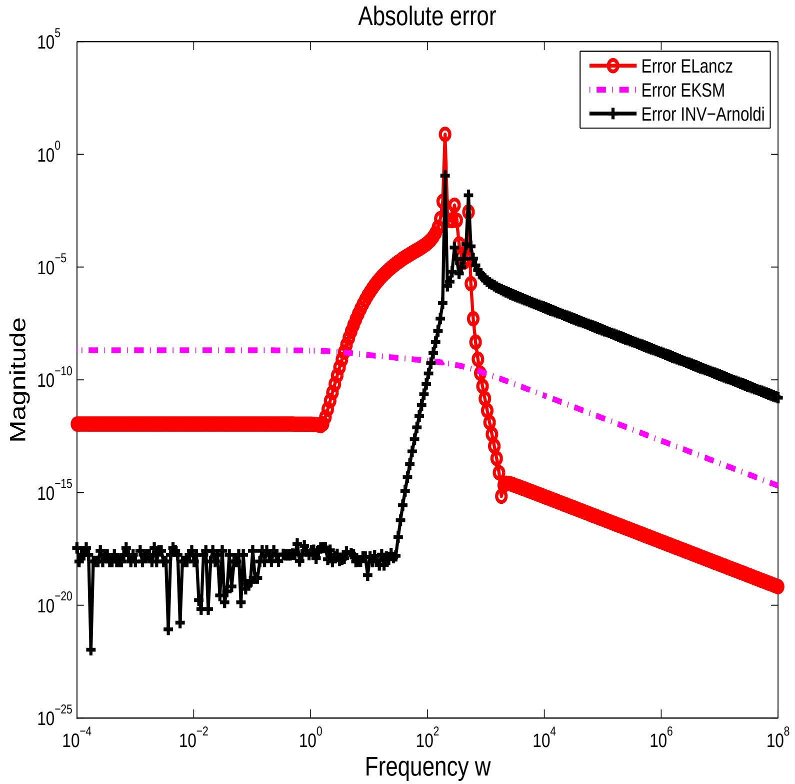
$$A_1 = \begin{bmatrix} -0.01 & -200 \\ 200 & 0.001 \end{bmatrix} \; A_2 = \begin{bmatrix} -0.2 & -300 \\ 300 & -0.1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -0.02 & -500 \\ 500 & 0 \end{bmatrix} \; A_4 = \begin{bmatrix} -0.01 & -520 \\ 520 & -0.01 \end{bmatrix}$$

and $D =$diag(1:400)

$B = C^\top$. Vector ($s = 1$) with large projection onto nonsym part.

Example. cont'ed. Error $|H(\sigma) - H_k(\sigma)|$

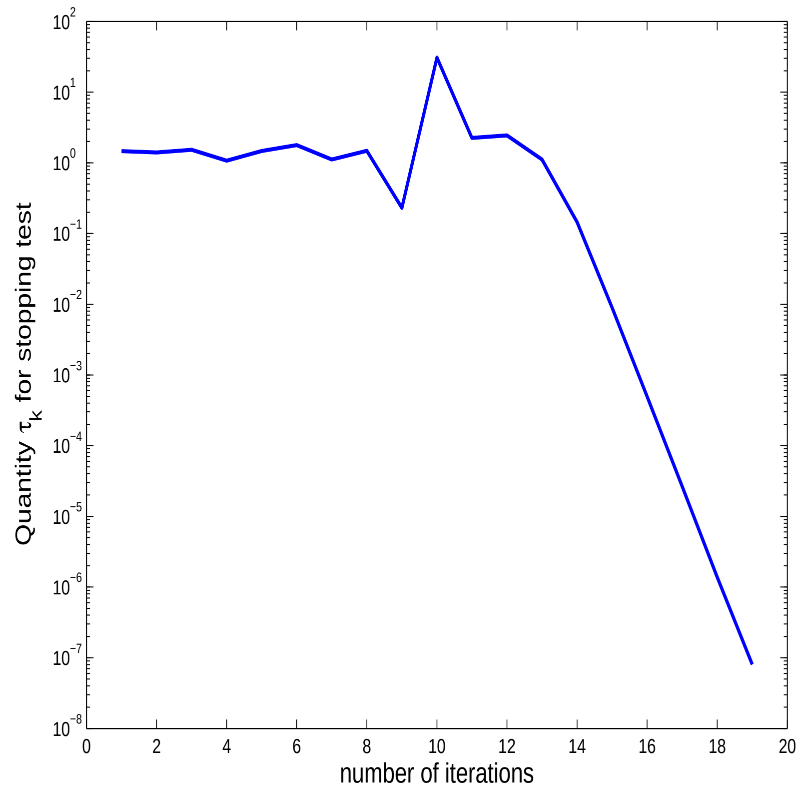


Balancing Ext'd Krylov: T_r of size 19 (space of max size 40)

Balancing Ext'd Lanczos: T_r of size 17 (left-right spaces of max size 20 each)

Inverted-Arnoldi: space of size 40

Convergence of Hankel singular values



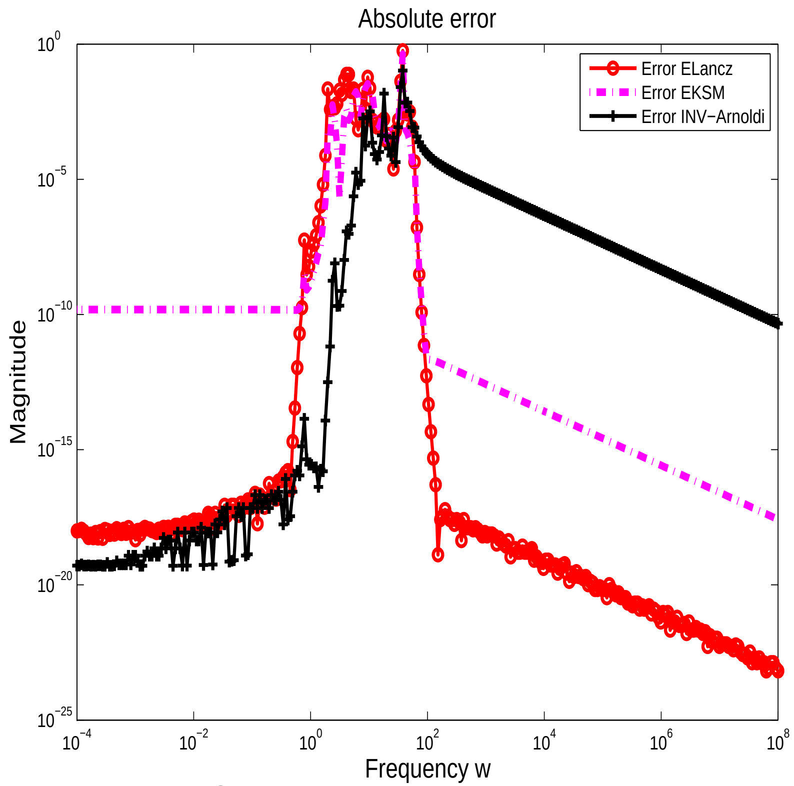
EK: At each iteration k

$$\tau_k = \sum_{j \in \mathcal{J}_k} \frac{|\sigma_j^{(k-1)} - \sigma_j^{(k)}|}{\sigma_1^{(k)}} \text{ where } \mathcal{J}_k = \{\text{index } j \, : \, \sigma_j/\sigma_1 > 10^{-10}\}$$

Residuals of Lyapunov equations: $\|R_k\| = \|S_k\| = O(10^{-3})$

One more example. Error $|H(\sigma) - H_k(\sigma)|$

ISS case. (tiny: 270×270), $B \neq C^\top$ vectors

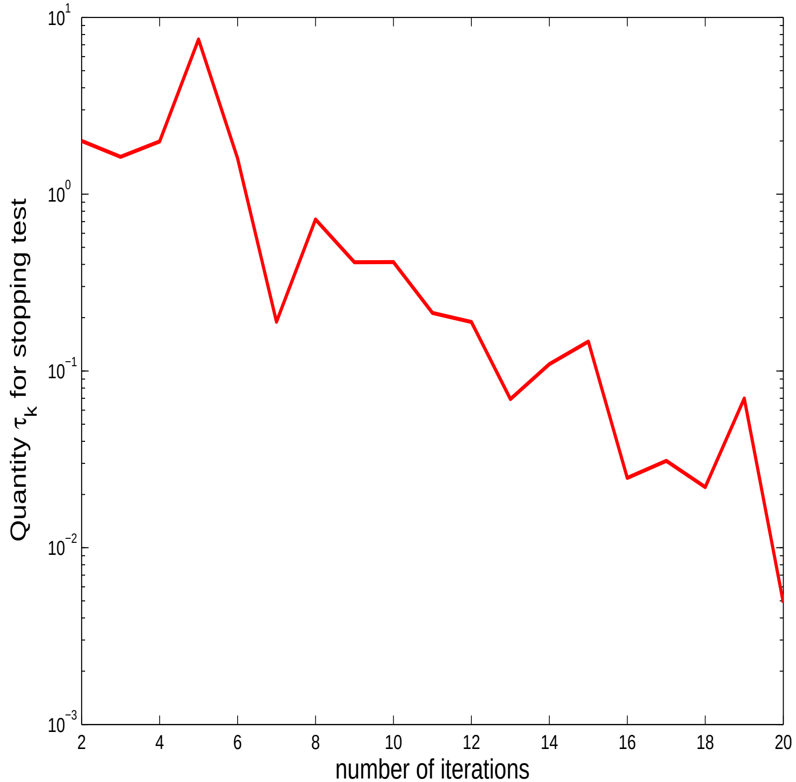


Balancing Ext'd Krylov: T_r of size 35 (space of max size 40)

Balancing Ext'd Lanczos: T_r of size 20 (left-right spaces of max size 20 each)

Inverted-Arnoldi: space of size 40

Convergence of Hankel singular values



EK Lanczos: At each iteration k

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Ext'd Lanczos. Residuals of Lyapunov equations: $\|R_k\| = O(1), \ \ \|S_k\| = O(10^3)$

Conclusions and Current Work

Extended Krylov Subspace approach :

- Efficient for the Lyapunov equation
- Promising results for more general MOR problems

More to be done:

- Complete understanding of convergence behavior
- Complete implementation for Balanced Truncation.