# Computational methods for large-scale matrix equations and application to PDEs

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# Some matrix equations

- Sylvester matrix equation

$$A\mathbf{X} + \mathbf{X}B + D = 0$$

Eigenvalue pbs and tracking, Control, MOR, Assignment pbs, Riccati eqn

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  Control, (Stochastic) PDEs, ...

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Focus: All or some of the matrices are large (and possibly sparse)

# Solving the Lyapunov equation. The problem

Approximate $\mathbf{X}$ in:

$$A\mathbf{X} + \mathbf{X}A^\top + BB^\top = 0$$

$A \in \mathbb{R}^{n \times n}$ neg.real $\qquad B \in \mathbb{R}^{n \times p}, \qquad 1 \leq p \ll n$

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Time-invariant linear system:

$$\mathbf{x}'(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \qquad \mathbf{x}(0) = x_0$$

Closed form solution:

$$\mathbf{X} = \int_0^\infty e^{-tA} BB^\top e^{-tA^\top} dt$$

$\Rightarrow \qquad \mathbf{X}$ symmetric semidef.

see, e.g., Antoulas '05, Benner '06

**Large linear systems:**

$$Ax = b, \quad A \in \mathbb{R}^{n \times n}$$

- Krylov subspace methods (CG, MINRES, GMRES, BiCGSTAB, etc.)

- Preconditioners: find $P$ such that

$$AP^{-1}\widetilde{x} = b \qquad x = P^{-1}\widetilde{x}$$

is easier and fast to solve

# Linear systems vs linear matrix equations

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**Large linear matrix equations:**

$$A\mathbf{X} + \mathbf{X}A^{\top} + BB^{\top} = 0$$

- No preconditioning - to preserve symmetry

- $\mathbf{X}$ is a large, dense matrix $\Rightarrow$ low rank approximation

$$\mathbf{X} \approx \widetilde{X} = ZZ^{\top}, \quad Z \text{ tall}$$

## Linear systems vs linear matrix equations

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**Large linear matrix equations:**

$$A\mathbf{X} + \mathbf{X}A^\top + BB^\top = 0$$

Kronecker formulation:

$$(A \otimes I + I \otimes A)x = b \qquad x = \text{vec}(\mathbf{X})$$

## Projection-type methods

Given an approximation space $\mathcal{K}$,

$$\mathbf{X} \approx X_m \qquad \mathrm{col}(X_m) \in \mathcal{K}$$

Galerkin condition: $\quad R := AX_m + X_m A^\top + BB^\top \quad \perp \quad \mathcal{K}$

$$V_m^\top R V_m = 0 \qquad\qquad \mathcal{K} = \mathrm{Range}(V_m)$$

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Assume $V_m^\top V_m = I_m$ and let $X_m := V_m Y_m V_m^\top$.

Projected Lyapunov equation:

$$V_m^\top (A V_m Y_m V_m^\top + V_m Y_m V_m^\top A^\top + B B^\top) V_m \;\; = \;\; 0$$

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(V_m^\top A V_m)Y_m + Y_m(V_m^\top A^\top V_m) + V_m^\top BB^\top V_m &= 0
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\end{aligned}
$$

Early contributions: Saad '90, Jaimoukha & Kasenally '94, for
$\mathcal{K} = \mathcal{K}_m(A, B) = \mathrm{Range}([B, AB, \ldots, A^{m-1}B])$

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# More recent options as approximation space

Enrich space to decrease space dimension

- Extended Krylov subspace

$$\mathcal{K} = \mathcal{K}_m(A, B) + \mathcal{K}_m(A^{-1}, A^{-1}B),$$

  that is, $\mathcal{K} = \mathrm{Range}([B, A^{-1}B, AB, A^{-2}B, A^2B, A^{-3}B, \ldots,])$

  (Druskin & Knizhnerman '98, Simoncini '07)

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- Rational Krylov subspace

$$\mathcal{K} = \mathbb{K} := \mathrm{Range}([B, (A - s_1 I)^{-1}B, \dots, (A - s_m I)^{-1}B])$$

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In both cases, for $\text{Range}(V_m) = \mathcal{K}$, projected Lyapunov equation:

$$(V_m^\top A V_m)Y_m + Y_m(V_m^\top A^\top V_m) + V_m^\top B B^\top V_m = 0$$

$X_m = V_m Y_m V_m^\top$

# Multiterm linear matrix equation

$$A_1 \mathbf{X} B_1 + A_2 \mathbf{X} B_2 + \ldots + A_\ell \mathbf{X} B_\ell = C$$

Applications:

- Control

- (Stochastic) PDEs

- Matrix least squares

- ...

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Main device: Kronecker formulation

$$\left( B_1^\top \otimes A_1 + \ldots + B_\ell^\top \otimes A_\ell \right) x = c$$

Iterative methods: matrix-matrix multiplications and rank truncation

(Benner, Breiten, Bouhamidi, Chehab, Damm, Grasedyck, Jbilou, Kressner, Matthies, Onwunta, Raydan, Stoll, Tobler, Zander, and **many** others...)

# Multiterm linear matrix equation

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**Alternative approaches:**

**low-rank approx in the problem space**. Some examples:

- Control problem

- PDEs on uniform discretizations

- Stochastic PDE

## PDEs on uniform grids and separable coeffs

$$-\varepsilon\Delta u + \phi_1(x)\psi_1(y)u_x + \phi_2(x)\psi_2(y)u_y + \gamma_1(x)\gamma_2(y)u = f \quad (x,y) \in \Omega$$

$\phi_i, \psi_i, \gamma_i,\ i = 1,2$ sufficiently regular functions + b.c.

Problem discretization by means of a tensor basis

Multiterm linear equation:

$$-\varepsilon T_1\mathbf{U} - \varepsilon\mathbf{U}T_2 + \Phi_1 B_1\mathbf{U}\Psi_1 + \Phi_2\mathbf{U}B_2^\top\Psi_2 + \Gamma_1\mathbf{U}\Gamma_2 = F$$

Finite Diff.: $\mathbf{U}_{i,j} = \mathbf{U}(x_i, y_j)$ approximate solution at the nodes

# PDEs with random inputs

Stochastic steady-state diffusion eqn: $Find\ u: D \times \Omega \to \mathbb{R}\ s.t.\ \mathbb{P}\text{-}a.s.,$

$$\begin{cases} -\nabla \cdot (a(\mathbf{x}, \omega)\nabla u(\mathbf{x}, \omega)) = & f(\mathbf{x}) \quad in\ \ D \\ u(\mathbf{x}, \omega) = & 0 \qquad on\ \ \partial D \end{cases}$$

$f$: deterministic;

$a$: random field, linear function of finite no. of real-valued random variables $\xi_r : \Omega \to \Gamma_r \subset \mathbb{R}$

Common choice: truncated Karhunen–Loève (KL) expansion,

$$a(\mathbf{x}, \omega) = \mu(\mathbf{x}) + \sigma \sum_{r=1}^{m} \sqrt{\lambda_r}\phi_r(\mathbf{x})\xi_r(\omega),$$

$\mu(\mathbf{x})$: expected value of diffusion coef. $\qquad\qquad$ $\sigma$: std dev.

$(\lambda_r, \phi_r(\mathbf{x}))$ eigs of the integral operator $\mathcal{V}$ wrto $V(\mathbf{x}, \mathbf{x}') = \frac{1}{\sigma^2}C(\mathbf{x}, \mathbf{x}')$

$(\lambda_r \searrow \qquad C : D \times D \to \mathbb{R}$ covariance fun. $)$

# Discretization by stochastic Galerkin

Approx with space in tensor product form[a] $\mathcal{X}_h \times S_p$

$$\mathcal{A}\mathbf{x} = \mathbf{b}, \qquad \mathcal{A} = G_0 \otimes K_0 + \sum_{r=1}^{m} G_r \otimes K_r, \quad \mathbf{b} = \mathbf{g}_0 \otimes \mathbf{f}_0,$$

$\mathbf{x}$: expansion coef. of approx to $u$ in the tensor product basis $\{\varphi_i \psi_k\}$

$K_r \in \mathbb{R}^{n_x \times n_x}$, FE matrices (sym)

$G_r \in \mathbb{R}^{n_\xi \times n_\xi}$, $r = 0, 1, \ldots, m$ Galerkin matrices associated w/ $S_p$ (sym.)

$\mathbf{g}_0$: first column of $G_0$

$\mathbf{f}_0$: FE rhs of deterministic PDE

$$n_\xi = \dim(S_p) = \frac{(m+p)!}{m!p!} \qquad \Rightarrow \boxed{n_x \cdot n_\xi} \text{ huge}$$

---

[a]$S_p$ set of multivariate polyn of total degree $\leq p$

# The matrix equation formulation

$$\left(G_0 \otimes K_0 + G_1 \otimes K_1 + \ldots + G_m \otimes K_m\right) \mathbf{x} = \mathbf{g}_0 \otimes \mathbf{f}_0$$

transforms into

$$K_0 \mathbf{X} G_0 + K_1 \mathbf{X} G_1 + \ldots + K_m \mathbf{X} G_m = F, \qquad F = \mathbf{f}_0 \mathbf{g}_0^\top$$

$(G_0 = I)$

Solution strategy. Conjecture:

- $\{K_r\}$ from trunc'd Karhunen–Loève (KL) expansion

$$\Downarrow$$

$$\mathbf{X} \approx \widetilde{X} \text{ low rank}, \ \widetilde{X} = X_1 X_2^T$$

(Possibly extending results of Gradesyk, 2004)

# Matrix Galerkin approximation of the deterministic part. 1

Approximation space $\mathcal{K}_k$ and basis matrix $V_k$: $\qquad \mathbf{X} \approx X_k = V_k Y$

$$V_k^\top R_k = 0, \qquad R_k := K_0 X_k + K_1 X_k G_1 + \ldots + K_m X_k G_m - \mathbf{f}_0 \mathbf{g}_0^\top$$

Computational challenges:

- Generation of $\mathcal{K}_k$ involved $m+1$ different matrices $\{K_r\}$ !

- Matrices $K_r$ have different spectral properties

- $n_x, n_\xi$ so large that $X_k, R_k$ should not be formed !

(Powell & Silvester & Simoncini, SISC 2017)

# Example. $-\nabla \cdot (a\nabla u) = 1, \quad D = (-1,1)^2$. KL expansion.

$\mu = 1$, $\xi_r \sim U(-\sqrt{3}, \sqrt{3})$ and $C(\vec{x}_1, \vec{x}_2) = \sigma^2 \exp\left(-\frac{\|\vec{x}_1 - \vec{x}_2\|_1}{2}\right)$, $n_x = 65{,}025$, $\sigma = 0.3$

| $m$ | $p$ | $n_\xi$ | $k$ | inner its | $n_k$ $\mathcal{K}_k$ | rank $\widetilde{\mathbf{X}}$ | time secs | CG time (its) |
|---|---|---|---|---|---|---|---|---|
| | 2 | 45 | 17 | 9.8 | 128 | 45 | 32.1 | 13.4  (8) |
| 8 | 3 | 165 | 21 | 12.2 | 160 | 129 | 41.4 | 56.6 (10) |
| 87% | 4 | 495 | 24 | 14.5 | 183 | 178 | 51.1 | 197.0 (12) |
| | 5 | 1,287 | 27 | 16.9 | 207 | 207 | 64.0 | 553.0 (13) |
| | 2 | 91 | 15 | 9.9 | 165 | 89 | 47.8 | 30.0  (8) |
| 12 | 3 | 455 | 18 | 12.2 | 201 | 196 | 61.6 | 175.0 (10) |
| 89% | 4 | 1,820 | 21 | 15.0 | 236 | 236 | 86.4 | 821.0 (12) |
| | 5 | 6,188 | 25 | 18.6 | 281 | 281 | 188.0 | 3070.0 (13) |
| | 2 | 231 | 16 | 9.4 | 281 | 206 | 111.0 | 94.7  (8) |
| 20 | 3 | 1,771 | 23 | 12.3 | 399 | 399 | 197.0 | 845.0 (10) |
| 93% | 4 | 10,626 | 26 | 15.4 | 454 | 454 | 556.0 | Out of Mem |

% of variance integral of $a$

# Bilinear systems of matrix equations

Find $\mathbf{X} \in \mathbb{R}^{n_1 \times n_2}$ and $\mathbf{P} \in \mathbb{R}^{m \times n_2}$ such that

$$
\begin{aligned}
A_1 \mathbf{X} + \mathbf{X} A_2 + B^T \mathbf{P} &= F_1 \\
B \mathbf{X} &= F_2
\end{aligned}
$$

with $A_i \in \mathbb{R}^{n_i \times n_i}$, $B \in \mathbb{R}^{m \times n_1}$, $F_1 \in \mathbb{R}^{n_1 \times n_2}$, $F_2 \in \mathbb{R}^{m \times n_2}$, $m \leq n_1$

### Emerging matrix formulation of different application problems

- Constraint control

- Mixed formulations of stochastic diffusion problems

- Discretized deterministic/stochastic (Navier-)Stokes equations

- ...

## An example. Mixed FE formulation of stochastic Galerkin diffusion pb

$$
\begin{aligned}
c^{-1}\vec{u} - \nabla p &= 0, \\
-\nabla \cdot \vec{u} &= f,
\end{aligned}
$$

Assume that $c^{-1} = c_0 + \sum_{r=1}^{\ell} \sqrt{\lambda_r} c_r(\vec{x}) \xi_r(\omega)$ and that an appropriate class of finite elements is used for the discretization of the problem

(see, e.g., the derivation in Elman-Furnival-Powell, 2010)

After discretization the problem reads:

$$
\begin{bmatrix}
G_0 \otimes K_0 + \sum_{r=1}^{\ell} \sqrt{\lambda} G_r \otimes K_r & G_0^T \otimes B_0^T \\
G_0 \otimes B_0 &
\end{bmatrix}
\begin{bmatrix}
u \\
p
\end{bmatrix}
=
\begin{bmatrix}
0 \\
f
\end{bmatrix}
$$

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$$c^{-1}\vec{u} - \nabla p = 0,$$
$$-\nabla \cdot \vec{u} = f,$$

Assume that $c^{-1} = c_0 + \sum_{r=1}^{\ell} \sqrt{\lambda_r} c_r(\vec{x}) \xi_r(\omega)$ and that an appropriate class of finite elements is used for the discretization of the problem

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$$\begin{bmatrix} G_0 \otimes K_0 + \sum_{r=1}^{\ell} \sqrt{\lambda} G_r \otimes K_r & G_0^T \otimes B_0^T \\ G_0 \otimes B_0 & \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

For $\ell = 1$ we obtain

$$K_0 \mathbf{X} G_0 + K_1 \mathbf{X} G_1 + B_0^T \mathbf{P} G_0 = 0, \qquad B_0 \mathbf{X} G_0 = F$$

# The bilinear case. Computational strategies

$$A_1 \mathbf{X} + \mathbf{X} A_2 + B^T \mathbf{P} \;\; = \;\; F_1$$

$$B \mathbf{X} \;\; = \;\; F_2$$

**Kronecker formulation:**

$$\begin{bmatrix} \mathcal{A} & \mathcal{B}^T \\ \mathcal{B} & \mathcal{O} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad \mathcal{A} = I \otimes A_1 + A_2^T \otimes I, \quad \mathcal{B} = B \otimes I$$

with $\mathbf{x} = \mathrm{vec}(\mathbf{X})$, $\mathbf{p} = \mathrm{vec}(\mathbf{P})$, $f_1 = \mathrm{vec}(F_1)$ and $f_2 = \mathrm{vec}(F_2)$

Extremely rich literature from saddle point algebraic linear systems

Problem: Coefficient matrix has size $(n_1 n_2 + m n_2) \times (n_1 n_2 + m n_2)$

## The bilinear case. Computational strategies. Cont'd

$$A_1\mathbf{X} + \mathbf{X}A_2 + B^T\mathbf{P} = F_1$$

$$B\mathbf{X} = F_2$$

⋆ Derive numerical strategies that directly work with the matrix equations:

- Small scale: Null space method

- Small and medium scale: Schur complement method
  (also directly applicable to trilinear case)

- Large scale: Iterative method for low rank $F_i$, $i = 1, 2$

**"Small and medium scale" actually means "Large scale" for the Kronecker form!**

$$A_1 \mathbf{X} + \mathbf{X} A_2 + B^T \mathbf{P} \quad = \quad F_1$$

$$B \mathbf{X} \qquad\qquad\qquad = \quad F_2$$

Rewrite as

$$\begin{bmatrix} A_1 & B^* \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{P} \end{bmatrix} + \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{P} \end{bmatrix} A_2 = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \quad \Leftrightarrow \quad \mathcal{M} \mathbf{Z} + \mathcal{D}_0 \mathbf{Z} A_2 = F$$

with

$\mathcal{M}, \mathcal{D}_0 \in \mathbb{R}^{(n_1+m) \times (n_1+m)}$

$A_2 \in \mathbb{R}^{n_2 \times n_2}$ nonsingular

$\mathcal{D}_0$ highly singular

If $F$ low rank, exploit projection-type strategies for Sylvester equations

$$\begin{bmatrix} A_1 & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{P} \end{bmatrix} + \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{P} \end{bmatrix} A_2 = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad \Leftrightarrow \quad \mathcal{M}\mathbf{Z} + \mathcal{D}_0 \mathbf{Z} A_2 = F$$

with $F$ low rank. We rewrite the matrix equation as

$$\mathbf{Z} A_2^{-1} + \mathcal{M}^{-1} \mathcal{D}_0 \mathbf{Z} = \widehat{F}, \qquad \widehat{F} = \mathcal{M}^{-1} F A_2^{-1},$$

which is a Sylvester equation with a singular coefficient matrix.

$$\Rightarrow \qquad \mathbf{Z} \approx \widetilde{\mathbf{Z}}_k = V_k \mathcal{Z}_k W_k^T$$

with $Range(V_k), Range(W_k)$ appropriate approximation spaces of small dimensions

$$\mathbf{Z}A_2^{-1} + \mathcal{M}^{-1}\mathcal{D}_0\mathbf{Z} = \widehat{F}_l\widehat{F}_r^T \quad \Rightarrow \quad \mathbf{Z} \approx \widetilde{\mathbf{Z}}_k = V_k\mathcal{Z}_kW_k^T$$

Choice of $V_k, W_k$. A possible strategy:

- $W_k = \mathbb{EK}_k(A_2^{-T}, \widehat{F}_r)$,    Extended Krylov subspace

- $V_k = K_k(\mathcal{M}^{-1}\mathcal{D}_0, \widehat{F}_l) \cup K_k((\mathcal{M}^{-1}\mathcal{D}_0 + \sigma I)^{-1}, \widehat{F}_l)$
  Augmented Krylov subspace, $\sigma \in \mathbb{R}$
  (see, e.g., Shank & Simoncini (2013))

Note: $\mathcal{M}$ has size $(n_1 + m) \times (n_1 + m)$

(Compare with $(n_1n_2 + mn_2) \times (n_1n_2 + mn_2)$ of the Kronecker form)

# Numerical experiments

$$A_1 \mathbf{X} - \mathbf{X} A_2 + B^T \mathbf{P} = 0, \qquad B\mathbf{X} = F_2 \qquad \text{vs} \qquad \mathcal{A}\mathbf{z} = f$$

$$A_1 \to \mathcal{L}_1 = -u_{xx} - u_{yy}$$

$$A_2 \to \mathcal{L}_1 = -(e^{-10xy} u_x)_x - (e^{10xy} u_y)_y + 10(x+y)u_x$$

$[F_1; F_2]$ rank-1 matrix

$$B = \mathrm{bidiag}(-1, \underline{1}) \in \mathbb{R}^{(n_2 - n_1) \times n_2}, \qquad \text{iterative: tol} = 10^{-6}, \ \sigma = 10^{-2}$$

## Elapsed time

| $n_1$ | $n_2$ | size($\mathcal{A}$) | Monolithic | Iterative $\mathbb{EK}$ |
|-------|-------|--------------------|------------|--------------------------|
| 400 | 100 | 79,000 | 6.9769e-02 | 3.1523e-02 (4) |
| 900 | 225 | 401,625 | 3.4808e-01 | 5.0447e-02 (4) |
| 1600 | 400 | 1272,000 | 1.1319e+00 | 7.8018e-02 (4) |
| 2500 | 625 | 3109,375 | 3.1212e+00 | 1.5282e-01 (5) |
| 3600 | 900 | 6453,000 | 1.0210e+01 | 2.8053e-01 (5) |
| 4900 | 1225 | 11,962,125 | 3.7699e+01 | 1.4754e+00 (5) |

$$\begin{bmatrix} \mathcal{H} & \mathcal{B}^T \\ \mathcal{B} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \qquad \mathcal{H} = (\nu_0 G_0 + \nu_1 G_1) \otimes A_x, \quad \mathcal{B} = G_0 \otimes B_x$$

where $\nu = \nu_0 + \nu_1 \xi(\omega)$ uncertain viscosity, $\xi$ random variable

Then

$$A_x \mathbf{X} G_0 \nu_0 + A_x \mathbf{X} G_1 \nu_1 + B_x^T \mathbf{P} G_0 = F_1, \quad B_x \mathbf{X} = F_2$$

with $G_0 = I$. This corresponds to

$$\begin{bmatrix} \nu_0 A_x & B_x^T \\ B_x & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{P} \end{bmatrix} + \begin{bmatrix} \nu_1 A_x & \\ & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{P} \end{bmatrix} G_1 = \begin{bmatrix} F1 \\ F2 \end{bmatrix}$$

that is

$$\mathcal{M}\mathbf{Z} + \mathcal{D}_0 \mathbf{Z} G_1 = F$$

$$\mathcal{M}\mathbf{Z} + \mathcal{D}_0\mathbf{Z}G_1 = F \qquad \text{vs} \qquad \mathcal{A}\mathbf{z} = f$$

| $n_1$ | $n_2$ | size($\mathcal{A}$) | Monolithic | Iterative $\mathbb{EK}$ |
|-------|-------|---------------------|------------|-------------------------|
| 1256  | 4     | 6,580               | 0.1852     | 0.12 (2)                |
| 3526  | 4     | 18,064              | 0.9063     | 0.56 (2)                |
| 9812  | 4     | 49,708              | 4.6418     | 2.19 (2)                |

$\nu_0 = 1/10, \nu_1 = 3\nu_0/10$      Powell-Silvester (2012)

$$\mathcal{M}\mathbf{Z} + \mathcal{D}_0\mathbf{Z}G_1 = F \qquad \text{vs} \qquad \mathcal{A}\mathbf{z} = f$$

| $n_1$ | $n_2$ | size($\mathcal{A}$) | Monolithic | Iterative $\mathbb{EK}$ |
|---|---|---|---|---|
| 1256 | 165 | 271,425 | 2.91 | 0.18 (2) |
| 3526 | 165 | 745,140 | 20.01 | 0.45 (2) |
| 9812 | 165 | 2050,455 | - | 1.87 (2) |

$\nu_0 = 1/10, \nu_1 = 3\nu_0/10$ \qquad Powell-Silvester (2012)

- $n_2$ could be much larger, $n_2 = O(10^3)$

- Memory requirements are very limited, $\widetilde{\mathbf{Z}} = Z_1 Z_2^T$ of very low rank

# Numerical experiments. 2D stochastic Stokes problem

$$\begin{bmatrix} \mathcal{H} & \mathcal{B}^T \\ \mathcal{B} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \qquad \begin{aligned} \mathcal{H} &= \text{blkdiag}((\nu_0 G_0 + \nu_1 G_1) \otimes A_x, (\nu_0 G_0 + \nu_1 G_1) \otimes A_y) \\ \mathcal{B} &= [G_0 \otimes B_x, \, G_0 \otimes B_y] \end{aligned}$$

$$\mathcal{M}\mathbf{Z} + \mathcal{D}_0 \mathbf{Z} G_1 = F \qquad \text{vs} \qquad \mathcal{A}\mathbf{z} = f$$

| $n_1$ | $n_2$ | size($\mathcal{A}$) | Monolithic | Iterative $\mathbb{EK}$ |
|---|---|---|---|---|
| 2512 | 4 | 11,604 | 0.55 | 0.28 (2) |
| 7052 | 4 | 32,168 | 3.73 | 1.26 (2) |
| 19624 | 4 | 88,956 | 11.93 | 3.95 (2) |

$\nu_0 = 1/10, \nu_1 = 3\nu_0/10$ 　　Powell-Silvester (2012)

# Not discussed but in this category

- Sylvester-like linear matrix equations

$$AX + f(X)B = C$$

typically (but not only!): $\quad f(X) = \bar{X},\ f(X) = X^\top,\ \text{or}\ f(X) = X^*$
(Bevis, Braden, Byers, Chiang, De Terán, Dopico, Duan, Feng, Gonzalez, Guillery, Hall, Hartwig, Ikramov, Kressner, Montealegre, Reyes, Schröder, Vorntsov, Watkins, Wu, ...)

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- Linear systems with complex tensor structure

$$\mathcal{A}\mathbf{x} = b \quad \text{with} \quad \mathcal{A} = \sum_{j=1}^{k} I_{n_1} \otimes \cdots \otimes I_{n_{j-1}} \otimes A_j \otimes I_{n_{j+1}} \cdots \otimes I_{n_k}.$$

Dolgov, Grasedyck, Khoromskij, Kressner, Oseledets, Tobler, Tyrtyshnikov, and many more...

# Conclusions

## Multiterm (Kron) linear equations is the new challenge

- Great advances in solving really large linear matrix equations

- Second order (matrix) challenges rely on strength and maturity of linear system solvers

- Low-rank tensor formats is the new generation of approximations

**Reference for linear matrix equations:**

⋆ V. Simoncini,

*Computational methods for linear matrix equations,*

SIAM Review, Sept. 2016.