Mathematical Models and Methods in Applied Sciences
Vol. 14, No. 12 (2004) 1735–1759

World Scientific
www.worldscientific.com

A Γ-CONVERGENCE APPROACH TO NON-PERIODIC HOMOGENIZATION OF STRONGLY ANISOTROPIC FUNCTIONALS

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Received 9 February 2004
Revised 31 March 2004
Communicated by M. Fabrizio

In this work we present a homogenization result for a class of degenerate elliptic functionals mimicking strongly anisotropic media. We study the limit as $\varepsilon \to 0$ of the functionals

$$\int \langle \alpha_\varepsilon(x, \nabla u) A_\varepsilon(x) \nabla u, \nabla u \rangle \, dx \,,$$

where, for any $\varepsilon > 0$, $\alpha_\varepsilon : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\alpha_\varepsilon(x, \xi) \approx \langle A_\varepsilon(x)\xi, \xi\rangle^{p/2-1}$, $A_\varepsilon \in M^{n\times n}(\mathbb{R})$ being measurable non-negative matrices such that $A_\varepsilon^t(x) = A_\varepsilon(x)$ almost everywhere, $p > 1$. To take into account the anisotropy of the media we consider two families of weight functions reasonably different, λ_ε and Λ_ε, possibly degenerate or singular, such that:

$$\lambda_\varepsilon^{2/p}(x)|\xi|^2 \le \langle A_\varepsilon(x)\xi, \xi\rangle \le \Lambda_\varepsilon^{2/p}(x)|\xi|^2 \,.$$

The convergence to the homogenized problem is obtained by a classical approach of Γ-convergence.

Keywords: Homogenization; anisotropic operators; Γ-convergence; weighted Sobolev spaces.

AMS Subject Classification: 35B27, 35J70

1. Introduction

Composites are materials containing more than one constituent, finely mixed.[21] They are widely used in several branches of industry, due to their interesting properties. Indeed it is known that in general a composite performs better than a material made of a single component since it combines the attributes of the constituent materials, as in the case for example of ceramics or reinforced concrete. Common examples of composites are bones, which are porous composites, porous rock, in which the pores are often filled with salt water or oil, construction materials, such as wood and concrete, martensite, which is typical of a shape memory material, with a laminar-type structure comprised of alternating layers of the two variants of martensite.

Usually in a composite the heterogeneities are very small compared with the global dimension of the sample. The smaller are the heterogeneities, the better is the mixture, which from a macroscopic point of view looks like a "homogeneous" material. It is then crucial to understand the relationship between the properties of the constituent materials, the underlying microstructure of a composite, and the overall effective (electrical, thermal, elastic) moduli which govern the macroscopic behavior. The aim of the homogenization theory is to describe the macroscopic properties of the composite by taking into account the properties of the microscopic structure. Homogenization covers a wide range of applications, such as the study of composites,[21] optimal design problems,[1] neutron transport problems[23] and many other fields, see for example Refs. 4 and 10 and references therein.

In this paper we are interested in the study of a homogenization problem for highly anysotropic non-periodic composites. We want to emphasis that we are considering a non-periodic modellization. Indeed in many situations a simple and appropriate way to model evenly distributed heterogeneities is to consider as a first approximation a periodic distribution, see the vast literature on the topic, e.g. Refs. 10, 13 and 25. However, this approximation is not always appropriate, for example to the case of random media.[27]

In many physical situation a certain "energy" (thermal, electric, elastic) of a system is modelled by a nonlinear integral of the form

$$F_\varepsilon(u) = \int_\Omega f_\varepsilon(x, \nabla u)\, dx\,, \tag{1.1}$$

where ε is a scale parameter, small compared to the size of the set $\Omega \subset \mathbb{R}^3$, describing in some way the heterogeneities of the medium, and u is a physical field such as for example the displacement or the temperature. The aim of the homogenization in this case is to describe the overall properties of the medium by a simpler homogenized energy integral of the form

$$F_{\mathrm{hom}}(u) = \int_\Omega f_{\mathrm{hom}}(\nabla u)\, dx\,, \tag{1.2}$$

obtained by appropriately taking the limit of F_ε for $\varepsilon \to 0$.

In the homogenization of multiple integrals, the right notion of variational convergence to be used turns out to be the Γ-convergence one, introduced in Ref. 16, whose natural framework is that of lower semicontinuous functionals. This is precisely the tool we use in this paper to obtain the convergence to the homogenized problem. For comprehensive accounts on this techique see e.g. Refs. 3 and 14.

We recall that there are many other different techniques available to treat homogenization problems which arise in other mathematical contexts, for example in the context of PDEs or of boundary values problems, for which we refer to Refs. 1, 10 and 13 and references therein.

In this paper we consider the following functionals,

$$J_\varepsilon(u) = \int_\Omega f_\varepsilon(x, \nabla u)\, dx + \int_\Omega g u\, dx\,, \tag{1.3}$$

defined on any bounded open set Ω with Lipschitz boundary in $\mathbb{R}^n$, for $\varepsilon > 0$ and $g \in L^\infty(\Omega)$, with u belonging to some space where ∇u exists. The family of functions $f_\varepsilon : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$ is defined by

$$f_\varepsilon(x, \xi) := \langle \alpha_\varepsilon(x, \xi) A_\varepsilon(x)\xi, \xi\rangle\,, \quad \varepsilon > 0\,, \tag{1.4}$$

where $A_\varepsilon : \mathbb{R}^n \to M^{n\times n}(\mathbb{R})$ is a family of measurable matrix-valued functions such that $A^t_\varepsilon(x) = A_\varepsilon(x) \geq 0$ a.e. in $\mathbb{R}^n$ for all $\varepsilon > 0$. We shall assume that there exist two families of weight functions (i.e. non-negative locally summable functions) $(\lambda_\varepsilon)_{\varepsilon>0}$, $(\Lambda_\varepsilon)_{\varepsilon>0}$ such that:

$$\lambda_\varepsilon^{2/p}(x)|\xi|^2 \leq \langle A_\varepsilon(x)\xi, \xi\rangle \leq \Lambda_\varepsilon^{2/p}(x)|\xi|^2 \tag{1.5}$$

for a.e. $x \in \mathbb{R}^n$ and for all $\xi \in \mathbb{R}^n$, with $p > 1$.

Moreover, we assume

$$\begin{cases} \text{(i)} & \alpha_\varepsilon(x, \cdot) : \mathbb{R}^n \to \mathbb{R} \text{ continuous for a.e. } x \in \mathbb{R}^n\,, \\ & \alpha_\varepsilon(\cdot, \xi) \text{ measurable on } \mathbb{R}^n \text{ for any } \xi \in \mathbb{R}^n\,; \\ \text{(ii)} & b_1\langle A_\varepsilon(x)\xi, \xi\rangle^{p/2-1} \leq \alpha_\varepsilon(x, \xi) \leq b_2\langle A_\varepsilon(x)\xi, \xi\rangle^{p/2-1}\,, \\ & \text{where } b_1 \text{ and } b_2 \text{ are positive constants};\\ \text{(iii)} & \langle \alpha_\varepsilon(x, \xi) A_\varepsilon(x)\xi, \xi\rangle \text{ convex in } \xi\,. \end{cases} \tag{1.6}$$

Up to changing the constants in (1.5), it holds

$$\lambda_\varepsilon(x)|\xi|^p \leq f_\varepsilon(x, \xi) \leq \Lambda_\varepsilon(x)|\xi|^p\,, \tag{1.7}$$

for a.e. $x \in \mathbb{R}^n$, for any $\xi \in \mathbb{R}^n$, $p > 1$.

Concerning the weights we will make the following assumptions:

(w_1) $\lambda_\varepsilon \in A_p(A), p > 1$, uniformly with respect to ε, i.e. there exists $A > 0$ such that

$$\left(\fint_I \lambda_\varepsilon\, dx\right)\left(\fint_I \lambda_\varepsilon^{-1/(p-1)} dx\right)^{p-1} \leq A \tag{1.8}$$

for all cubes I with faces parallel to the coordinate planes and for all $\varepsilon > 0$; in addition, for any given cube $I \subset \mathbb{R}^n$, there exists a constant $C_1 = C_1(I) > 0$ such that

$$\fint_I \lambda_\varepsilon \geq C_1 \tag{1.9}$$

for all $\varepsilon > 0$;

(w_2) $\Lambda_\varepsilon \in L^{1+\mu}_{\rm loc}(\mathbb{R}^n)$, $\mu > 0$, uniformly with respect to ε, i.e. for any given cube $I \subset \mathbb{R}^n$ there exists a constant $C_2 = C_2(I) > 0$ such that

$$\left(\int_I \Lambda_\varepsilon^{1+\mu} dx\right)^{\frac{1}{1+\mu}} \le C_2 \tag{1.10}$$

for all $\varepsilon > 0$;

(w_3) for every cube $I \subseteq \mathbb{R}^n$ there exists a constant $K = K(I) > 0$ such that

$$\int_I \frac{\Lambda_\varepsilon(x)}{\lambda_\varepsilon(x)}\, dx \le K\,, \tag{1.11}$$

uniformly with respect to ε.

We give some examples of admissible weights at the end of the section.

Given the functionals $J_\varepsilon(u) = \int_\Omega \langle \alpha_\varepsilon(x, \nabla u) A_\varepsilon(x)\nabla u, \nabla u\rangle\, dx + \int_\Omega gu\, dx$ we want to prove that there exists a subsequence $\langle \alpha_{\varepsilon_k}(x,\xi) A_{\varepsilon_k}(x)\xi, \xi\rangle$ and a function $f_\infty(x,\xi)$, whose properties will be specified later, such that the functionals $J_{\varepsilon_k}(u)$ Γ-converge in the $L^1(\Omega)$ topology to a functional that can be written, for suitably regular functions u, as $\int_\Omega f_\infty(x,\nabla u)\, dx + \int_\Omega gu\, dx$. In addition we will show that

$$\min\{J_{\varepsilon_k}(u) : u = 0 \text{ on } \partial\Omega\}$$

converge in the $L^1(\Omega)$ topology to

$$\min\left\{\int_\Omega f_\infty(x,\nabla u)\, dx + \int_\Omega gu\, dx : u = 0 \text{ on } \partial\Omega\right\} \tag{1.12}$$

with the minima taken in suitable function spaces.

Functionals as in (1.3), with a weighted growth condition on the energy density such as the one considered in (1.7), can be used to describe some fine properties of a wide class of degenerate anisotropic structures. Notice that we do not make any periodicity requirement on the energy density, in order to be able to describe non-periodic structures. For example one could think of models of porous media,[11] or models of mixtures of materials with different nonlinearities, showing a particular different behavior along preferred directions.[20]

As a simple model case of a physical situation with an energy density such as (1.4) we can consider an electromagnetic material in $\mathbb{R}^3$ with a mesoscopic structure given by alternate layers of two materials, a metal with high conductivity and a plastic that is electrically insulating. The constitutive equation for this medium is given by $\mathbf{j}(x) = \sigma(x)\mathbf{e}(x)$, $x = (x_1, x_2, x_3)$, where $\mathbf{j}(x)$ is the current field, $\mathbf{e}(x)$ is the electric field and the conductivity tensor field $(\sigma_{i,j}(x))_{i,j=1,2,3}$ is such that $\sigma_{1,3}(x) = \sigma_{3,1}(x) = \sigma_{2,3}(x) = \sigma_{3,2}(x) = 0$ for any $x \in \mathbb{R}^3$, and $\sigma_{3,3}(x) = \lambda(x_3)$, λ being a periodic A_2-weight vanishing at a finite number of points. The energy density of such a medium is modelled by $f_\varepsilon(x,\nabla u) = \langle \sigma_\varepsilon(x)\nabla u, \nabla u\rangle$ and the energy is described by the functional

$$J_\varepsilon(u) = \int_\Omega \left\{\sum_{i,j=1}^{2} \sigma_{i,j}\partial_{x_i}u\partial_{x_j}u + \lambda_\varepsilon(x_3)|\partial_{x_3}u|^2\right\} dx + \int_\Omega gu\, dx\,, \tag{1.13}$$

where Ω is a bounded connected open subset of $\mathbb{R}^3$, the sub-matrix $(\sigma_{i,j}(x))_{i,j=1,2}$ is bounded and positively definite (again to simplify the picture) and the functions λ_ε have the form $\lambda_\varepsilon(x_3) = \lambda(\frac{x_3}{\varepsilon})$.

The homogenization process leads to a highly anisotropic composite that has the conducting properties of the metal in the directions parallel to the layers and the insulating properties of the plastic normal to the layers.

Functionals like (1.13) can model the energy of several different physical situations in addition to the one mentioned above, like for instance thermal conduction in a body containing at the microscopic scale a horizontal bundle of thin layers characterized by a very low thermal conductivity, or again the behavior of heterogeneous materials with some kind of "stiff degeneracies" along a given direction. Clearly, in (1.13), Λ_ε is a constant, but we might easily modify the model to take into account opposite behavior of the inclusions, taking now $\lambda_\varepsilon \equiv 1$ and possibly $\Lambda_\varepsilon = +\infty$ along some direction.

Concerning previous studies of models of possibly degenerate anisotropic situations as the ones we are interested in here, we are basically aware of two papers. In Ref. 15 the author consider functionals with density energy f_ε with a growth condition as in (1.7), restricted to the case $\Lambda_\varepsilon = c\lambda_\varepsilon$, that is, describing possibly degenerate phenomena basically isotropic in all directions. On the contrary, since we are interested in studying strongly anisotropic situations, we consider two different families of weights λ_ε and Λ_ε, which can differ each other in the sense that the ratios $\Lambda_\varepsilon/\lambda_\varepsilon$ can blow-up at some point, provided (1.11) is satisfied.

A possibly degenerate anisotropic nonlinear elliptic problem with the same structure of (1.4) and with the same structural hypotheses was treated in Ref. 17, with a completely different approach based on weighted compensated compactness techniques. However, in Ref. 17 the authors were somehow forced to restrict themselves to the periodic context.

We want to stress explicitly that the anisotropy of the problem treated here, i.e. the presence of two different families of weights controlling the structure of the functionals, leads to face several problems that we try to explain now.

We have to be vague for a while (for precise definitions see the next section). Let Ω be a bounded open set, and $p > 1$. We denote by $W^{1,p}_{A_\varepsilon}(\Omega)$ a two-weight Sobolev space related to the matrix A_ε (satisfying (1.5) for any $\varepsilon > 0$), endowed with a norm "induced" by A_ε, in a sense specified in Definition 2.3 below. Then we consider the space given by the closure of regular functions in Ω with respect to that norm, and we denote it by $H^{1,p}_{A_\varepsilon}(\Omega)$. Now, a Meyers–Serrin type result reading "$H^{1,p}_{A_\varepsilon}(\Omega) = W^{1,p}_{A_\varepsilon}(\Omega)$" is not true in the two-weight case (see Ref. 9 and also some examples in Sec. 6). This means that $H^{1,p}_{A_\varepsilon}(\Omega)$ does not coincide with the finiteness domain of the functional J_ε, and therefore a Lavrentev-type phenomenon can occur.[22]

In addition, when dealing with the $\Gamma\text{-}\lim_{\varepsilon\to 0} J_\varepsilon$, it is not even obvious in which way to define the spaces of type H and W connected with the finiteness domain of the Γ-limit. One of the features of this note is to produce consistent definitions of suitable H and W-type space where to represent the Γ-limit. This problem, representing one of the main difficulties of this work, is extensively discussed in Secs. 4 and 5. Moreover, in Sec. 6 we show that these spaces H and W might be different.

Examples of admissible weights

The simplest situation one could think of is the case of equal periodical A_p weights $\lambda = \Lambda$, if we set $\lambda_\varepsilon(x) = \lambda(\frac{x}{\varepsilon})$, which clearly satisfy hypotheses $(w1)$, $(w2)$ and $(w3)$ (notice that a periodic extension of an A_p weight is again in the A_p class).

A less trivial example (however still periodic) is provided by a pair (λ, Λ) of periodic weights with the same period, if we put $\lambda_\varepsilon(x) = \lambda(\frac{x}{\varepsilon})$, $\Lambda_\varepsilon(x) = \Lambda(\frac{x}{\varepsilon})$, with $\lambda \in A_p$ and $\Lambda \in L^{1+\mu}_{\mathrm{loc}}(\mathbb{R}^n)$ provided $\Lambda/\lambda \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

In order to provide simple examples of non-periodic weights, consider first for instance the case $\lambda_\varepsilon \equiv 1$. Then (w_1) is obvious, whereas (w_2) and (w_3) read straightforwardly as $\Lambda_\varepsilon \in L^{1+\mu}_{\mathrm{loc}}(\mathbb{R}^n)$ uniformly in ε. On the other hand, it is easy to produce non-periodic A_p weights with controlled behavior in ε as required in (w_1) and (w_3) just starting from periodic A_p weights with non-rationally comparable periods.

We stress also explicitly that, even if we assume λ_ε, Λ_ε periodic, the function f_ε might not be periodic.

Our notation is standard. The Lebesgue measure of sets is denoted by $|\ |$ and the scalar product in $\mathbb{R}^n$ by $\langle\, ,\rangle$. For any Lebesgue measurable set E, if $\omega \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ we denote by $\omega(E) := \int_E \omega(x)\,dx$, and if S denotes a set in Ω, u_S is the average of the function u in the set S, i.e. $u_S = \fint_S u(y)\,dy = \frac{1}{|S|}\int_S u(y)\,dy$. We denote by $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$ the space of locally Lipschitz functions on $\mathbb{R}^n$ and, for any bounded open set Ω we denote by $\mathrm{Lip}_0(\Omega)$ the space of locally Lipschitz functions with support compactly contained in Ω.

The plan of the work is the following. Section 2 briefly recalls the definition of Γ-convergence and some basic properties of A_p weights. In Sec. 3 we prove some preliminary results about the Γ-limit of the functionals J_ε. Section 4 is devoted to the definition of suitable Sobolev-type spaces, denoted by $\overset{\circ}{H}_\infty(\Omega)$ and $\overset{\circ}{W}_\infty(\Omega)$, for studying problem (1.12). In Sec. 5 we obtain a representation theorem for the Γ-limit of J_ε in $\overset{\circ}{H}_\infty$ and we give a result concerning the convergence of the minimum points of J_ε. In Sec. 6 we produce two examples concerning the relations between the spaces H and W used in the paper. In the first example we provide weights such that $H^{1,p}_{A_\varepsilon}(\Omega) \neq W^{1,p}_{A_\varepsilon}(\Omega)$ but when we pass to the Γ-limit $\overset{\circ}{H}_\infty(\Omega) = \overset{\circ}{W}_\infty(\Omega)$. On the contrary, in the second one we choose weight functions so that $H^{1,p}_{A_\varepsilon}(\Omega) = W^{1,p}_{A_\varepsilon}(\Omega)$ but $\overset{\circ}{H}_\infty(\Omega) \neq \overset{\circ}{W}_\infty(\Omega)$.

2. Basic Tools

The notion of convergence used in this paper is the so-called Γ-convergence (see Ref. 16).

Definition 2.1. Let X be a metric space and for $\varepsilon > 0$ consider the functional $F_\varepsilon : X \to [-\infty, +\infty]$. We say that F_ε Γ-converges to F on X as ε goes to zero if the following two conditions hold:

(1) for every $u \in X$ and every sequence (u_ε) which converges to u in X there holds

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \geq F(u)\,; \tag{2.1}$$

(2) for every $u \in X$ there exists a sequence (u_ε) which converges to u in X and

$$\lim_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) = F(u)\,. \tag{2.2}$$

Many properties about Γ-convergence can be found for example in Refs. 3–5 and 14. In particular we remark explicitly that, if the functionals F_ε are convex, the Γ-limit F is convex. Moreover Γ-convergence turns out to be stable with respect to perturbation with a continuous function and composition with an increasing function.

We briefly recall some results on Muckenhoupt weights:

Definition 2.2. [24] We say that a non-negative mesurable function λ on $\mathbb{R}^n$ is a weight in the class $A_p(A), p > 1$, for a given constant $A \geq 1$, if it satisfies the following condition:

$$\left(\fint_I \lambda \, dx \right) \left(\fint_I \lambda^{-1/(p-1)} dx \right)^{p-1} \leq A\,,$$

for all cubes $I \subset \mathbb{R}^n$. Moreover, we denote by $A_p := \bigcup_{A \geq 1} A_p(A)$.

We need the following result, see Refs. 12, 15 and 18:

Theorem 2.1. *Let $\lambda \in A_p(A), p > 1$. Then there exist two positive constants $\sigma = \sigma(n, p, A)$ and $C = C(n, p, A)$ such that*

$$\left(\fint_I \lambda^{1+\sigma} dx \right)^{1/(1+\sigma)} \leq C \fint_I \lambda \, dx\,, \tag{2.3}$$

$$\left(\fint_I \lambda^{-(1+\sigma)/(p-1)} dx \right)^{1/(1+\sigma)} \leq C \fint_I \lambda^{-1/(p-1)} dx \tag{2.4}$$

for every cube I with faces parallel to the coordinate planes.

Let M be the Hardy–Littlewood maximal function operator defined by $Mu(x) = \sup_{I \ni x} \fint_I |u(y)| \, dy$, where u is a locally integrable function and I is a cube as above. The following weighted norm inequality for M holds:

Theorem 2.2. [24] *Let $1 < p < \infty$. Given a weight function $\lambda \in A_p(A)$ there exists a constant $c = c(A)$ such that*

$$\int_{\mathbb{R}^n} (M|u|)^p \lambda(x) \, dx \leq c \int_{\mathbb{R}^n} |u|^p \lambda(x) \, dx \quad \forall\, u \in L^p(\mathbb{R}^n, \lambda)\,, \tag{2.5}$$

where $L^p(\mathbb{R}^n, \lambda) = \{u \in L^1_{\text{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} |u(x)|^p \lambda(x) \, dx < \infty\}$.

Let us now introduce some function spaces suitable to our problem. Let Ω be a bounded open set in $\mathbb{R}^n$. If $p > 1$ and $\lambda \in A_p$ we denote by $W^{1,p}(\Omega, \lambda) = \{u \in L^p(\Omega, \lambda) \cap W^{1,1}_{\mathrm{loc}}(\Omega) : |\nabla u| \in L^p(\Omega, \lambda)\}$, by $\overset{\circ}{W}{}^{1,p}(\Omega, \lambda) := \overset{\circ}{W}{}^{1,1}(\Omega) \cap W^{1,p}(\Omega, \lambda)$, by $H^{1,p}(\Omega, \lambda)$ the closure of $C^1(\Omega) \cap W^{1,p}(\Omega, \lambda)$ in $W^{1,p}(\Omega, \lambda)$ and by $\overset{\circ}{H}{}^{1,p}(\Omega, \lambda)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega, \lambda)$. In addition, if Ω has Lipschitz boundary, then the following characterization has been proved (see Ref. 15 and also Ref. 9):

$$\overset{\circ}{H}{}^{1,p}(\Omega, \lambda) = \overset{\circ}{W}{}^{1,p}(\Omega, \lambda) .$$

Let A_ε be the matrix defined in (1.5) and the pair of weights $(\lambda_\varepsilon, \Lambda_\varepsilon)$ satisfying hypothesis (w_1), (w_2) and (w_3) listed in the Introduction. In complete analogy with Ref. 17, we define the following spaces:

Definition 2.3. If $1 < p < \infty$, we define for all $\varepsilon > 0$

$$W^{1,p}_{A_\varepsilon}(\Omega) = \left\{ u \in L^p(\Omega, \lambda_\varepsilon) \cap W^{1,1}_{\mathrm{loc}}(\Omega) : \int_\Omega \langle A_\varepsilon \nabla u, \nabla u \rangle^{p/2} dx < +\infty \right\}, \tag{2.6}$$

endowed with the norm

$$\|u\|_{W^{1,p}_{A_\varepsilon}(\Omega)} = \left(\int_\Omega |u|^p \lambda_\varepsilon \, dx \right)^{1/p} + \left(\int_\Omega \langle A_\varepsilon \nabla u, \nabla u \rangle^{p/2} dx \right)^{1/p} . \tag{2.7}$$

With suitable minor changes in the proof of Theorem 1 in Ref. 17, it is possible to show that $W^{1,p}_{A_\varepsilon}(\Omega)$ is a reflexive Banach space continuously embedded in $W^{1,p}(\Omega, \lambda_\varepsilon)$ and hence continuously embedded in $W^{1,1}(\Omega)$. We stress explicitly that the embedding in $W^{1,1}(\Omega)$ is uniform in ε since the constant controlling the norm $W^{1,1}(\Omega)$ depends only on the constant A of Definition 2.2.

Definition 2.4. If $1 < p < \infty$, we define for all $\varepsilon > 0$

$$\overset{\circ}{H}{}^{1,p}_{A_\varepsilon}(\Omega) = \overline{\mathrm{Lip}_0(\Omega)}^{\|\cdot\|_{W^{1,p}_{A_\varepsilon}(\Omega)}} . \tag{2.8}$$

Due to the assumptions on f_ε, the functionals J_ε defined in (1.3) have minima in $\overset{\circ}{W}{}^{1,p}_{A_\varepsilon}(\Omega) := \overset{\circ}{W}{}^{1,1}(\Omega) \cap \overset{\circ}{W}{}^{1,p}_{A_\varepsilon}(\Omega)$ for any $\varepsilon > 0$ (see e.g. Ref. 19). In general $\overset{\circ}{W}{}^{1,p}_{A_\varepsilon}(\Omega)$ and $\overset{\circ}{H}{}^{1,p}_{A_\varepsilon}(\Omega)$ do not coincide, as shown in Sec. 6. Therefore the result concerning the convergence of minimum points of J_ε when passing to the Γ-limit is restricted to only the spaces $\overset{\circ}{H}{}^{1,p}_{A_\varepsilon}(\Omega)$ (see Theorem 5.2 for a detailed statement).

3. Preliminary Results

The following theorem states that for functions in $\mathrm{Lip}_{\mathrm{loc}}$ the Γ-limit of (1.3) has an integral representation. More precisely we have:

Theorem 3.1. (Theorem 3.4 of Ref. 7) *Let* (f_ε) *be the family of functions defined in* (1.4) *satisfying hypothesis* (1.6), *and* (Λ_ε) *be a family of weights satisfying* (1.10). *Then there exists a weight* $\Lambda_\infty \in L^{1+\mu}_{\mathrm{loc}}(\mathbb{R}^n)$ *such that, up to subsequences,*

$$\Lambda_\varepsilon \overset{w-L^{1+\mu}(I)}{\longrightarrow} \Lambda_\infty , \tag{3.1}$$

for every given cube I in $\mathbb{R}^n$. Moreover, there exist a subfamily $(f_{\varepsilon_k}) \subset (f_\varepsilon)$ and a function $f_\infty(x,\xi)$ convex in ξ such that

$$0 \le f_\infty(x,\xi) \le \Lambda_\infty(x)|\xi|^p\,, \quad \text{a.e. } x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n \tag{3.2}$$

and

$$\Gamma\text{-}\lim_{k\to\infty} \int_\Omega f_{\varepsilon_k}(x,\nabla u)\,dx = \int_\Omega f_\infty(x,\nabla u)\,dx < +\infty\,, \tag{3.3}$$

in the $L^1(\Omega)$ topology, for any bounded open set Ω, $u \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$.

Now if we assume a uniform A_p condition and a local uniform lower bound on the weights λ_ε, it is possible to obtain, locally, a sharper estimate for f_∞. This result is largely inspired by Proposition 4.1 in Ref. 15.

Theorem 3.2. *Let (f_ε) be the family of functions defined in (1.4) satisfying hypothesis (1.6). Let I be a given cube, and (λ_ε) and (Λ_ε) be as in (1.8)–(1.10) respectively. Then there exist a subfamily $(f_{\varepsilon_k}) \subset (f_\varepsilon)$ and two weights $\Lambda_\infty \in L^{1+\mu}_{\mathrm{loc}}(\mathbb{R}^n)$ and λ_∞ in A_p such that,*

$$\Lambda_\varepsilon \overset{w-L^{1+\mu}(I)}{\longrightarrow} \Lambda_\infty\,, \tag{3.4}$$

and there exists $\sigma > 0$ such that

$$\lambda_\varepsilon^{-1/(p-1)} \overset{w-L^{1+\sigma}(I)}{\longrightarrow} \lambda_\infty^{-1/(p-1)}\,. \tag{3.5}$$

Moreover, there exists a function $f_\infty(x,\xi)$, convex in ξ, satisfying

$$\lambda_\infty(x)|\xi|^p \le f_\infty(x,\xi) \le \Lambda_\infty(x)|\xi|^p \tag{3.6}$$

for almost every $x \in I$, for any $\xi \in \mathbb{R}^n$ and such that

$$\Gamma\text{-}\lim_{k\to\infty} \int_\Omega f_{\varepsilon_k}(x,\nabla u)\,dx = \int_\Omega f_\infty(x,\nabla u)\,dx \tag{3.7}$$

in the $L^1(\Omega)$-topology, for every open set $\Omega \subset\subset I$ and $u \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Let I be a given cube such that (1.9) holds. By hypothesis (1.10), we have, up to a subsequence,

$$\Lambda_\varepsilon \overset{w-L^{1+\mu}(I)}{\longrightarrow} \Lambda_\infty\,. \tag{3.8}$$

By Theorem 3.1 the estimate (3.2) holds almost everywhere in I, and we have the related Γ-convergence result (3.3) for all bounded open set $\Omega \subset\subset I$ for every $u \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$.

Due to hypotheses (1.8) and (1.9), and by Theorem 2.1 there exist $\sigma > 0$ and $C_3 > 0$ such that, for all $\varepsilon > 0$, we have

$$\left(⨍_I \lambda_\varepsilon^{-(1+\sigma)/(p-1)}\right)^{1/(1+\sigma)} \le C_3 \left(\frac{A}{C_1}\right)^{1/(p-1)}\,; \tag{3.9}$$

therefore, up to a subsequence, there exists a function $\lambda_\infty^{-1/(p-1)} \in L^{1+\sigma}(I)$ such that

$$\lambda_\varepsilon^{-1/(p-1)} \overset{w-L^{1+\sigma}(I)}{\longrightarrow} \lambda_\infty^{-1/(p-1)}. \tag{3.10}$$

As for the proof that $\lambda_\infty \in A_p$, we argue as in the proof of Proposition 4.1 in Ref. 15, since by (1.10), we have $⨍_I \lambda_\varepsilon \, dx \le C_2 |I|^{\frac{\mu-1}{1+\mu}}$.

Now, to get the estimate from below in (3.6) we proceed as follows. By Hölder's inequality, if $u \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$ is given, it is trivial to see that

$$\left(\int_\Omega |\nabla u| \, dx\right)^p \le \left(\int_\Omega |\nabla u|^p \lambda_\varepsilon \, dx\right) \left(\int_\Omega \lambda_\varepsilon^{-1/p-1} dx\right)^{p-1}.$$

Applying the lower estimate in (1.7) to the right-hand side, it follows that

$$\left(\int_\Omega |\nabla u| \, dx\right)^p \le \left(\int_\Omega f_\varepsilon(x, \nabla u) \, dx\right) \left(\int_\Omega \lambda_\varepsilon^{-1/p-1} dx\right)^{p-1}. \tag{3.11}$$

By Theorem 3.1 there exists $u_\varepsilon \to u$ in $L^1(\Omega)$, such that $\lim_{\varepsilon\to 0} \int_\Omega f_\varepsilon(x, \nabla u_\varepsilon) \, dx = \int_\Omega f_\infty(x, \nabla u) \, dx$. On the other hand, the functional $\int_\Omega |\nabla u| \, dx$ is $L^1(\Omega)$-lower semicontinuous on $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ (and hence in $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$), hence $\int_\Omega |\nabla u| \, dx \le \liminf_{\varepsilon\to 0} \int_\Omega |\nabla u_\varepsilon| \, dx$. Therefore, using (3.11) for u_ε, we have $(\int_\Omega |\nabla u| \, dx)^p \le \liminf_\varepsilon (\int_\Omega |\nabla u_\varepsilon| \, dx)^p \le \liminf_\varepsilon (\int_\Omega f_\varepsilon(x, \nabla u_\varepsilon) \, dx)(\int_\Omega \lambda_\varepsilon^{-1/p-1} dx)^{p-1}$, and we get

$$\left(\int_\Omega |\nabla u| \, dx\right)^p \le \left(\int_\Omega f_\infty(x, \nabla u) \, dx\right) \left(\int_\Omega \lambda_\infty^{-1/p-1} dx\right)^{p-1}.$$

Now take $u(x) = \langle \xi, x\rangle$. Choosing $\Omega = B(\bar{x}, r)$ and letting $r \to 0$ we get the estimate $\lambda_\infty(\bar{x})|\xi|^p \le f_\infty(\bar{x}, \xi)$ at any Lebesgue point $\bar{x}$ of $\lambda_\infty^{-1/p-1}$ and $f_\infty(\cdot, \xi)$. □

Remark 3.1. If we choose $\alpha_\varepsilon(x, \xi) = \langle A_\varepsilon(x)\xi, \xi\rangle^{p/2-1}$, we can apply Theorem 3.2 to the functions $f_\varepsilon(x, \xi) = \langle A_\varepsilon(x)\xi, \xi\rangle^{p/2}$. More precisely, there exists a function $Q(x, \xi)$, convex in ξ, satisfying

$$\lambda_\infty(x)|\xi|^p \le Q(x, \xi) \le \Lambda_\infty(x)|\xi|^p \tag{3.12}$$

for almost every $x \in I$, for any $\xi \in \mathbb{R}^n$ and such that

$$\Gamma\text{-}\lim_{k\to\infty} \int_\Omega \langle A_{\varepsilon_k}(x)\nabla u, \nabla u\rangle^{p/2} dx = \int_\Omega Q(x, \nabla u) \, dx \tag{3.13}$$

in the $L^1(\Omega)$ topology, for every open set $\Omega \subset\subset I$ and $u \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$.

Even if the functions f_ε have a polynomial structure, as in previous remark, the the Γ-limit might have a different structure (for a counterexample see Ref. 6). Nevertheless, if $f_\varepsilon(x, \xi) = \langle A_\varepsilon(x)\xi, \xi\rangle^{p/2}$, we show in the next section that $\int_\Omega Q(x, \nabla u) \, dx$ produces a norm in $\mathrm{Lip}_0(\Omega)$.

4. Definition of the Space $\overset{\circ}{H}_\infty$

In the sequel I is a given cube, where hypotheses of Theorem 3.2 are satisfied and we consider a Lipschitz domain Ω, compactly contained in I. For $\varepsilon > 0$, we consider the functionals

$$\hat{G}_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \langle A_\varepsilon(x)\nabla u, \nabla u)\rangle^{p/2} dx & u \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)\,, \\ +\infty & u \in (W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n) - \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n))\,. \end{cases} \tag{4.1}$$

By a Γ-convergence property, up to subsequences, it is known that the Γ-limit in the $L^1(\Omega)$ topology is well defined for all $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$; we denote it by $\hat{G}_\Gamma(u)$. Using Remark 3.1, and because of Proposition 6.15 in Ref. 14 the previous Γ-limit is represented by the functionals

$$\hat{G}_\Gamma(u) = \int_\Omega Q(x, \nabla u)\, dx \tag{4.2}$$

for every $u \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$, where Q satisfies (3.12).

We introduce the following notation:

Definition 4.1. For $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$, we denote by $[u]$ the following Γ-limit

$$[u] := \Gamma\text{-}\lim_{\varepsilon\to 0} \left(\int_\Omega |u|\, dx + (\hat{G}_\varepsilon(u))^{1/p} \right) \tag{4.3}$$

in the $L^1(\Omega)$ topology.

Recalling that Γ-convergence is stable under continuous perturbations and under composition by an increasing function, the following proposition follows easily:

Proposition 4.1. *For $u \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$,*

$$[u] = \int_\Omega |u|\, dx + \left(\int_\Omega Q(x, \nabla u)\, dx \right)^{1/p}\,, \tag{4.4}$$

where $Q(x,\xi)$ is given by (3.13) *and satisfies* (3.12).

We now restrict ourselves to the space $\mathrm{Lip}_0(\Omega)$ and show that $[\cdot]$ is a norm on $\mathrm{Lip}_0(\Omega)$. To achieve this result we first recall the following statement proved in Ref. 8, Theorem 1.2:

Proposition 4.2. *Let Ω be a bounded Lipschitz domain, $u \in W^{1,1}_{\mathrm{loc}}(\Omega) \cap L^1(\Omega)$, and $\lambda_\varepsilon \in A_p(A)$, then*

$$\int_\Omega |u - u_\Omega|^p \lambda_\varepsilon\, dx \le C_\Omega \int_\Omega |\nabla u|^p \lambda_\varepsilon\, dx, \tag{4.5}$$

where $u_\Omega = ⨍_\Omega |u|\, dx$ and the constant $C_\Omega = C_\Omega(A)$ is independent of u and ε.

Let us introduce a new norm on $W^{1,p}_{A_\varepsilon}(\Omega)$ equivalent to (2.7):

Lemma 4.1. *Let* (λ_ε), (Λ_ε) *satisfying* (1.8) *and* (1.10); *in addition let* I *be a given cube, assume that* (1.9) *holds in* I *for all* $\varepsilon > 0$. *Let* Ω *be a bounded Lipschitz domain,* $\Omega \subset\subset I$. *The norm* (2.7) *and the following*

$$|||u|||_{W^{1,p}_{A_\varepsilon}(\Omega)} := \int_\Omega |u|\,dx + \left(\int_\Omega \langle A_\varepsilon \nabla u, \nabla u\rangle^{p/2} dx\right)^{1/p} \tag{4.6}$$

are equivalent on $W^{1,p}_{A_\varepsilon}(\Omega)$, *and the constants appearing in the equivalence are independent of* ε.

Proof. By Hölder's inequality, since λ_ε are uniformly A_p and (1.9) holds, we get $\int_\Omega |u|\,dx \le c(\int_\Omega |u|^p \lambda_\varepsilon\,dx)^{1/p}$ with c independent of ε.

On the other hand, using Poincaré inequality (4.5) and (1.10), we easily obtain

$$\begin{aligned}\left(\int_\Omega |u|^p \lambda_\varepsilon\,dx\right)^{1/p} &\le \left(\int_\Omega |u-u_\Omega|^p \lambda_\varepsilon\,dx\right)^{1/p} + |u_\Omega|\lambda_\varepsilon(\Omega)^{1/p} \\ &\le C_\Omega^{1/p}\left(\int_\Omega |\nabla u|^p \lambda_\varepsilon\,dx\right)^{1/p} + \frac{\Lambda_\varepsilon(\Omega)^{1/p}}{|\Omega|}\int_\Omega |u|\,dx \\ &\le C_\Omega^{1/p}\left(\int_\Omega \langle A_\varepsilon \nabla u, \nabla u\rangle^{p/2} dx\right)^{1/p} + C'_\Omega \int_\Omega |u|\,dx\,,\end{aligned}$$

where C'_Ω is a constant independent of ε. $\square$

By previous lemma $[u]$ can be seen as a Γ-limit of norms, hence we prove:

Proposition 4.3. $[\cdot]$ *is a norm on* $\mathrm{Lip}_0(\Omega)$.

Proof. First $[u] < +\infty$ on $\mathrm{Lip}_0(\Omega)$; indeed

$$\int_\Omega Q(x,\nabla u)\,dx \le \int_\Omega \Lambda_\infty(x)|\nabla u|^p dx \le \|u\|^p_{\mathrm{Lip}_0(\Omega)} \int_\Omega \Lambda_\infty(x)\,dx \le C_2^{1+\mu}\|u\|^p_{\mathrm{Lip}_0(\Omega)}$$

by (3.12) and since $\Lambda_\infty \in L^{1+\mu}_{\mathrm{loc}}(\mathbb{R}^n)$.

To simplify notations in the remaining of the proof we will indicate the norm (4.6) by $|||\cdot|||_\varepsilon$.

Let u be in $\mathrm{Lip}_0(\Omega)$ and $t \in \mathbb{R}$, $t \neq 0$. By definition of Γ-convergence, there exists a sequence of functions $\{v_\varepsilon\}$ in $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$v_\varepsilon \to tu \text{ in } L^1(\Omega)\,, \quad |||v_\varepsilon|||_\varepsilon \to [tu]\,.$$

Without loss of generality we can assume $v_\varepsilon = tu_\varepsilon$, with $u_\varepsilon \to u$ in $L^1(\Omega)$. Then

$$\begin{aligned}[u] &\le \liminf_{\varepsilon\to 0} |||u_\varepsilon|||_\varepsilon = \frac{1}{|t|}\liminf_{\varepsilon\to 0} |t|\,|||u_\varepsilon|||_\varepsilon \\ &= \frac{1}{|t|}\liminf_{\varepsilon\to 0} |||tu_\varepsilon|||_\varepsilon = \frac{1}{|t|}[tu]\,.\end{aligned}$$

Hence, for $t \neq 0$, $|t|[u] \le [tu]$ and analogously $\frac{1}{|t|}[tu] \le [u]$, which imply $[tu] = |t|[u]$.

As for the triangular inequality, let u, v be in $\mathrm{Lip}_0(\Omega)$. Then there exists two sequences of functions $\{u_\varepsilon\}$ and $\{u_\varepsilon\}$ in $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$u_\varepsilon \to u \text{ in } L^1(\Omega)\,, \qquad |||u_\varepsilon|||_\varepsilon \to [u]\,,$$
$$v_\varepsilon \to v \text{ in } L^1(\Omega)\,, \qquad |||v_\varepsilon|||_\varepsilon \to [v]\,.$$

Since $u_\varepsilon + v_\varepsilon \to u + v$ in $L^1(\Omega)$, we have

$$\begin{aligned}[u+v] &\le \liminf_{\varepsilon\to 0} |||u_\varepsilon + v_\varepsilon|||_\varepsilon \\ &\le \liminf_{\varepsilon\to 0}(|||u_\varepsilon|||_\varepsilon + |||v_\varepsilon|||_\varepsilon) \\ &\le \limsup_{\varepsilon\to 0}(|||u_\varepsilon|||_\varepsilon) + \limsup_{\varepsilon\to 0}(|||v_\varepsilon|||_\varepsilon) \\ &= [u] + [v]\end{aligned}$$

and we are done. □

Thanks to previous results it makes sense to define:

Definition 4.2. If $1 < p < \infty$, we define the space

$$\overset{\circ}{H}_\infty(\Omega) = \overline{\mathrm{Lip}_0(\Omega)}^{[\cdot]}\,. \tag{4.7}$$

The space $\overset{\circ}{H}_\infty(\Omega)$ will turn out to be the suitable one where to treat the minimum problem (1.12) stated in the Introduction and proved in Theorem 5.2 of Sec. 5.

In the sequel we show how to represent the Γ-limit (4.3) in $\overset{\circ}{H}_\infty(\Omega)$. First we prove that

Lemma 4.2. *The space $\overset{\circ}{H}_\infty(\Omega)$ is continuously embedded in $\overset{\circ}{W}{}^{1,1}(\Omega)$.*

Proof. Let $u \in \overset{\circ}{H}_\infty(\Omega)$, then there exists a sequence of functions $u_n \in \mathrm{Lip}_0(\Omega)$ such that $[u_n - u] \to 0$. This implies $u_n \to u$ in $L^1(\Omega)$. On the other hand, for any $v \in \mathrm{Lip}_0(\Omega)$, by (3.12) and (4.5) we have

$$\begin{aligned}[v] &\ge \left(\int_\Omega \lambda_\infty(x)|\nabla v|^p dx\right)^{1/p} + \int_\Omega |v|\,dx \ge C\int_\Omega |\nabla v|\,dx + \int_\Omega |v|\,dx \\ &\ge c\|v\|_{\overset{\circ}{W}{}^{1,1}(\Omega)}\end{aligned} \tag{4.8}$$

since $\lambda_\infty \in A_p$. The sequence u_n is a Cauchy sequence in $\overset{\circ}{H}_\infty(\Omega)$, hence by (4.8) it is a Cauchy sequence in $\overset{\circ}{W}{}^{1,1}(\Omega)$, thus $u_n \to \tilde{u}$ in $\overset{\circ}{W}{}^{1,1}(\Omega)$. By uniqueness of the limit $\tilde{u} = u$, that is $\overset{\circ}{H}_\infty(\Omega) \hookrightarrow \overset{\circ}{W}{}^{1,1}(\Omega)$. □

We show now how to represent the norm $[\cdot]$ in $\overset{\circ}{H}_\infty(\Omega)$; indeed we show that the representation result given in Proposition 4.1 holds not only on $\mathrm{Lip}_0(\Omega)$ but also in $\overset{\circ}{H}_\infty(\Omega)$.

Theorem 4.1. *If $u \in \overset{\circ}{H}_\infty(\Omega)$, then we have $[u] = \int_\Omega |u|\,dx + (\int_\Omega Q(x, \nabla u)\,dx)^{1/p}$.*

Proof. Since $u \in \overset{\circ}{H}_\infty(\Omega)$ there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathrm{Lip}_0(\Omega)$ such that $u_n \to u$ in $\overset{\circ}{H}_\infty(\Omega)$ and hence, by Lemma 4.2, in $\overset{\circ}{W}{}^{1,1}(\Omega)$. Thus, without loss of generality, we may assume that $u_n(x) \to u(x)$, $\nabla u_n(x) \to \nabla u(x)$ for a.e. $x \in \Omega$ and in particular we notice that $Q(x, \nabla u) = \lim_{n\to\infty} Q(x, \nabla u_n)$ since Q is continuous in the second variable. Hence, $\int_\Omega Q(x, \nabla u)\, dx \le \liminf_n \int_\Omega Q(x, \nabla u_n)\, dx \le \liminf_n [u_n] < \infty$, by Fatou's lemma and since $(u_n)_{n\in\mathbb{N}}$ is bounded in $\overset{\circ}{H}_\infty(\Omega)$. Hence $\int_\Omega Q(x, \nabla u)\, dx < +\infty$.

We show now that, if $u_n \in \mathrm{Lip}_0(\Omega)$, $u_n \to u$ in $\overset{\circ}{H}_\infty(\Omega)$ then

$$[u_n] \to \int_\Omega |u|\, dx + \left(\int_\Omega Q(x, \nabla u)\, dx \right)^{1/p} \quad \text{as } n \to \infty\,. \tag{4.9}$$

To prove (4.9) it is enough to show that the assertion holds for a subsequence. Let $u \in \overset{\circ}{H}_\infty(\Omega)$ be given, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{Lip}_0(\Omega)$ converging to u in $\overset{\circ}{H}_\infty(\Omega)$. Again, without loss of generality, we may assume that $u_n(x) \to u(x)$, $\nabla u_n(x) \to \nabla u(x)$ for a.e. $x \in \Omega$. On the other hand, mimicking the proof of Riesz–Fischer theorem, we choose a subsequence $(u_{n_i})_{i\in\mathbb{N}}$ such that $[u_{n_{i+1}} - u_{n_i}] < 1/2^i$. Now we can write (putting $u_{n_0} = 0$)

$$\nabla u_{n_m}(x) = \sum_{i=1}^{m} (\nabla u_{n_i} - \nabla u_{n_{i-1}})\,.$$

Thus for a.e. $x \in \Omega$

$$\begin{aligned} Q(x, \nabla u_{n_m}(x))^{1/p} &\le \sum_{i=1}^{m} Q(x, \nabla u_{n_i}(x) - \nabla u_{n_{i-1}}(x))^{1/p} \\ &\le \sum_{i=1}^{\infty} Q(x, \nabla u_{n_i}(x) - \nabla u_{n_{i-1}}(x))^{1/p}\,, \end{aligned} \tag{4.10}$$

where the first inequality is proved in detail in Remark 4.1 below. Since $[u_{n_m}] = (\int_\Omega Q(x, \nabla u_{n_m}(x))\, dx)^{1/p} + \int_\Omega |u_{n_m}(x)|\, dx$, (4.9) follows by dominated convergence theorem once we prove that

$$\sum_{i=1}^{\infty} Q(\cdot, \nabla u_{n_i}(\cdot) - \nabla u_{n_{i-1}}(\cdot))^{1/p} \in L^p(\Omega)\,.$$

This holds since

$$\begin{aligned} &\left(\int_\Omega \left(\sum_{i=1}^{\infty} Q(x, \nabla u_{n_i}(x) - \nabla u_{n_{i-1}}(x))^{1/p} \right)^p dx \right)^{1/p} \\ &\le \sum_{i=1}^{\infty} \left(\int_\Omega Q(x, \nabla u_{n_i}(x) - \nabla u_{n_{i-1}}(x))\, dx \right)^{1/p} \\ &\le \sum_{i=1}^{\infty} [u_{n_i} - u_{n_{i-1}}] \le \sum_{i=1}^{\infty} \frac{1}{2^i} = 1\,. \end{aligned}$$

Now the representation result follows easily. Indeed, let $u \in \overset{\circ}{H}_\infty(\Omega)$, by definition there is a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathrm{Lip}_0(\Omega)$ converging to u, hence, in particular, $[u_n] \to [u]$ as $n \to \infty$. By uniqueness of the limit (in $\mathbb{R}$), (4.9) the representation of $[u]$ in $\overset{\circ}{H}_\infty(\Omega)$. ☐

In the following remark we show in detail how to get formula (4.10) used in the previous proof.

Remark 4.1. For a.e. $\bar{x} \in \Omega$, for any $\xi, \eta \in \mathbb{R}^n$ it holds

$$Q(\bar{x}, \xi + \eta)^{1/p} \leq Q(\bar{x}, \xi)^{1/p} + Q(\bar{x}, \eta)^{1/p} . \tag{4.11}$$

Proof. Let $B = B(\bar{x}, r)$ be a ball centered in $\bar{x}$ with radius r, $B \subset \Omega$. Let u_ε, $v_\varepsilon \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$ such that $u_\varepsilon \to \langle \cdot, \xi\rangle$ $v_\varepsilon \to \langle \cdot, \eta\rangle$ in $L^1(B)$ and

$$\left(\int_B \langle A_\varepsilon(x)\nabla u_\varepsilon, \nabla u_\varepsilon\rangle^{p/2} dx\right)^{1/p} \to \left(\int_B Q(x,\xi)\, dx\right)^{1/p} ,$$

$$\left(\int_B \langle A_\varepsilon(x)\nabla v_\varepsilon, \nabla v_\varepsilon\rangle^{p/2} dx\right)^{1/p} \to \left(\int_B Q(x,\eta)\, dx\right)^{1/p} ;$$

this is possible since the Γ-limit has the integral representation (3.13), the functions $\langle \cdot, \xi\rangle$ $\langle \cdot, \eta\rangle$ being in $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$. For the same reason,

$$\begin{aligned}\left(\int_B Q(x,\xi+\eta)\, dx\right)^{1/p} &\leq \liminf_{\varepsilon\to 0} \left(\int_B \langle A_\varepsilon(x)\nabla(u_\varepsilon + v_\varepsilon), \nabla(u_\varepsilon + v_\varepsilon)\rangle^{p/2} dx\right)^{1/p} \\ &\leq \left(\int_B Q(x,\xi)\, dx\right)^{1/p} + \left(\int_B Q(x,\eta)\, dx\right)^{1/p} .\end{aligned}$$

Letting $r \to 0$, (4.11) is proved if $\bar{x}$ if a Lebesgue point of $Q(\cdot, \xi+\eta)$, $Q(\cdot,\xi)$ $Q(\cdot,\eta)$. On the other hand, the set

$$\Omega_0 = \{\bar{x} \in \Omega; \bar{x} \text{ is a Lebesgue point of } Q(\cdot,\xi+\eta), Q(\cdot,\xi), Q(\cdot,\eta) \text{ for any } \xi,\eta \in \mathbb{Q}^n\}$$

is such that $|\Omega \setminus \Omega_0| = 0$. Eventually (4.11) holds for $\xi, \eta \in \mathbb{R}^n$ and $x \in \Omega\setminus\Omega_0$, thanks to a limit argument. ☐

Let Ω be as above. The representation Theorem 4.1 allows us to give a representation result for the Γ-limit of the following functionals defined for functions $u_0 \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$,

$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega f_\varepsilon(x, \nabla u)\, dx & u \in u_0 + \mathrm{Lip}_0(\Omega) , \\ +\infty & u \in u_0 + (\overset{\circ}{W}{}^{1,1}(\Omega) - \mathrm{Lip}_0(\Omega)) . \end{cases} \tag{4.12}$$

By a property of Γ-convergence, up to a subsequence, the Γ-limit in $L^1(\Omega)$ of (4.12) is well defined for all $u \in u_0 + \overset{\circ}{W}{}^{1,1}(\Omega)$; we denote it by $F_\Gamma(u)$. The problem we are concerned with in next section is to find a representation for $F_\Gamma(u)$.

Let us first define the space $\overset{\circ}{W}_\infty(\Omega) := \{u \in \overset{\circ}{W}{}^{1,1}(\Omega); [u] < +\infty\}$; clearly $\overset{\circ}{H}_\infty(\Omega) \subset \overset{\circ}{W}_\infty(\Omega)$ but in general the two spaces do not coincide (see counter-examples in Sec. 6). The lack of equality between $\overset{\circ}{H}_\infty(\Omega)$ and $\overset{\circ}{W}_\infty(\Omega)$ involves a representation problem for the Γ-limit. Indeed, it is not possible to prove that

$$F_\Gamma(u) = \begin{cases} \displaystyle\int_\Omega f_\infty(x, \nabla u)\, dx & u \in u_0 + \overset{\circ}{H}_\infty(\Omega)\,, \\ +\infty & u \in u_0 + (\overset{\circ}{W}{}^{1,1}(\Omega) - \overset{\circ}{H}_\infty(\Omega))\,, \end{cases} \tag{4.13}$$

for every $u \in u_0 + \overset{\circ}{W}{}^{1,1}(\Omega)$ with $u_0 \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$, since as we already noticed the finiteness domain of F_Γ is in general bigger than $\overset{\circ}{H}_\infty(\Omega)$. For this reason we can get a representation theorem only in $\overset{\circ}{H}_\infty(\Omega)$, i.e. we prove that $F_\Gamma(u) = \int_\Omega f_\infty(x, \nabla u)\, dx$ for $u \in u_0 + \overset{\circ}{H}_\infty(\Omega)$.

5. Representation Results on Γ-Limits

At the end of previous section we explained the problem of representation of the limit of the functionals (4.12). As already mentioned, their Γ-limit $F_\Gamma(u)$ is well defined for all $u \in u_0 + \overset{\circ}{W}{}^{1,1}(\Omega)$. Recalling Theorem 3.2, it is easy to prove (see Ref. 14, Proposition 6.15) that F_Γ is represented by the functional

$$F_\Gamma(u) = \int_\Omega f_\infty(u, \nabla u)\, dx\,, \tag{5.1}$$

for every $u \in u_0 + \mathrm{Lip}_0(\Omega)$, where f_∞ satisfies (3.6).

In fact, we prove in Theorem 5.1 that $F_\Gamma(u) = \int_\Omega f_\infty(u, \nabla u)\, dx$ for every $u \in u_0 + \overset{\circ}{H}_\infty(\Omega)$.

Let us introduce the functionals

$$\hat{F}_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega f_\varepsilon(u, \nabla u)\, dx & u \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)\,, \\ +\infty & u \in (W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n) - \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n))\,. \end{cases} \tag{5.2}$$

Again up to subsequences, it is known that the Γ-limit in the $L^1(\Omega)$ topology is well defined for all $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$; we denote it by $\hat{F}_\Gamma(u)$. Using Remark 3.1, and because of Proposition 6.15 in Ref. 14 the previous Γ-limit is represented by the functional

$$\hat{F}_\Gamma(u) = \int_\Omega f_\infty(x, \nabla u)\, dx\,, \tag{5.3}$$

for every $u \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$, where f_∞ satisfies (3.6).

The proof of Theorem 5.1 is based on Theorem 3.1 and an approximation result stated in Proposition 5.1 below will be overcome via an approximation.

Proposition 5.1. *Let $p > 1$ and let Ω be a bounded open set compactly contained in a cube I. Let $(f_\varepsilon(x, \xi))$ be a family of functions measurable in x and convex in ξ satisfying $\lambda_\varepsilon(x)|\xi|^p \le f_\varepsilon(x, \xi) \le \Lambda_\varepsilon(x)|\xi|^p$ for a.e. $x \in \mathbb{R}^n$ for any $\xi \in \mathbb{R}^n$. Let (λ_ε) be as in* (1.8) *and* (1.9); *let (Λ_ε) satisfy* (1.10); *in addition suppose that the pair $(\lambda_\varepsilon, \Lambda_\varepsilon)$ satisfies* (1.11).

Consider $u \in \overset{\circ}{W}{}^{1,1}(\Omega)$ and a sequence $(u_\varepsilon) \in \mathrm{Lip}_0(\Omega)$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$ and

$$\int_\Omega \langle A_\varepsilon(x)\nabla u_\varepsilon, \nabla u_\varepsilon\rangle^{p/2} dx < C_4 \tag{5.4}$$

for all $\varepsilon > 0$.

Then for every $u_0 \in \mathrm{Lip}_0(\Omega)$ and $\tau > 0$ there exist $\Omega_\tau \subset \Omega$ so that $|\Omega - \Omega_\tau| < 3\tau$, $\beta_\tau > 0$, $\alpha_\tau > 0$, such that $\beta_\tau \to 0$, $\alpha_\tau \to \infty$, as $\tau \to 0$, and there exist a sequence $(v_{\varepsilon,\tau}) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$, and a function $v_\tau \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$ verifying $\|\nabla v_{\varepsilon,\tau}\|_{L^\infty(\mathbb{R}^n)} \le c(n)\alpha_\tau$, $v_{\varepsilon,\tau} \to v_\tau$ in $L^\infty(\Omega)$ as $\varepsilon \to 0$ and

$$\liminf_{\varepsilon\to 0} \int_\Omega f_\varepsilon(x, \nabla u_0 + \nabla u_\varepsilon)\, dx \ge \liminf_{\varepsilon \to 0} \int_{\Omega_\tau} f_\varepsilon(x, \nabla u_0 + \nabla v_{\varepsilon,\tau})\, dx - \beta_\tau\,. \tag{5.5}$$

In addition, setting

$$V_\tau := \{x \in \mathbb{R}^n : v_\tau(x) \ne u(x)\}\,, \tag{5.6}$$

we have $|V_\tau| < 2\tau$.

Proof. The scheme of the proof is essentially the one of Theorem 3.1 in Ref. 15. However since we are dealing with two different weights, appropriate changes must be taken into account. For the reader convenience we will report the proof, obviously stressing the differences. To avoid cumbersome notation we will use the same label for sequences and subsequences.

Without loss of generality, consider $(u_\varepsilon) \in C_0^\infty(\Omega)$ and identify u and (u_ε) with their zero extension outside Ω.

For $N > 0$, let us define the sets $C_N^\varepsilon = \{x \in \mathbb{R}^n : \frac{\Lambda_\varepsilon}{\lambda_\varepsilon} > N\} \cap \Omega$. By hypothesis (1.11), it follows that

$$N|C_N^\varepsilon| = N\int_{C_N^\varepsilon} dx \le \int_I \frac{\Lambda_\varepsilon}{\lambda_\varepsilon}\, dx < K \tag{5.7}$$

uniformly in ε.

Then, for a fixed $\tau > 0$ there exists a N_τ such that, for $N > N_\tau$, $|C_N^\varepsilon| < \tau$, uniformly in ε.

On the other hand, since the weights λ_ε are in the $A_p(A)$ class we can apply Theorem 2.2 to the functions ∇u_ε which gives

$$\int_{\mathbb{R}^n} (M|\nabla u_\varepsilon|)^p \lambda_\varepsilon(x)\, dx \le c\int_\Omega |\nabla u_\varepsilon|^p \lambda_\varepsilon(x)\, dx\,.$$

By (1.5) and (5.4) the integrals $\int_{\mathbb{R}^n}(M|\nabla u_\varepsilon|)^p\lambda_\varepsilon(x)\,dx$ are equibounded, therefore the sequence $((M|\nabla u_\varepsilon|)^p\lambda_\varepsilon(x))$ is bounded in $L^1(\mathbb{R}^n)$. Then, as proved in Ref. 2, for every $\eta>0$ we can find a set S_η with $|S_\eta|<\eta$ and a constant δ_η such that, up to a subsequence

$$\int_B (M|\nabla u_\varepsilon|)^p\lambda_\varepsilon(x)\,dx<\eta \tag{5.8}$$

for any measurable set B with $B\cap S_\eta=\emptyset$ and $|B|<\delta_\eta$. By Hölder's inequality, the uniform condition A_p on λ_ε and (1.5) yield $(\int_\Omega|\nabla u_\varepsilon|\,dx)^p=(\int_\Omega|\nabla u_\varepsilon|\lambda_\varepsilon^{1/p}\lambda_\varepsilon^{-1/p}dx)^p\le(\int_\Omega\lambda_\varepsilon^{-1/(p-1)}dx)^{p-1}\int_\Omega\langle A_\varepsilon(x)\nabla u_\varepsilon,\nabla u_\varepsilon\rangle^{p/2}dx<C_4\frac{A}{C_1}$ where in the last inequality we also use (1.9) and (5.4). By the one-to-one weak type property of the maximal function (see e.g. Theorem 1 of Ref. 26), there exists a constant $c(n)$ depending only on n such that $|\{x\in\mathbb{R}^n:(M|\nabla u_\varepsilon|)>\gamma\}|\le\frac{c(n)}{\gamma}\|\nabla u_\varepsilon\|_{L^1(\mathbb{R}^n)}$. Therefore we can choose $\alpha_\eta\ge\frac{c(n)}{\eta}(\frac{AC_4}{C_1})^{1/p}$ such that

$$|\{x\in\mathbb{R}^n:(M|\nabla u_\varepsilon|)\ge\alpha_\eta\}|\le\min\{\eta,\delta_\eta\}\,. \tag{5.9}$$

We define the sets

$$Z_\varepsilon^\eta=\{x\in\mathbb{R}^n:(M|\nabla u_\varepsilon|)\le\alpha_\eta\}\,;$$

we have $u_\varepsilon\in\mathrm{Lip}(Z_\varepsilon^\eta)$ with Lipschitz constant $c(n)\alpha_\eta$ (see Lemma II-11 of Ref. 2). We denote by $v_{\varepsilon,\eta}$ the Lipschitz extension of u_ε out of $Z_\varepsilon^\eta\cap\Omega$ to the whole $\mathbb{R}^n$, with the same Lipschitz constant $c(n)\alpha_\eta$. In summary we have

$$v_{\varepsilon,\eta}(x)=u_\varepsilon(x)\quad\text{and}\quad\nabla v_{\varepsilon,\eta}(x)=\nabla u_\varepsilon(x)\text{ a.e. in }Z_\varepsilon^\eta\cap\Omega \tag{5.10}$$

and

$$\|\nabla v_{\varepsilon,\eta}\|_{L^\infty(\mathbb{R}^n)}\le c(n)\alpha_\eta\,; \tag{5.11}$$

moreover, without loss of generality we can assume that $v_{\varepsilon,\eta}(x)=0$ if $\mathrm{dist}(x,\Omega)>1$. Up to a subsequence, it holds that

$$v_{\varepsilon,\eta}\to v_\eta\text{ in }L^\infty(\Omega)\quad\text{and}\quad\|\nabla v_\eta\|_{L^\infty(\mathbb{R}^n)}\le c(n)\alpha_\eta\,,$$

$|\{x\in\Omega:v_\eta(x)\ne u(x)\}|\le2\eta$, and by (5.9) $|(\Omega-S_\eta)-Z_\varepsilon^\eta|<\min\{\eta,\delta_\eta\}$ (see Ref. 15 for a detailed proof).

Finally, we can proceed as follows:

$$\begin{aligned}\int_\Omega f_\varepsilon(x,\nabla u_0+\nabla u_\varepsilon)\,dx&\\&=\int_{\Omega\cap C_N^\varepsilon}f_\varepsilon(x,\nabla u_0+\nabla u_\varepsilon)\,dx+\int_{\Omega-C_N^\varepsilon}f_\varepsilon(x,\nabla u_0+\nabla u_\varepsilon)\,dx\\&\ge\int_{\Omega-C_N^\varepsilon}f_\varepsilon(x,\nabla u_0+\nabla u_\varepsilon)\,dx\,.\end{aligned} \tag{5.12}$$

Let us set $\tilde{\Omega} = \Omega - C_N^\varepsilon$, clearly $|(\tilde{\Omega} - S_\eta) - Z_\varepsilon^\eta| < \min\{\eta, \delta_\eta\}$ uniformly in ε; then,

$$\int_{\tilde{\Omega}} f_\varepsilon(x, \nabla u_0 + \nabla u_\varepsilon)\, dx \geq \int_{(\tilde{\Omega}-S_\eta)\cap Z_\varepsilon^\eta} f_\varepsilon(x, \nabla u_0 + \nabla v_{\varepsilon,\eta})\, dx$$

$$= \int_{(\tilde{\Omega}-S_\eta)} f_\varepsilon(x, \nabla u_0 + \nabla v_{\varepsilon,\eta})\, dx - \int_{(\tilde{\Omega}-S_\eta)-Z_\varepsilon^\eta} f_\varepsilon(x, \nabla u_0 + \nabla v_{\varepsilon,\eta})\, dx\,. \quad (5.13)$$

We want to show that the second integral in the last line is small with respect to τ. We stress that the following chain of inequalities involves two weight estimates, requiring the use of our assumptions on the functionals, listed in the order they have to be used ((1.7), (5.10), (5.11), (5.9), (1.10), (5.7) (5.8)).

Let $L_0 = \|\nabla u_0\|_{L^\infty(\Omega)}$, we have:

$$\begin{aligned}
&\int_{(\tilde{\Omega}-S_\eta)-Z_\varepsilon^\eta} f_\varepsilon(x, \nabla u_0 + \nabla v_{\varepsilon,\eta})\, dx \\
&\quad\leq \int_{(\tilde{\Omega}-S_\eta)-Z_\varepsilon^\eta} |\nabla u_0 + \nabla v_{\varepsilon,\eta}|^p \Lambda_\varepsilon(x)\, dx \\
&\quad\leq 2^{p-1} L_0^p \int_{\tilde{\Omega}-Z_\varepsilon^\eta} \Lambda_\varepsilon(x)\, dx + 2^{p-1}(c(n)\alpha_\eta)^p \int_{(\tilde{\Omega}-S_\eta)-Z_\varepsilon^\eta} \Lambda_\varepsilon(x)\, dx \\
&\quad\leq 2^{p-1} L_0^p \left(\int_{\tilde{\Omega}-Z_\varepsilon^\eta} \Lambda_\varepsilon^{1+\mu}(x)\, dx\right)^{\frac{1}{1+\mu}} |\tilde{\Omega} - Z_\varepsilon^\eta|^{\frac{\mu}{1+\mu}} \\
&\qquad + 2^{p-1} c(n)^p \int_{(\tilde{\Omega}-S_\eta)-Z_\varepsilon^\eta} (M|\nabla u_\varepsilon|)^p \Lambda_\varepsilon(x)\, dx \\
&\quad\leq 2^{p-1} L_0^p C_2 |\tilde{\Omega} - Z_\varepsilon^\eta|^{\frac{\mu}{1+\mu}} + 2^{p-1} c(n)^p \int_{(\tilde{\Omega}-S_\eta)-Z_\varepsilon^\eta} (M|\nabla u_\varepsilon|)^p N \lambda_\varepsilon(x)\, dx \\
&\quad\leq 2^{p-1} L_0^p C_2 \eta^{\frac{\mu}{1+\mu}} + 2^{p-1} c(n)^p \eta N\,. \qquad (5.14)
\end{aligned}$$

Set $\gamma_\eta := 2^{p-1} L_0^p C_2 \eta^{\frac{\mu}{1+\mu}} + 2^{p-1} c(n)^p \eta N$.

For a fixed $\tau > 0$ there exists $\eta = \eta(\tau)$ such that $\gamma_\eta < \tau$ for $\eta < \eta(\tau)$. Since from now on $\eta = \eta(\tau)$ we will denote the quantities $v_{\varepsilon,\eta}$, v_η, α_η, γ_η, S_η by $v_{\varepsilon,\tau}$, v_τ, α_τ, γ_τ, S_τ to stress their τ dependence. Clearly $\gamma_\tau \to 0$ as $\tau \to 0$.

For any measurable subset E of I we have

$$\int_E \Lambda_\varepsilon(x)\, dx \leq \left(\int_I \Lambda_\varepsilon(x)^{1+\mu} dx\right)^{\frac{1}{1+\mu}} |E|^{\frac{\mu}{1+\mu}} \leq C_2 |E|^{\frac{\mu}{1+\mu}}\,.$$

This inequality and (5.11) allows us to choose an open set Ω_τ such that $\tilde{\Omega} - S_\tau \subset \Omega_\tau$ for which

$$\left|\int_{\Omega_\tau} f_\varepsilon(x, \nabla u_0 + \nabla v_{\varepsilon,\tau})\, dx - \int_{\tilde{\Omega}-S_\tau} f_\varepsilon(x, \nabla u_0 + \nabla v_{\varepsilon,\tau})\, dx\right| < \tau\,; \quad (5.15)$$

it is easy to see that $|\Omega - \Omega_\tau| < 3\tau$.

Using (5.12)–(5.15) we have

$$\int_{\Omega} f_\varepsilon(x, \nabla u_0 + \nabla u_\varepsilon)\, dx \geq \int_{\Omega_\tau} f_\varepsilon(x, \nabla u_0 + \nabla v_{\varepsilon,\tau})\, dx - \gamma_\tau - \tau\,.$$

We define $\beta_\tau := \gamma_\tau + \tau$. Obviously $\beta_\tau \to 0$ as $\tau \to 0$ and the thesis follows. □

Theorem 5.1. *Let Ω be an open set with Lipschitz boundary, compactly contained in a cube I. Let f_ε be the family of functions defined in* (1.4) *satisfying hypothesis* (1.6). *Let (λ_ε) and (Λ_ε) be as in* (1.8)–(1.10) *respectively.*

Then for all $u \in u_0 + \overset{\circ}{H}_\infty(\Omega)$ with u_0 in $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$, it holds

$$F_\Gamma(u) = \int_\Omega f_\infty(x, \nabla u)\, dx\,, \tag{5.16}$$

where F_Γ is the Γ-limit of F_ε and F_ε are defined in (4.12).

Proof. In the sequel we always argue up to subsequences. Let u_0 be in $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$. Let us prove that

$$F_\Gamma(u) \geq \int_\Omega f_\infty(x, \nabla u)\, dx \tag{5.17}$$

for all $u \in u_0 + \overset{\circ}{H}_\infty(\Omega)$.

As remarked above, F_Γ is well defined for any $u \in u_0 + \overset{\circ}{W}{}^{1,1}(\Omega)$. Hence if we consider $u, u_\varepsilon \in \overset{\circ}{H}_\infty(\Omega) (\subset \overset{\circ}{W}{}^{1,1}(\Omega))$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$ and

$$\limsup F_\varepsilon(u_0 + u_\varepsilon) \leq F_\Gamma(u_0 + u) < \infty\,, \tag{5.18}$$

by definition (4.12) we deduce that $u_\varepsilon \in \mathrm{Lip}_0(\Omega)$ for all ε and so $F_\varepsilon(u_\varepsilon) = \int_\Omega f_\varepsilon(x, \nabla u_\varepsilon)\, dx$.

Therefore

$$F_\Gamma(u_0 + u) \geq \liminf_{\varepsilon \to 0} \int_\Omega f_\varepsilon(x, \nabla(u_0 + u_\varepsilon))\, dx\,. \tag{5.19}$$

By Proposition 5.1, for a fixed $\tau > 0$, there exist $\Omega_\tau \subset \Omega$, $\beta_\tau > 0$, and $(v_{\varepsilon,\tau})$, $v_\tau \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$ such as in Proposition 5.1, so that

$$\begin{aligned} \liminf_{\varepsilon \to 0} \int_\Omega f_\varepsilon(x, \nabla(u_0 + u_\varepsilon))\, dx &\geq \liminf_{\varepsilon \to 0} \int_{\Omega_\tau} f_\varepsilon(x, \nabla(u_0 + v_{\varepsilon,\tau}))\, dx - \beta_\tau \\ &\geq \hat{F}_\Gamma(u_0 + v_\tau) - \beta_\tau \\ &= \int_{\Omega_\tau} f_\infty(x, \nabla u_0 + \nabla v_\tau)\, dx - \beta_\tau\,, \end{aligned} \tag{5.20}$$

where the functional $\hat{F}_\Gamma$ is now restricted to Ω_τ and the last equality follows by the representation given in (5.3). Let V_τ be defined as in (5.6), by Proposition 5.1 we know that $|V_\tau| < 2\tau$. Moreover

$$\int_{\Omega_\tau} f_\infty(x, \nabla u_0 + \nabla v_\tau)\, dx - \beta_\tau \geq \int_{\Omega_\tau - V_\tau} f_\infty(x, \nabla u_0 + \nabla v_\tau)\, dx - \beta_\tau$$

$$= \int_{\Omega_\tau - V_\tau} f_\infty(x, \nabla u_0 + \nabla u)\, dx - \beta_\tau\,. \quad (5.21)$$

In conclusion, for all $u \in \overset{\circ}{H}_\infty(\Omega)$ such that (5.18) holds we find that

$$F_\Gamma(u_0 + u) \geq \int_{\Omega_\tau - V_\tau} f_\infty(x, \nabla u_0 + \nabla u)\, dx - \beta_\tau\,. \quad (5.22)$$

Since $|\Omega - (\Omega_\tau - V_\tau)| \leq 5\tau$ we get (5.17) from (5.22) letting $\tau \to 0$.

Let us prove now that $F_\Gamma(u) \leq F(u)$ in $u_0 + \overset{\circ}{H}_\infty(\Omega)$.

Let $u \in u_0 + \overset{\circ}{H}_\infty(\Omega)$ and $u_k \in u_0 + \mathrm{Lip}_0(\Omega)$ such that $[(u_k - u_0) - (u - u_0)] \to 0$ in $\overset{\circ}{H}_\infty(\Omega) (\subset \overset{\circ}{W}{}^{1,1}(\Omega))$. In particular $u_k \to u$ in $L^1(\Omega)$ and $\nabla u_k \to \nabla u$ in $L^1(\Omega)$, thus, without loss of generality we may assume that $\nabla u_k(x) \to \nabla u(x)$ for a.e. $x \in \Omega$. By the convexity of f_∞ this implies $f_\infty(x, \nabla u_k(x)) \to f_\infty(x, \nabla u(x))$ for $k \to \infty$ for a.e. $x \in \Omega$. By the very definition of f_ε we have the estimate

$$b_1 \langle A_\varepsilon(x)\xi, \xi\rangle^{p/2} \leq f_\varepsilon(x, \xi) \leq b_2 \langle A_\varepsilon(x)\xi, \xi\rangle^{p/2}\,. \quad (5.23)$$

Using (3.3) and (3.13), passing to Γ-limits we obtain

$$b_1 \int_\Omega Q(x, \nabla u)\, dx \leq \int_\Omega f_\infty(x, \nabla u)\, dx \leq b_2 \int_\Omega Q(x, \nabla u)\, dx \quad (5.24)$$

for every $u \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$. Since for $u \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$ both $f_\infty(x, \nabla u)$ and $Q(x, \nabla u)$ belong to $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ (by (3.2) and (3.12)), passing to the Lebesgue points we obtain

$$b_1 Q(x, \nabla u) \leq f_\infty(x, \nabla u) \leq b_2 Q(x, \nabla u) \quad \text{a.e. } x \in \mathbb{R}^n \quad (5.25)$$

for every $u \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$. On the other hand, recalling that $[\cdot]$ is a norm on $\overset{\circ}{H}_\infty(\Omega)$, we have $\int_\Omega Q(x, \nabla u_k)\, dx \to \int_\Omega Q(x, \nabla u)$. Hence, there exits a subsequence, still denoted by $Q(x, \nabla u_k)$ and a function $h \in L^1(\Omega)$ such that $|Q(x, \nabla u_k)| \leq h(x)$ a.e. in Ω.

By (5.25) we have $f_\infty(x, \nabla u_k(x)) \leq b_2 Q(x, \nabla u_k(x)) \leq b_2 h(x)$. The dominated convergence theorem implies

$$\int_\Omega f_\infty(x, \nabla u)\, dx = \lim_k \int_\Omega f_\infty(x, \nabla u_k)\, dx\,.$$

By (5.1) $F_\Gamma(u_k) = \int_\Omega f_\infty(x, \nabla u_k)\, dx$. Eventually, using the lower semicontinuity property of the Γ-limit, we get

$$\int_\Omega f_\infty(x, \nabla u)\, dx = \lim_{k\to\infty} \int_\Omega f_\infty(x, \nabla u_k)\, dx$$
$$= \lim_{k\to\infty} F_\Gamma(u_k) \geq F_\Gamma(u)\,,$$

that implies together with (5.17) the desired result (5.16). □

Thanks to result above we can state the following theorem.

Theorem 5.2. *Let $p > 1$, I a cube in $\mathbb{R}^n$ and f_ε be the family of functions defined in* (1.4) *satisfying hypothesis* (1.6). *Let (λ_ε) and (Λ_ε) be as in* (1.8)–(1.10) *respectively, and satisfying* (1.11). *Then there exist a subfamily $(f_{\varepsilon_k}) \subset (f_\varepsilon)$ and two weights $\Lambda_\infty \in L^{1+\mu}_{\mathrm{loc}}(\mathbb{R}^n)$ and $\lambda_\infty \in A_p$ and a function f_∞ satisfying* (3.6), *such that for any open set $\Omega \subset\subset I$ with Lipschitz boundary, $g \in L^\infty(\Omega)$ and $u_0 \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$ the sequence of solutions of the problem*

$$\min\left\{\int_\Omega f_\varepsilon(x, \nabla u)\, dx + \int_\Omega gu\, dx : u \in u_0 + \overset{\circ}{H}{}^{1,p}_{A_\varepsilon}(\Omega)\right\}$$

converge in $L^1(\Omega)$ to the solution of the problem

$$\min\left\{\int_\Omega f_\infty(x, \nabla u)\, dx + \int_\Omega gu\, dx : u \in u_0 + \overset{\circ}{H}_\infty(\Omega)\right\};$$

and the convergence of the minimum values holds.

Proof. Using Theorem 5.1 the proof is the same as the one in Theorem 4.6 of Ref. 15. □

6. Examples and Counterexamples

In this section we produce two examples concerning the relations between the spaces H and W used in the paper. Let Ω be a bounded open set in $\mathbb{R}^2$, with Lipschitz boundary, and $p > 2$. In the first example we provide weights such that $\overset{\circ}{H}{}^{1,p}_{A_\varepsilon}(\Omega) \neq \overset{\circ}{W}{}^{1,p}_{A_\varepsilon}(\Omega)$ but when we pass to the limit $\overset{\circ}{H}_\infty(\Omega) = \overset{\circ}{W}_\infty(\Omega)$. In the second one we consider weights such that $\overset{\circ}{H}{}^{1,p}_{A_\varepsilon}(\Omega) = \overset{\circ}{W}{}^{1,p}_{A_\varepsilon}(\Omega)$ but $\overset{\circ}{H}_\infty(\Omega) \neq \overset{\circ}{W}_\infty(\Omega)$.

Example 6.1. Let I_0 be the unit cube $I_0 =]-1/2, 1/2[\times]-1/2, 1/2[$ in $\mathbb{R}^2$. In Ref. 9, Example 2.2, the authors construct an example of a weight λ such that $H^{1,p}(I_0, \lambda) \neq W^{1,p}(I_0, \lambda)$, where λ is defined as follows. Let $p, \alpha, \beta \in \mathbb{R}$, with $p > 2$ and $0 < \alpha < \beta < 2(p-1)$. The authors define a suitable π-periodic smooth function $k : \mathbb{R} \to [\alpha, \beta]$ (see Ref. 9 for details), and they define the weight $\lambda : I_0 \to [0, +\infty[$ as

$$\lambda(x) = \begin{cases} |x|^{k(\arccos\frac{x_1}{|x|})} & \text{if } |x| \neq 0\,, \\ 0 & \text{if } |x| = 0\,. \end{cases} \tag{6.1}$$

It is clear that

$$|x|^\beta \leq \lambda(x) \leq |x|^\alpha \quad \text{for every } x \in I_0\,. \tag{6.2}$$

Then they define $u : I_0 \to \mathbb{R}$ as

$$u(x) = \begin{cases} 1 & \text{if } x_1,\ x_2 > 0\,, \\ 0 & \text{if } x_1,\ x_2 < 0\,, \\ \dfrac{x_2}{|x|} & \text{if } x_1 < 0 < x_2\,, \\ \dfrac{x_1}{|x|} & \text{if } x_2 < 0 < x_1 \end{cases} \tag{6.3}$$

and show that $u \in W^{1,p}(I_0, \lambda)$ but $u \notin H^{1,p}(I_0, \lambda)$ for $\beta > p - 2$ and $0 < \alpha < \beta < 2(p-1)$. We now consider a smooth function ψ, with $\psi \equiv 1$ in $]-1/4, 1/4[\times]-1/4, 1/4[$ and supp $\psi \subset\subset I_0$, and we define $U : I_0 \to \mathbb{R}$ as $U = \psi u$. Arguing as in Ref. 9 with the same choice of α and β, we can show that $U \in \overset{\circ}{W}{}^{1,p}(I_0, \lambda)$ but $U \notin \overset{\circ}{H}{}^{1,p}(I_0, \lambda)$.

Let us take the matrix $A_\varepsilon(x) := \lambda_{\#}^{2/p}(\frac{x}{\varepsilon})\mathbb{I}$, where the symbol $\#$ denotes the periodic extension of λ to the whole plane and $\mathbb{I}$ is the unit matrix in $\mathbb{R}^2$. With this definition and by (6.2), the assumption (1.5) becomes $|\frac{x}{\varepsilon}|^{2\beta/p}|\xi|^2 \le \langle A_\varepsilon(x)\xi, \xi\rangle \le |\frac{x}{\varepsilon}|^{2\alpha/p}|\xi|^2$ a.e. in I_0. We now extend to the whole $\mathbb{R}^2$ the functions $|\frac{x}{\varepsilon}|^\beta$ and $|\frac{x}{\varepsilon}|^\alpha$, by periodicity and we denote the periodic extension adding the symbol $\#$ to both functions. We choose the weights $\lambda_\varepsilon(x) = |\frac{x}{\varepsilon}|^\beta_{\#}$ and $\Lambda_\varepsilon(x) = |\frac{x}{\varepsilon}|^\alpha_{\#}$. By periodicity, it is easy to show that λ_ε and Λ_ε satisfy conditions (w_1) and (w_2) in the Introduction. As for condition (w_3) to hold, we need further relation $\alpha > \beta - 2$. By a careful analysis of the proof given in Ref. 9 we can also show that for this choice of α and β we still have $U \in \overset{\circ}{W}{}^{1,p}(I_0, \lambda)$ but $U \notin \overset{\circ}{H}{}^{1,p}(I_0, \lambda)$.

Let $\Omega = I_0$ for simplicity. We claim that the periodic function $u_\varepsilon(x) := U_{\#}(\frac{x}{\varepsilon})$ is in $W^{1,p}_{A_\varepsilon}(\Omega)$ but not in $H^{1,p}_{A_\varepsilon}(\Omega)$.

The first assertion is easy to be verified. Indeed, since supp $U \subset\subset I_0$ then $\nabla U_{\#} = (\nabla U)_{\#}$ and since the number of ε-cells recovering Ω is of the order $1/\varepsilon^2$ and $u \in W^{1,p}(I_0, \lambda)$, for every $\varepsilon > 0$ we have

$$\begin{aligned} \int_\Omega \langle A_\varepsilon(x)\nabla u_\varepsilon, \nabla u_\varepsilon\rangle^{p/2} &= \int_\Omega \lambda_{\#}\left(\frac{x}{\varepsilon}\right)|\nabla u_\varepsilon|^p dx \\ &= \frac{1}{\varepsilon}\int_\Omega \lambda_{\#}\left(\frac{x}{\varepsilon}\right)\left|\nabla(\psi u)_{\#}\left(\frac{x}{\varepsilon}\right)\right|^p dx \\ &= \frac{1}{\varepsilon^3}\int_{\frac{1}{\varepsilon}\Omega} \lambda_{\#}(x)|\nabla(\psi u)_{\#}(x)|^p dx \\ &\le \frac{1}{\varepsilon}\int_\Omega \lambda(x)|\nabla(\psi u)(x)|^p dx \\ &\le \frac{1}{\varepsilon}\left(\int_\Omega \lambda(x)|u|^p dx + \int_\Omega \lambda(x)|\nabla u|^p dx\right) < \infty\,. \end{aligned}$$

To show that $u_\varepsilon \notin H^{1,p}_{A_\varepsilon}(\Omega)$, we follows the same argument used in Ref. 9. By contradiction, suppose there exists a sequence $(u_\varepsilon)_k \in C^1(\Omega)$ which converges to u_ε in

$W^{1,p}_{A_\varepsilon}(\Omega)$. In particular the convergence holds in the ε-cell $I_\varepsilon = \frac{1}{\varepsilon}\Omega$ (which can be supposed centered in the origin). In particular we have $\lim_{k\to\infty}\int_{I_\varepsilon}\lambda_\#(\frac{x}{\varepsilon})|\nabla(u_\varepsilon)_k - \nabla u_\varepsilon|^p dx = 0$. Up to rescaling, we have in fact shown that U can be approximated by a C^1-function in Ω and we have a contradiction since $\nabla u_\varepsilon(x) = (\nabla U)_\#(\frac{x}{\varepsilon})$ and $U \notin H^{1,p}(\Omega,\lambda)$.

In addition, if we choose a cutoff function Φ with support contained in Ω, and we consider $\bar{u}_\varepsilon := \Phi u_\varepsilon$, we see that $\bar{u}_\varepsilon \in \overset{\circ}{W}{}^{1,p}_{A_\varepsilon}(\Omega)$ but $\bar{u}_\varepsilon \notin \overset{\circ}{H}{}^{1,p}_{A_\varepsilon}(\Omega)$.

We want to show that the two families of admissible weights describing the problem in this example induce, via a Γ-limit argument, $\overset{\circ}{H}_\infty(\Omega) = \overset{\circ}{W}_\infty(\Omega)$. Indeed, by Remark 3.1, there exist a function $Q(x,\xi)$ such that $\lambda_\infty(x)|\xi|^p \le Q(x,\xi) \le \Lambda_\infty(x)|\xi|^p$ for almost every $x \in \Omega$, for any $\xi \in \mathbb{R}^n$, where the two weights λ_∞ and Λ_∞ are weak limits of λ_ε and Λ_ε respectively. On the other hand, since the weights λ_ε and Λ_ε are periodic they converges to their averages (and hence to two positive constants a, b). By unicity of the limit, $\lambda_\infty = a$ and $\Lambda_\infty = b$. Therefore the ellipticity condition $a|\xi|^p \le Q(x,\xi) \le b|\xi|^p$ implies $\overset{\circ}{H}_\infty(\Omega) = \overset{\circ}{W}_\infty(\Omega) = \overset{\circ}{W}{}^{1,1}(\Omega)$.

Example 6.2. Let $\Omega = I_0$ and λ as in (6.1). Define $\tilde{\lambda}_\varepsilon = \max\{\lambda,\varepsilon\}$, clearly $\tilde{\lambda}_\varepsilon \ge \varepsilon$; moreover by (6.2) $\tilde{\lambda}_\varepsilon \le 1$ in Ω. Define $A_\varepsilon(x) = \tilde{\lambda}_\varepsilon(x)^{2/p}\mathbb{I}$, since for any $\varepsilon > 0$ the ellipticity growth condition $\varepsilon|\xi|^p \le \langle A_\varepsilon(x)\xi,\xi\rangle^{p/2} \le |\xi|^p$ holds for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^n$, we have $\overset{\circ}{H}{}^{1,p}_{A_\varepsilon}(\Omega) = \overset{\circ}{W}{}^{1,p}_{A_\varepsilon}(\Omega)$.

On the other hand, in Theorem 4.1 (where we take $u_0 = 0$) it is shown that, up to subsequences, $[u] = \Gamma - \lim(\int_\Omega |u|\,dx + (\int_\Omega \langle A_\varepsilon(x)\nabla u, \nabla u\rangle^{p/2}dx)^{1/p}) = (\int_\Omega |u|\,dx + (\int_\Omega Q(x,\nabla u)\,dx)^{1/p})$ for $u \in \overset{\circ}{H}_\infty(\Omega)$. By definition of $\tilde{\lambda}_\varepsilon$, the sequence $(\langle A_\varepsilon(x)\nabla u, \nabla u\rangle^{p/2})_\varepsilon$ is non-increasing when $\varepsilon \to 0$, hence using the monotone convergence theorem, $\lim_{\varepsilon\to 0}(\int_\Omega \langle A_\varepsilon(x)\nabla u, \nabla u\rangle^{p/2}dx)^{1/p} = (\int_\Omega |\nabla u|^p\lambda\,dx)^{1/p}$ on $\mathrm{Lip}_0(\Omega)$. Since $(\int_\Omega |\nabla u|^p\lambda\,dx)^{1/p}$ is $L^1(\Omega)$-lower semicontinuous on $\mathrm{Lip}_0(\Omega)$, by Theorem 5.7 and Proposition 6.1 in Ref. 14, the norm $[\cdot]$ coincides with the norm $\|\cdot\|_{W^{1,p}(\Omega,\lambda)}$ in $\mathrm{Lip}_0(\Omega)$ and hence $\overset{\circ}{H}_\infty(\Omega) = \overset{\circ}{H}{}^{1,p}(\Omega,\lambda)$. Now take U as in Example 6.1, and recall that $U \notin \overset{\circ}{H}{}^{1,p}(\Omega,\lambda)$ but $U \in \overset{\circ}{W}{}^{1,p}(\Omega,\lambda) \subset \overset{\circ}{W}_\infty(\Omega)$. Hence $\overset{\circ}{H}_\infty(\Omega) \neq \overset{\circ}{W}_\infty(\Omega)$.

Acknowledgments

It is a pleasure to thank M. Fabrizio for fruitful discussions and many valuable suggestions. Investigation supported by University of Bologna. Funds for selected research topics. Partially supported by MIUR, Italy, and by GNAMPA og INDAM, Italy.

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