NoDEA
Nonlinear differ. equ. appl. 00 (2001) 00–00
1021–9722/01/00000–00 $ 1.50+0.20/0

Nonlinear Differential Equations
and Applications NoDEA

Homogenization for strongly anisotropic nonlinear elliptic equations*

Bruno FRANCHI[†], Maria Carla TESI[‡]
Dipartimento di Matematica
Piazza di Porta S. Donato, 5
40127 Bologna, Italy
e-mail: franchib@dm.unibo.it; tesi@dm.unibo.it

Abstract. In this paper we present homogenization results for elliptic degenerate differential equations describing strongly anisotropic media. More precisely, we study the limit as $\epsilon \to 0$ of the following Dirichlet problems with rapidly oscillating periodic coefficients:

$$\begin{cases} -\mathrm{div}\,(\alpha(\frac{x}{\epsilon}, \nabla u) A(\frac{x}{\epsilon})\nabla u) = f(x) \in L^\infty(\Omega) \\ u = 0 \text{ su } \partial\Omega \end{cases}$$

where $p > 1$, $\alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\alpha(y,\xi) \approx \langle A(y)\xi, \xi\rangle^{p/2-1}, A \in M^{n\times n}(\mathbb{R})$, A being a measurable periodic matrix such that $A^t(x) = A(x) \geq 0$ almost everywhere.

The anisotropy of the medium is described by the following structure hypotheses on the matrix A:

$$\lambda^{2/p}(x)|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda^{2/p}(x)|\xi|^2,$$

where the weight functions λ and Λ (satisfying suitable summability assumptions) can vanish or blow up, and can also be "moderately" different. The convergence to the homogenized problem is obtained by a classical compensated compactness argument, that had to be extended to two-weight Sobolev spaces.

Mathematics Subject Classification: 35J70. 35B40.
Key words: Homogenization, weighted Sobolev spaces, compensated compactness.

*Investigation supported by University of Bologna. Funds for selected research topics
[†]Partially supported by MURST, Italy, and by GNAFA of INdAM, Italy.
[‡]Supported by a post-doctoral joint grant of MURST and University of Bologna.

1 Introduction

Throughout this paper $Y = (0,1)^n$ will denote the open unit cube in $\mathbb{R}^n$. Let $A : \mathbb{R}^n \to M^{n\times n}(\mathbb{R})$ be a Y-periodic measurable matrix-valued function such that $A^t(y) = A(y) \geq 0$ a.e. in Y. We shall assume that there exist two Y-periodic weight functions (i.e. non negative locally summable periodic functions) λ, Λ such that:

$$\lambda^{2/p}(y)|\xi|^2 \leq \langle A(y)\xi, \xi\rangle \leq \Lambda^{2/p}(y)|\xi|^2 \tag{1}$$

for a.e. $y \in Y$ and for all $\xi \in \mathbb{R}^n$, where $p > 1$. Let Ω be a regular bounded open subset of $\mathbb{R}^n$; in this paper we shall describe the asymptotic behavior, as $\epsilon \to 0^+$, of the solutions of the Dirichlet problem in Ω for the nonlinear degenerate or singular elliptic equation

$$-\mathrm{div}\left(a\left(\frac{x}{\epsilon}, \nabla u\right)\right) := -\mathrm{div}\left(\alpha\left(\frac{x}{\epsilon}, \nabla u\right) A\left(\frac{x}{\epsilon}\right)\nabla u\right) = f(x) \in L^\infty(\Omega), \tag{2}$$

where $\alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies suitable structure conditions that will be specified below (see conditions $H_1 - H_4$ at the end of this section). Roughly speaking, we shall assume that $\alpha(y,\xi) \approx \langle A(y)\xi,\xi\rangle^{p/2-1}$. For the moment, we can think typically of the generalized p-Laplace operator of the form

$$-\mathrm{div}\left(\left|\sqrt{A}\left(\frac{x}{\epsilon}\right)\nabla u\right|^{p-2} A\left(\frac{x}{\epsilon}\right)\nabla u\right) = f, \tag{3}$$

where A has a strongly anisotropic behavior, i.e. the ratio of the upper and the lower eigenvalues is not bounded. When $\lambda \equiv \lambda_0 > 0$ and $\Lambda \equiv \Lambda_0 < \infty$, i.e. when the differential operator (2) is uniformly elliptic in Ω, this problem has been largely studied in the last few years. For a detailed survey about the subject (the so-called homogenization theory), together with physical motivations and applications, the reader can refer to [1], [18], [14], and to the recent monographies [2] and [7]. The result we want to extend in this note states, roughly speaking, that the solutions of Dirichlet problems associated with equations with rapidly oscillating periodic coefficients (like (2)) converge to the solution of a new Dirichlet problem, the so-called homogenized problem, that can be explicitly written. If $\lambda \equiv \Lambda$, then the problem has been fully solved by De Arcangelis and Serra Cassano in [9] for λ belonging to the Muckenhoupt's class A_p. A precise definition of this condition is given below, but for the moment we want to stress the fact that to check such a condition we need a precise description of the local behavior of the coefficients at any scale. Moreover, the assumption $\lambda \equiv \Lambda$ implies that the equation describes phenomena that are basically isotropic in all directions. However, the main feature of the present result will consist of the fact that the weight functions λ and Λ need not to be either bounded or bounded away from zero, and that they can differ each other, in the sense that the ratio Λ/λ can blow-up at some point. Think for instance of an homogeneous material with inclusions producing a strongly anisotropic behavior; our model example will be provided by a, say linear, equation in the plane of the form

$$\frac{\partial}{\partial x_1}\left(\lambda\left(\frac{x_1}{\epsilon},\frac{x_2}{\epsilon}\right)\frac{\partial}{\partial x_1}\right)u+\frac{\partial}{\partial x_2}\left(\Lambda\left(\frac{x_1}{\epsilon},\frac{x_2}{\epsilon}\right)\frac{\partial}{\partial x_2}\right)u=f(x_1,x_2).$$

The key tool we shall use to prove our result will be a two-weight compensated compactness theorem. We stress that the main difference between our result and the classical one (see e.g. [17], [20]), or the weighted version of it proved in [9], relies on the fact that the basic estimate and the dual estimate are assumed here to hold with respect to different weight functions; in this way we keep into account the anisotropy of our situation. In turn, our approach relies on an approximation argument based on Poincaré inequality ([12], [16], [19]) and on two-weights Poincaré inequalities proved by S. Chanillo and R.L. Wheeden in [4]. In fact, our approach has been largely inspired by [4, 5], where the authors study systematically linear anisotropic operators whose quadratic form is controlled in terms of a couple of weights, like in assumption (1). Obviously, in this paper, to prove the convergence to the homogenized problem we have to impose that the two weights, that take into account our anisotropic behavior, are in a sense not "too bad" and not "too different". To state our assumptions in a concise way, we introduce some notations that will be used throughout this paper. If ω is a weight function and E is a measurable subset of $\mathbb{R}^n$, we shall write

$$\omega(E):=\int_E\omega(x)\,dx,\qquad \fint_E\omega(x)\,dx:=\frac{1}{|E|}\int_E\omega(x)\,dx,$$

where $|E|$ is the Lebesgue measure of E.

We shall assume the following conditions are satisfied. The weight functions λ and Λ are not "too bad" in the sense that

I) $\lambda\in A_p$ (the Muckenhoupt's class), i.e. λ is a non-negative measurable function on $\mathbb{R}^n$ such that

$$K:=\ \sup_Q\ \left(\fint_Q\lambda dy\right)\cdot\left(\fint_Q\lambda^{-1/(p-1)}dy\right)^{p-1}<\infty, \tag{4}$$

where the supremum is taken on all cubes $Q=Q(x,r)=\{y=(y_1,\dots,y_n)\in\mathbb{R}^n;\max_j|x_j-y_j|<r\}$, where $x=(x_1,\dots,x_n)\in\mathbb{R}^n, r\in(0,r_0], r_0>0$. We shall call x and r respectively the center and the radius of $Q(x,r)$.

II) $\Lambda\in L^{1+\mu}(Y),\mu>0$, and it is a doubling weight, i.e.

$$D_\Lambda=\sup_Q\Lambda(2Q)/\Lambda(Q)<\infty,$$

where the supremum is taken on all cubes $Q=Q(x,r),x\in\mathbb{R}^n,r\in(0,r_0]$ and $2Q=Q(x,2r)$.

It is well known that condition (4) implies that also λ is a doubling weight, with doubling constant depending only on K (see [13]). The two weights are not "too different" in the sense that

III) there exist $q > p$ and $C > 0$ such that, if I, J are cubes of radius $r(I), r(J)$ respectively, such that $I \subseteq J$ and $r(I), r(J) \le r_0$, then

$$\frac{r(I)}{r(J)} \left(\frac{\Lambda(I)}{\Lambda(J)}\right)^{1/q} \le C \left(\frac{\lambda(I)}{\lambda(J)}\right)^{1/p}. \tag{5}$$

The role of assumptions I), II), and III) above will become clear in Proposition 4 below, since they will yield a scale-invariant two-weight Sobolev–Poincaré inequality that in turn will provide the key tool for a weighted compensated compactness theorem (see Theorem 6).

Example 1 Suppose $\lambda \equiv 1$ and let Λ satisfy (II). Then (III) reads

$$\left(\frac{r(I)}{r(J)}\right)^{1-n/p} \left(\frac{\Lambda(I)}{\Lambda(J)}\right)^{1/q} \le C, \tag{6}$$

that is satisfied for instance if

$$1 - \frac{n}{p} + \frac{1}{p} \log_2 D_\Lambda > 0.$$

Example 2 Suppose $\lambda \equiv 1$ and $\Lambda(x) = |x|^{-\alpha}, 0 < \alpha < n$ on Y, continued by periodicity on all $\mathbb{R}^n$. Assumption (II) is satisfied since Λ is an A_1-weight. An elementary computation shows that

$$\frac{\Lambda(I)}{\Lambda(J)} \le C \left(\frac{r(I)}{r(J)}\right)^{n-\alpha},$$

so that (6) holds provided there exists $q > p$ such that

$$1 - \frac{n}{p} + \frac{1}{q}(n - \alpha) \ge 0;$$

in particular, such a q exists if

$$\alpha < \min\{p, n\}.$$

Example 3 Suppose $\lambda \equiv 1$ and $\mathbb{R}^n = \mathbb{R}^m_x \times \mathbb{R}^{n-m}_y$, and let $\Lambda(x, y) = |x|^{-\alpha}, 0 < \alpha < m$, continued by periodicity. Again we see that condition (III) is satisfied if $\alpha < \min\{p, m\}$ (condition (II) still holds since $\Lambda \in A_1$).

Example 4 If $\Lambda \approx \lambda$, then condition (III) follows by doubling. Finally, let us list precisely our (fairly standard) structure assumptions on the operator in (2). We assume that

H$_1$) $\alpha(y,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$is continuous for a.e. $y \in \mathbb{R}^n$,

H$_2$) $\alpha(\cdot,\xi)$ is measurable and Y-periodic on $\mathbb{R}^n, \forall \xi \in \mathbb{R}^n$,

H$_3$) $c_1\langle A(y)\xi,\xi\rangle^{p/2-1} \le \alpha(y,\xi) \le c_2\langle A(y)\xi,\xi\rangle^{p/2-1}$ where c_1 and c_2 are positive constants,

H$_4$) $\langle \alpha(y,\xi_1)A(y)\xi_1 - \alpha(y,\xi_2)A(y)\xi_2, \xi_1-\xi_2\rangle > 0$ for a.e. $y \in \mathbb{R}^n$ and for every ξ_1 and ξ_2 in $\mathbb{R}^n$ with $\xi_1 \neq \xi_2$.

The paper will be organized as follows: Section 2 will deal with function spaces and will contain the statement of the main theorem. In Section 3 we shall prove the two-weight compensated compactness theorem, and finally Section 4 contains the proof of the convergence theorem.

2 Preliminaries

Let us start by proving that, by periodicity and by doubling, the local assumptions I), II) and III) are in fact global, i.e. they hold for any $r > 0$.

Lemma 1 *If λ and Λ satisfy assumptions* I), II) *and* III) *for $0 < r < r_0$, then they satisfy the same assumptions for any $r > 0$.*

Proof. Write $Y = \cup_{j=1}^{N_0} Q_j$, with $r(Q_j) := r_1 < \frac{1}{2}r_0$ for $j = 1,\dots,N_0$. From now on, r_1 and N_0 will be *fixed geometric constants*. By doubling[1], if Q_i and Q_j are contiguous cubes, then $\lambda(Q_i)$ and $\lambda(Q_j)$ are equivalent[2], as well as $\Lambda(Q_i)$ and $\Lambda(Q_j)$. Thus $\lambda(Q_j) \approx \lambda_0$ and $\Lambda(Q_j) \approx \Lambda_0$, for $j = 1,\dots,N_0$. By periodicity, all the space can be covered by a countable family of congruent cubes $\{Q_j, j = 1,\dots\}$ of radius r_1 enjoying the same property. In addition, again by doubling, if $\tilde{Q}$ is any cube with radius between r_1 and r_0, then $\lambda(\tilde{Q}) \approx \lambda_0$ and $\Lambda(\tilde{Q}) \approx \Lambda_0$. Let now $Q = Q(x,r)$ be any cube with $r \ge r_0$. If we denote by $\mathcal{Q}_*$ and $\mathcal{Q}^*$ respectively the subfamily of cubes of $\{Q_j\}$ contained in Q and of cubes having nonempty intersection with Q, we have

$$\#\mathcal{Q}_* \ge \left(\left[\frac{2r}{r_0}\right] - 1\right)^n, \qquad \#\mathcal{Q}^* \le \left(\left[\frac{2r}{r_0}\right] + 1\right)^n,$$

so that

$$\lambda(Q) \approx \left[\frac{2r}{r_0}\right]^n \lambda_0, \quad \Lambda(Q) \approx \left[\frac{2r}{r_0}\right]^n \Lambda_0, \quad |Q| \approx \left[\frac{2r}{r_0}\right]^n.$$

[1] By saying 'by doubling' we shall mean that the assertion relies on the doubling property of Λ specified in II) and on the doubling property of λ that follows from I), and that all constants depend only on K, D_Λ and on other geometric constants.

[2] As above 'equivalent' ($\approx$) will mean that the ratio $\lambda(Q_i)/\lambda(Q_j)$ is bounded and bounded away from zero by positive constants that depend only on geometric constants.

Thus, clearly, I) and II) hold for any $r > 0$, as well as III) when both $r(I)$, $r(J) \geq r_0$. As for III), if $r(I) < r_0 \leq r(J)$, we can argue as follows. Let $\tilde{J}$ be the cube with the same center as I and radius r_0. By doubling, $\lambda(\tilde{J}) \approx [\frac{2r(J)}{r_0}]^n \lambda(J)$ and $\Lambda(\tilde{J}) \approx [\frac{2r(J)}{r_0}]^n \Lambda(J)$, and then

$$\begin{aligned} &\frac{r(I)}{r(J)} \left(\frac{\Lambda(I)}{\Lambda(J)}\right)^{1/q} \left(\frac{\lambda(I)}{\lambda(J)}\right)^{-1/p} \\ &\qquad \approx \frac{r(I)}{r(\tilde{J})} \left(\frac{\Lambda(I)}{\Lambda(\tilde{J})}\right)^{1/q} \left(\frac{\lambda(I)}{\lambda(\tilde{J})}\right)^{-1/p} \left[\frac{2r}{r_0}\right]^{n/p-n/q-1} \leq C, \end{aligned}$$

by III), since $r(I), r(\tilde{J}) \leq r_0$, and we can always assume q close to p such that $n/p - n/q - 1 \leq 0$ (keep in mind $\frac{2r(J)}{r_0} \geq 2$). $\square$

We now define some function spaces suitable to our problem. If $s > 1$ and $\omega \in A_s$ we shall denote by $L^s(\Omega, \omega) = \{u \in L^1_{loc}(\Omega); \omega^{1/s} u \in L^s(\Omega)\}$, and by $W^{1,s}(\Omega, \omega) = \{u \in W^{1,1}_{loc}(\Omega); \omega^{1/s} u \in L^s(\Omega), \omega^{1/s} \nabla u \in (L^s(\Omega))^n\}$. Moreover, $\overset{\circ}{W}{}^{1,s}(\Omega, \omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,s}(\Omega, \omega)$.

Definition 1 If $1 < p < \infty$, we put

$$W_A^{1,p}(\Omega) = \left\{ u \in L^p(\Omega, \lambda) \cap W^{1,1}_{loc}(\Omega); \int_\Omega \langle A\nabla u, \nabla u \rangle^{p/2} dx < +\infty \right\}, \tag{7}$$

endowed with the norm

$$\|u\|_{W_A^{1,p}(\Omega)} = \left(\int_\Omega \langle A\nabla u, \nabla u \rangle^{p/2} dx \right)^{1/p} + \left(\int_\Omega |u|^p \lambda dx \right)^{1/p}. \tag{8}$$

We shall show below (Theorem 1) that $W_A^{1,p}(\Omega)$ is continuously embedded in $W^{1,1}(\Omega)$. Thus we can define

$$\overset{\circ}{W}{}_A^{1,p}(\Omega) := W_A^{1,p}(\Omega) \cap \overset{\circ}{W}{}^{1,1}(\Omega).$$

Remark 1 If $\lambda \approx \Lambda$ then $\overset{\circ}{W}{}_A^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W_A^{1,p}(\Omega)$ (see [9], Remark 3.6).

Theorem 1 *Suppose λ satisfies hypothesis* I) *in the Introduction. Then we have:*

i) *$W_A^{1,p}(\Omega)$ is a reflexive Banach space;*

ii) *$W_A^{1,p}(\Omega)$ is continuously embedded in $W^{1,p}(\Omega, \lambda)$ and hence continuously embedded in $W^{1,1}(\Omega)$. Moreover $\overset{\circ}{W}{}_A^{1,p}(\Omega)$ is continuously embedded in $\overset{\circ}{W}{}^{1,p}(\Omega, \lambda)$;*

iii) $[u]^p_{W^{1,p}_A(\Omega)} := \int_\Omega \langle A\nabla u, \nabla u\rangle^{p/2} dx$ *is a norm on* $\overset{\circ}{W}{}^{1,p}_A(\Omega)$. *More precisely,* $\|u\|_{L^p(\Omega,\lambda)} \le C[u]_{W^{1,p}_A(\Omega)}$, *where the constant* C *is a geometric constant depending only on* K, *the* A_p *constant of* λ *in* (4);

iv) $(W^{1,p}_A(\Omega))^* = \{u \in \mathcal{D}'(\Omega), u = g_0\lambda^{1/p} + div(\sqrt{A}g)$, *with* $(g_0, g) \in (L^{p'}(\Omega))^{n+1}$, $1/p + 1/p' = 1\}$, *and the action of* u *on* $\varphi \in W^{1,p}_A(\Omega)$ *is given by*

$$\int_\Omega \left[g_0\varphi\lambda^{1/p} + \langle\sqrt{A}g, \nabla\varphi\rangle\right] dx;$$

v) $(\overset{\circ}{W}{}^{1,p}_A(\Omega))^* = \{u \in \mathcal{D}'(\Omega), u = div(\sqrt{A}g)$, *with* $g \in (L^{p'}(\Omega))^n, 1/p + 1/p' = 1\}$, *and the action of* u *on* $\varphi \in \overset{\circ}{W}{}^{1,p}_A(\Omega)$ *is given by*

$$\int_\Omega \langle\sqrt{A}g, \nabla\varphi\rangle dx.$$

Proof. The first statement in ii) follows straightforwardly from (1) and from well known properties of A_p-weights. Moreover, by [9], Remark 3.6,

$$\begin{aligned}\overset{\circ}{W}{}^{1,p}_A(\Omega) &= W^{1,p}_A(\Omega) \cap \overset{\circ}{W}{}^{1,1}(\Omega) = W^{1,p}_A(\Omega) \cap W^{1,p}(\Omega,\lambda) \cap \overset{\circ}{W}{}^{1,1}(\Omega) \\ &= W^{1,p}_A(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega,\lambda) \hookrightarrow \overset{\circ}{W}{}^{1,p}(\Omega,\lambda)\end{aligned}$$

To prove i), let us show that $W^{1,p}_A(\Omega)$ is linearly isometric to a closed subspace of $(L^p(\Omega))^{n+1}$. Indeed, consider the map $T : W^{1,p}_A(\Omega) \to (L^p(\Omega))^{n+1}$ defined by

$$Tu = (\lambda^{1/p}u, \sqrt{A}\nabla u).$$

Clearly, T is a linear isometry, and we have only to show that its range is closed. Thus, take $(f_0, f) = (f_0, f_1, \dots, f_n)$ such that there exist $u_k \in W^{1,p}_A(\Omega)$ such that $\lambda^{1/p}u_k$ and $\sqrt{A}\nabla u_k$ converge in $L^p(\Omega)$ to f_0 and f respectively. By ii), there exists $u_0 \in W^{1,1}(\Omega)$ such that $u_k \to u_0$ in $W^{1,1}(\Omega)$; moreover it is easy to see that $\lambda^{1/p}u_0 = f_0$ and $u_k \to u_0$ in $L^p(\Omega,\lambda)$ as $k \to \infty$. To achieve the proof, let us notice that, without loss of generality we may assume that $\nabla u_k \to \nabla u_0$ a.e. in Ω, and that there exists $h \in L^p(\Omega)$ such that $|\sqrt{A}\nabla u_k| \le h$ a.e. in Ω. Thus, by dominate convergence theorem, $\sqrt{A}\nabla u_k \to \sqrt{A}\nabla u_0$, that implies that $f = \sqrt{A}\nabla u_0$, so that $(f_0, f) \in T(W^{1,p}_A(\Omega))$, and we are done. We can prove now iv) by noticing that, thanks to Hahn–Banach theorem, a functional $F \in (W^{1,p}_A(\Omega))^*$ can be written as

$$F(\varphi) = \int_\Omega \langle(g_0, g), T(\varphi)\rangle\, dx,$$

where $(g_0, g) \in (L^{p'}(\Omega))^{n+1}$, $1/p + 1/p' = 1$. To prove iii) holds, it is enough to show that the $L^p(\Omega, \lambda)$-norm of $u \in \overset{\circ}{W}{}^{1,p}_{A}(\Omega)$ can be controlled by $[u]^p_{W^{1,p}_A(\Omega)}$. But this follows straightforwardly thanks to (1) and ii), by known Poincaré type inequalities for A_p weights (see [10]), since u belongs to $\overset{\circ}{W}{}^{1,p}(\Omega, \lambda)$ and then can be approximated in this last norm by smooth functions supported in Ω to whom Poincaré inequality applies. Finally, v) can be proved as iv), keeping in mind iii). □

Let us now recall the following result ([15], III Corollary 1.8, page 87):

Theorem 2 *Let X be a Banach space, $K \subseteq X$ be a nonempty closed convex subset and let $A : K \to X^*$ be monotone, coercive and continuous on finite dimensional subspaces. Then there exists $u \in K$ such that*

$$\langle Au, v - u\rangle_{X^*,X} \geq 0 \text{ for any } v \in K. \tag{9}$$

If $\epsilon > 0$ we put

$$A_\epsilon(x) = A\left(\frac{x}{\epsilon}\right).$$

Theorem 3 *If $f \in L^\infty(\Omega)$ then there exists a unique $u_\epsilon \in \overset{\circ}{W}{}^{1,p}_{A_\epsilon}(\Omega)$ such that*

$$\int_\Omega \left\langle a\left(\frac{x}{\epsilon}, \nabla u_\epsilon\right), \nabla\varphi\right\rangle dx = \int_\Omega f\varphi dx \tag{10}$$

for all $\varphi \in \overset{\circ}{W}{}^{1,p}_{A_\epsilon}(\Omega)$.

Proof. Consider the operator $\mathcal{A} : \overset{\circ}{W}{}^{1,p}_{A_\epsilon}(\Omega) \to (\overset{\circ}{W}{}^{1,p}_{A_\epsilon}(\Omega))^*$ defined by $\mathcal{A}(u) = \operatorname{div}(a(\cdot, \nabla u)) - f$. It can be easily verified that $\mathcal{A}(u) \in (\overset{\circ}{W}{}^{1,p}_{A_\epsilon}(\Omega))^*$ by using (H_3). Moreover $\mathcal{A}$ is monotone, coercive and weakly continuous, so that Theorem 2 can be applied. □

We shall denote by $W^{1,p}_{A,\#}(Y)$ the set of real functions $u \in W^{1,1}_{loc}(\mathbb{R}^n)$, u Y-periodic, $u \in W^{1,p}_A(Y)$.

Theorem 4 *If $\xi \in \mathbb{R}^n$, then there exists a unique function $v \in \langle\xi, \cdot\rangle + W^{1,p}_{A,\#}(Y)$ with $\int_Y v dy = 0$ such that*

$$\int_Y \langle a(y, \nabla v(y)), \nabla w(y)\rangle dy = 0 \quad \forall w \in W^{1,p}_{A,\#}(Y) \tag{11}$$

Proof. The existence of the solution is a consequence of Theorem 2, with $X = W_A^{1,p}(Y), K = \{u = \langle \xi, \cdot \rangle + \tilde{u}, \tilde{u} \in W_{A,\#}^{1,p}(Y), \int_Y \tilde{u} dy = 0\}$ and $Au = div(a(\cdot, \nabla u))$. Indeed Theorem 2 implies that

$$\int_Y \langle a(y, \nabla v(y)), \nabla \varphi(y) - \nabla v(y) \rangle dy \geq 0$$

for any $\varphi \in K$. Thus, if $w \in W_{A,\#}^{1,p}$ and $v = \langle \xi, \cdot \rangle + \tilde{v}$ with $\tilde{v} \in W_{A,\#}^{1,p}$, to obtain (11) it is enough to choose $\varphi = \pm w + \tilde{v} + \langle \xi, \cdot \rangle \mp \int_Y w dy$. Uniqueness is due to the strict monotonicity of a, see property H_4. □

Arguing as in [9], if $\xi \in \mathbb{R}^n$ we put

$$b(\xi) = \int_Y a(y, \nabla v(y)) dy, \tag{12}$$

where v is the solution of (11). In fact b will define our homogenized operator. We have:

Proposition 1 *Let b be defined by* (12), *then*

i) $|b(\xi)| \leq c_1 |\xi|^{p-1} \quad \forall \xi \in \mathbb{R}^n$;

ii) $\langle b(\xi), \xi \rangle \geq c_2 |\xi|^p \quad \forall \xi \in \mathbb{R}^n$;

iii) $\langle b(\xi_1) - b(\xi_2), \xi_1 - \xi_2 \rangle > 0 \quad \forall \xi_1, \xi_2 \in \mathbb{R}^n, \xi_1 \neq \xi_2$;

iv) *b is continuous on $\mathbb{R}^n$.*

Proof. Just repeat more or less verbatim the proofs of Lemmas 3.3, 3.4 and 3.5 in [9]. □

We can state now the desired convergence result:

Theorem 5 *Let $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a function satisfying the following structure properties:*

$$\begin{cases} \text{i)} & a(y, \cdot) : \mathbb{R}^n \to \mathbb{R}^n \textit{ is continuous } \forall y \in \mathbb{R}^n, \\ \text{ii)} & a(\cdot, \xi) \textit{ is measurable and } Y \textit{ periodic on } \mathbb{R}^n, \forall \xi \in \mathbb{R}^n, \\ \text{iii)} & a(y, \xi) \textit{ is of the form } a(y, \xi) = \alpha(y, \xi) A(y) \xi, \\ & \textit{where } \alpha(y, \xi) \in \mathbb{R}, A(y) \in M^{n \times n}(\mathbb{R}), y, \xi \in \mathbb{R}^n, \\ & \textit{and moreover } A = A^t, \alpha(y, \xi) \approx \langle A(y) \xi, \xi \rangle^{p/2-1}; \end{cases} \tag{S_1}$$

$$\lambda^{2/p}(y) |\xi|^2 \leq \langle A(y) \xi, \xi \rangle \leq \Lambda^{2/p}(y) |\xi|^2 \tag{S_2}$$

where the weights λ *and* Λ *are Y-periodic and satisfies conditions* I), II) *and* III) *in the Introduction;*

$$\langle \alpha(y,\xi_1)A(y)\xi_1 - \alpha(y,\xi_2)A(y)\xi_2, \xi_1 - \xi_2\rangle > 0 \tag{S_3}$$

for a.e. $y \in \mathbb{R}^n$ *and for every* ξ_1 *and* ξ_2 *in* $\mathbb{R}^n$ *with* $\xi_1 \neq \xi_2$. *Let* $\epsilon > 0, \Omega$ *be a bounded open set of* $\mathbb{R}^n$ *and* $f \in L^\infty(\Omega)$. *Let* u_ϵ *be the weak solution of the Dirichlet problem* (P_ϵ)*:*

$$\begin{cases} -div(\alpha(\frac{x}{\epsilon},\nabla u)A(\frac{x}{\epsilon})\nabla u) = f & on\ \Omega \\ u \in \overset{\circ}{W}{}^{1,p}_{A_\epsilon}(\Omega) \end{cases} \tag{P_ϵ}$$

where $A_\epsilon(x) = A(\frac{x}{\epsilon})$, *and let* u_0 *be the solution of the Dirichlet problem* (P_0)*:*

$$\begin{cases} -div(b(\nabla u)) = f & on\ \Omega \\ u \in \overset{\circ}{W}{}^{1,p}(\Omega) \end{cases} \tag{P_0}$$

Then, for $\epsilon \to 0$, *we have*

$$\begin{array}{rl} u_\epsilon \to u_0 & in\ \overset{\circ}{W}{}^{1,1}(\Omega) - weak \\ \alpha(\frac{x}{\epsilon},\nabla u_\epsilon)A(\frac{x}{\epsilon})\nabla u_\epsilon \to b(\nabla u_0) & in\ (L^1(\Omega))^n - weak. \end{array}$$

Remark 2 We recall that every problem $(P_\epsilon)_{\epsilon\geq 0}$ has a unique weak solution, see Theorem 3.

The proof of the above theorem will be the content of Section 4.

3 Compensated compactness

Our compensated compactness result (that is the main result of the present paper) reads as follows.

Theorem 6 *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$, *and let* ν *be an* A_p *weight in* $\mathbb{R}^n$. *Moreover let* $\{\lambda_\epsilon\}$ *and* $\{\Lambda_\epsilon\}$ *be two families of weight functions defined in* $\mathbb{R}^n$ *satisfying the following conditions:*

1. *there exists* $q > p > 1$ *and* $C > 0$ *such that, if* I, J *are cubes of radius respectively* $r(I)$ *and* $r(J)$ *and such that* $I \subseteq J$, *then*

$$\frac{r(I)}{r(J)}\left(\frac{\Lambda_\epsilon(I)}{\Lambda_\epsilon(J)}\right)^{1/q} \leq C\left(\frac{\lambda_\epsilon(I)}{\lambda_\epsilon(J)}\right)^{1/p}, \tag{13}$$

   where $C > 0$ *is independent of* $\epsilon \in (0,1)$.

2. $\{\Lambda_\epsilon\}$ *is uniformly doubling and* $\{\lambda_\epsilon\}$ *is uniformly* A_p, *i.e. there exist* $D, K > 0$ *such that*

$$\Lambda_\epsilon(2I) \leq D\Lambda_\epsilon(I), \qquad ⨍_I \lambda_\epsilon dx \left(⨍_I \lambda_\epsilon^{-1/(p-1)} dx \right)^{p-1} \leq K \tag{14}$$

for all cubes I *and for all* $\epsilon \in (0,1)$.

Let now $\{u_\epsilon\}$ *be a family of functions such that*

$$\begin{cases} u_\epsilon \in W^{1,p}(\Omega, \lambda_\epsilon) & & \text{(a)} \\ \|u_\epsilon\|_{W^{1,p}(\Omega,\lambda_\epsilon)} \leq \alpha_1 & \textit{for all } \epsilon \in (0,1) & \text{(b)} \\ \textit{there exists } u \in W^{1,p}(\Omega, \nu); & u_\epsilon \to u \textit{ in } L^1(\Omega). & \text{(c)} \end{cases}$$

Moreover, let $\{a_\epsilon\}$ *be a family of vector-valued functions such that*

$$\begin{cases} \langle a_\epsilon, \nabla u_\epsilon \rangle \in L^1(\Omega) & \textit{for all } \epsilon \in (0,1) & \text{(d)} \\ \int_\Omega |a_\epsilon|^{p'} \Lambda_\epsilon^{-1/(p-1)} dx \leq \alpha_2^{p'} & \textit{for all } \epsilon \in (0,1) & \text{(e)} \\ div(a_\epsilon) = f \in L^\infty \textit{ on } C_0^1(\Omega) & \textit{for all } \epsilon \in (0,1) & \text{(f)} \\ \textit{there exists } a \in (L^{p'}(\Omega, \nu^{-1/(p-1)})^n; & a_\epsilon \to a \textit{ in } (L^1(\Omega))^n - \textit{weak}. & \text{(g)} \end{cases}$$

Then

$$\langle a_\epsilon, \nabla u_\epsilon \rangle \to \langle a, \nabla u \rangle \textit{ in } \mathcal{D}'(\Omega). \tag{15}$$

Remark 3 Note that the limit $\langle a, \nabla u \rangle \in L^1(\Omega)$, by (c) and (g).

Remark 4 As we shall see, assumptions 1) and 2) in Theorem 6 are in fact stronger than we really need. Indeed, we might restrict ourselves to assume 1) and 2) hold not globally, but locally, in the sense that the result would follow by assuming that for any open set $\Omega' \subset\subset \Omega$ 1) and 2) hold for cubes $I, J \subseteq \Omega'$, with constants depending on Ω'.

Remark 5 In is easy to see that assumption 1) and 2) of the above theorem are satisfied by choosing

$$\lambda_\epsilon(x) = \lambda\left(\frac{x}{\epsilon}\right), \quad \Lambda_\epsilon(x) = \Lambda\left(\frac{x}{\epsilon}\right). \tag{16}$$

This choice of λ_ϵ and Λ_ϵ will be used to prove later the main convergence theorem.

Let us state now a preliminary result. The first part is well known, whereas the last statement is basically contained in [21].

Proposition 2 *Suppose* Λ *satisfies hypothesis* II) *in the Introduction. If* I *is any cube and* $t \in (0,1)$, *then*

$$\Lambda_\epsilon(tI) \geq \frac{1}{D} t^{\log_2 D} \Lambda_\epsilon(I).$$

Moreover, if we put $\tilde{D} = 1 + \frac{1}{D} 5^{-\log_2 D} > 1$ *we have*

$$\Lambda_\epsilon(2I) \geq \tilde{D}\Lambda_\epsilon(I)$$

for any cube I*, and hence*

$$\Lambda_\epsilon(tI) \leq \tilde{D} t^{\log_2 \tilde{D}} \Lambda_\epsilon(I)$$

for all $t \in (0,1)$ *and* $\epsilon \in (0,1)$*.*

The following result will be crucial in the proof of Theorem 6.

Proposition 3 *With the notations of Theorem 6, for any* $\eta > 0$ *there exists* $r(\eta, \Omega) > 0$ *such that, for any cube* $Q = Q(x,r), r < r(\eta, \Omega)$ *we have*

$$\left(\frac{\Lambda_\epsilon(Q)}{\lambda_\epsilon(Q)}\right)^{1/p} r < \eta \tag{17}$$

for all $\epsilon \in (0,1)$*.*

Proof. Let $\{J_1, \dots, J_\ell\}$ be a fixed covering of $\bar{\Omega}$, where $J_k = Q(x_k, 1)$, $k = 1, \dots, \ell$. Let now $Q = Q(x,r)$ be any cube with $x \in \Omega$ and $r \leq 1$, and let k be such that $x \in J_k$, so that $Q \subseteq 2J_k$ and we can apply (13) with $J = 2J_k$ and $I = Q$. After few elementary computations we get

$$\begin{aligned} r\left(\frac{\Lambda_\epsilon(Q)}{\lambda_\epsilon(Q)}\right)^{1/p} &\leq 2C\left(\frac{\Lambda_\epsilon(2J_k)}{\lambda_\epsilon(2J_k)}\right)^{1/p}\left(\frac{\Lambda_\epsilon(Q)}{\Lambda_\epsilon(2J_k)}\right)^{1/p-1/q} \\ &\leq 2C\max_i\left(\frac{\Lambda_\epsilon(2J_i)}{\lambda_\epsilon(2J_i)}\right)^{1/p}\left(\frac{\Lambda_\epsilon(Q)}{\Lambda_\epsilon(2J_k)}\right)^{1/p-1/q} \\ &= \left(\frac{\Lambda_\epsilon(Q)}{\Lambda_\epsilon(2J_k)}\right)^{1/p-1/q}. \end{aligned}$$

On the other hand, $Q(x,1) \subseteq Q(x_k, 2) = 2J_k$, so that $\Lambda_\epsilon(2J_k) \geq \Lambda_\epsilon(Q(x,1))$. Hence, by Proposition 2,

$$\frac{\Lambda_\epsilon(Q)}{\Lambda_\epsilon(2J_k)} \leq \frac{\Lambda_\epsilon(Q(x,r))}{\Lambda_\epsilon(Q(x,1))} \leq \tilde{D} r^{\log_2 \tilde{D}},$$

and the assertion follows, since $1/p - 1/q > 0$. □

The proof of Theorem 6 relies on the following two-weight Sobolev–Poincaré inequality. This result is basically proved in [4].

Proposition 4 *Under the assumptions of Theorem* 6*, if B is an Euclidean ball, $B \subseteq \Omega$, and $u \in W^{1,p}(\Omega, \lambda_\epsilon)$, then for any $s \in [p,q]$*

$$\left(\frac{1}{\Lambda_\epsilon(B)}\int_B |u-u_B|^s \Lambda_\epsilon dx\right)^{1/s} \le C_s r(B)\left(\frac{1}{\lambda_\epsilon(B)}\int_B |\nabla u|^p \lambda_\epsilon dx\right)^{1/p}, \qquad (18)$$

where $u_B = \fint_B u(x)dx$ is the (Lebesgue) average of u on B, and the constant $C_s > 0$ is a geometric constant depending only on n, s, and on the constants of assumptions 1) *and* 2) *of Theorem* 6*. In particular, C is independent of $\epsilon \in (0,1)$. Moreover, if $u \in \overset{\circ}{W}{}^{1,p}(\Omega, \lambda_\epsilon)$, then*

$$\left(\frac{1}{\Lambda_\epsilon(B)}\int_B |u|^s \Lambda_\epsilon dx\right)^{1/s} \le C_s r(B)\left(\frac{1}{\lambda_\epsilon(B)}\int_B |\nabla u|^p \lambda_\epsilon dx\right)^{1/p}, \qquad (19)$$

where again C_s is independent of $\epsilon \in (0,1)$.

Proof. By [4], (18) holds when $u \in Lip_{loc}(\Omega)$. If now $u \in W^{1,p}(\Omega, \lambda_\epsilon)$ there exists a sequence $u_k \in C^\infty(\Omega) \cap W^{1,p}(\Omega, \lambda_\epsilon), u_k \to u$ in $W^{1,p}(\Omega, \lambda_\epsilon)$, by [6]. In particular, $u_k \to u$ in $L^1_{loc}(\Omega)$ and a.e. in Ω, so that $(u_k)_B \to u_B$ and the we can conclude by Fatou's lemma. The proof of the second statement is analogous. □

Remark 6 Through a covering argument it follows easily from (19) that, if $\Omega' \subset\subset \Omega$, then for every $u \in W^{1,p}(\Omega, \lambda_\epsilon)$

$$\|u\|_{L^p(\Omega', \Lambda_\epsilon)} \le C_{\Omega'} \|u\|_{W^{1,p}(\Omega, \lambda_\epsilon)} \qquad (20)$$

where $C_{\Omega'}$ is independent of $\epsilon \in (0,1)$.

In order to prove the theorem we need the following approximation lemma:

Lemma 2 *Let λ_ϵ, Λ_ϵ, u and u_ϵ be as in Theorem* 6*. Then for any $\Omega' \subset\subset \Omega$ and for any $\eta > 0$*

- *for any $\epsilon \in (0,1)$ there exists $u_{\epsilon,\eta} \in C^\infty(\Omega)$ such that*

$$\int_{\Omega'} |u_\epsilon - u_{\epsilon,\eta}|^p \Lambda_\epsilon dx \le \eta^p \int_\Omega |\nabla u_\epsilon|^p \lambda_\epsilon dx; \qquad (21)$$

- *there exist $u_\eta \in C^\infty(\Omega)$ such that*

$$\int_{\Omega'} |u - u_\eta|^p \nu dx \le \eta^p \int_\Omega |\nabla u|^p \nu dx. \qquad (22)$$

Moreover for any $\eta > 0$

$$u_{\epsilon,\eta} \to u_\eta \quad \text{as} \quad \epsilon \to 0^+ \qquad (23)$$

in $L^\infty(\Omega')$.

Proof. Arguing as in [12], if $\delta \in (0, 1/10)$ there exist a countable family $\{B_j^\delta = B(x_j^\delta, r_{j\delta}), j \in N\}$ of balls, a countable family of smooth functions $\{\varphi_j^\delta, j \in N\}$, and a geometric constant $c > 0$ such that:

1. $\bigcup_j \frac{3}{4}B_j^\delta = \Omega$, $\frac{1}{4}B_j^\delta \cap \frac{1}{4}B_i^\delta = \emptyset, \quad i \neq j$;
2. $r_{j\delta} = \delta\, \mathrm{dist}\,(\frac{1}{4}B_j^\delta, \partial\Omega)$;
3. $\sum_j \chi_{B_j^\delta} \leq c\chi_\Omega$;
4. if $B_j^\delta \cap B_i^\delta \neq \emptyset$, then $\frac{1}{2}r_{i\delta} \leq r_{j\delta} \leq 2r_{i\delta}$;
5. $\varphi_j^\delta \in C^\infty(\mathbb{R}^n)$;
6. $\sum_j \varphi_j^\delta \equiv 1$ on $\overline{\Omega}$ and $\varphi_j^\delta \geq 0$ for $j \in N$;
7. $\mathrm{supp}\,\varphi_j^\delta \subseteq B_j^\delta$ for $j \in N$;
8. $|\nabla\varphi_j^\delta| \leq c/r_{j\delta}$ for $j \in N$.

Now we put

$$u_{\epsilon,\eta} = \sum_j c(u_\epsilon, B_j^\delta)\varphi_j^\delta$$

where $\delta = \delta(\eta)$ will be fixed later, and

$$c(u_\epsilon, B_j^\delta) = \fint_{B_j^\delta} u_\epsilon dx$$

(note that $u_\epsilon, u \in L^1_{loc}(\Omega)$). Let us prove for instance that (21) holds. Arguing as in [12], page 108 and keeping in mind that, by (3), for any $x \in \Omega$ $\varphi_j^\delta(x) \neq 0$ for at most c values of j, we have

$$\begin{aligned}
\int_{\Omega'} |u_\epsilon - u_{\epsilon,\eta}|^p \Lambda_\epsilon dx &= \int_{\Omega'} \left| \sum_j \left(u_\epsilon - \fint_{B_j^\delta} u_\epsilon dx \right) \varphi_j^\delta \right|^p \Lambda_\epsilon dx \\
&\leq C \sum_{B_j^\delta \cap \Omega' \neq \emptyset} \int_{B_j^\delta \cap \Omega'} \left| u_\epsilon - \fint_{B_j^\delta} u_\epsilon dx \right|^p \Lambda_\epsilon dx \\
&\leq C \sum_{B_j^\delta \cap \Omega' \neq \emptyset} \frac{\Lambda_\epsilon(B_j^\delta)}{\lambda_\epsilon(B_j^\delta)} r_{j\delta}^p \int_{B_j^\delta} |\nabla u_\epsilon|^p \lambda_\epsilon dx
\end{aligned}$$

where C is a geometric constant and the last inequality follows from Proposition 4. Thus, since by (2) $r_{j\delta} \leq \delta\, \mathrm{diam}(\Omega)$, if δ is sufficiently small, then $r_{j\delta} < r(\eta, \Omega)$, where $r(\eta, \Omega)$ is defined in Proposition 3, so that the last sum is bounded by

$$C\eta^p \sum_{B_j^\delta \cap \Omega' \neq \emptyset} \int_{B_j^\delta} |\nabla u_\epsilon|^p \lambda_\epsilon dx \leq C\eta^p \int_\Omega |\nabla u_\epsilon|^p \lambda_\epsilon dx, \tag{24}$$

again by property (3) of our covering. On the other hand, to prove (22) it is enough to repeat the same argument keeping in mind that $\nu \in A_p$. Hence

$$\begin{aligned}\int_{\Omega'} |u - u_\eta|^p \nu dx &\leq C \sum_{B_j^\delta \cap \Omega'} \int_{B_j^\delta} \left| u - \fint_{B_j^\delta} u dx \right|^p \nu dx \\ &\leq C \sum_{B_j^\delta \cap \Omega'} r_{j\delta}^p \int_{B_j^\delta} |\nabla u_\epsilon|^p \nu dx \\ &\leq C\delta^p (\mathrm{diam}\Omega)^p \int_\Omega |\nabla u|^p \nu dx,\end{aligned}$$

by [10], Theorem 45. Thus we have only to prove that $u_{\epsilon,\eta} \to u_\eta$ as $\epsilon \to 0^+$ in $L^\infty(\Omega')$. Note now that, if we put $J_\delta(\Omega') = \{j \in N; B_j^\delta \cap \Omega' \neq \emptyset\}$, then $\# J_\delta(\Omega') < \infty$ (arguing as in [12], page 108). We have for $x \in \Omega'$ and $\delta = \delta(\eta)$:

$$\begin{aligned}|u_{\epsilon,\eta}(x) - u_\eta(x)| &\leq \sum_{j \in J_\delta(\Omega')} \varphi_i^\delta(x) \left| \fint_{B_j^\delta} (u_\epsilon - u) dx \right| \\ &\leq \# J_\delta(\Omega') \max_{j \in J_\delta(\Omega')} \fint_{B_j^\delta} |u_\epsilon - u| dx \to 0\end{aligned}$$

as $\epsilon \to 0$ since $u_\epsilon \to u$ in $L^1(\Omega)$. □

Proof of Theorem 6. We must show that for any $\varphi \in \mathcal{D}(\Omega)$ we have

$$\langle a_\epsilon, \nabla u_\epsilon \rangle(\varphi) \to \langle a, \nabla u \rangle(\varphi) \tag{25}$$

as $\epsilon \to 0$. Let $\Phi \in \mathcal{D}(\Omega)$ be such that $\Phi \equiv 1$ in a neighborhood $\mathcal{U}_\varphi$ of $\mathrm{supp}\,\varphi$, and put $\overline{u}_\epsilon = u_\epsilon \Phi, \overline{u} = u\Phi$; $\overline{u}_\epsilon$ and $\overline{u}$ can be considered continued by zero on all of $\mathbb{R}^n$. Now for $\eta > 0$ we have

$$\begin{aligned}\int_\Omega \langle a_\epsilon, \nabla\varphi \rangle u_\epsilon dx &= \int_{\mathrm{supp}\,\Phi} \langle a_\epsilon, \nabla\varphi \rangle \Lambda_\epsilon^{-1/p} (\overline{u}_\epsilon - \overline{u}_{\epsilon,\eta}) \Lambda_\epsilon^{1/p} dx \\ &\quad + \int_{\mathrm{supp}\,\Phi} \langle a_\epsilon, \nabla\varphi \rangle \overline{u}_{\epsilon,\eta} dx = I_1^\epsilon + I_2^\epsilon,\end{aligned}$$

where, for sake of simplicity, we write $\overline{u}_{\epsilon,\eta}$ instead of $(\overline{u}_\epsilon)_\eta$. By Hölder inequality and Lemma 2 we now have:

$$|I_1^\epsilon| \leq \left(\int_\Omega |a_\epsilon|^{p'} \Lambda_\epsilon^{-1/(p-1)} |\nabla\varphi|^{p'} \right)^{1/p'} \eta \|\nabla \overline{u}_\epsilon\|_{L^p(\Omega,\lambda_\epsilon)} \leq C_\varphi \alpha_1 \alpha_2 \eta, \tag{26}$$

provided ϵ is sufficiently small. Hence

$$\begin{aligned}&\left| \int_\Omega \langle a_\epsilon, \nabla\varphi \rangle u_\epsilon dx - \int_\Omega \langle a, \nabla\varphi \rangle u dx \right| = \left| \int_\Omega \langle a_\epsilon, \nabla\varphi \rangle \overline{u}_\epsilon dx - \langle a, \nabla\varphi \rangle u dx \right| \\ &\leq C_\varphi \alpha_1 \alpha_2 \eta + \left| \int_\Omega \langle a_\epsilon, \nabla\varphi \rangle \overline{u}_{\epsilon,\eta} dx - \langle a, \nabla\varphi \rangle u dx \right|.\end{aligned}$$

Now, by Lemma 2, since $\overline{u}_\epsilon \to \overline{u}$ in $L^1(\Omega)$, $\overline{u}_{\epsilon,\eta} \to \overline{u}_\eta$ on $L^\infty(supp\,\Phi)$ as $\epsilon \to 0^+$. On the other hand, supp $\langle a_\epsilon, \nabla\varphi\rangle \subseteq \mathcal{U}_\varphi$, where $\Phi \equiv 1$, and hence, if η is sufficiently small, $\overline{u}_\eta = u_\eta$ on supp φ, so that, keeping in mind that $a_\epsilon \to a$ weakly in $(L^1(\Omega))^n$,

$$\int_\Omega \langle a_\epsilon, \nabla\varphi\rangle \overline{u}_{\epsilon,\eta} dx \overset{\epsilon\to 0^+}{\to} \int_\Omega \langle a, \nabla\varphi\rangle \overline{u}_\eta dx \equiv \int_\Omega \langle a, \nabla\varphi\rangle u_\eta dx. \tag{27}$$

Therefore

$$\begin{aligned} &\limsup_{\epsilon\to 0^+} \left| \int_\Omega \langle a_\epsilon, \nabla\varphi\rangle u_\epsilon dx - \int_\Omega \langle a, \nabla\varphi\rangle u dx \right| \\ &\quad \le C\eta + \left| \int_\Omega \langle a, \nabla\varphi\rangle u_\eta dx - \langle a, \nabla\varphi\rangle u dx \right| \le C\eta, \end{aligned}$$

since

$$\left| \int_\Omega \langle a, \nabla\varphi\rangle (u_\eta - u) dx \right| \le C_\varphi \left(\int_\Omega |a|^{p'} \nu^{-1/(p-1)} dx \right)^{1/p'} \left(\int_\Omega |\nabla u|^p \nu dx \right)^{1/p} \eta. \tag{28}$$

Hence

$$\int_\Omega \langle a_\epsilon, \nabla\varphi\rangle u_\epsilon dx \to \int_\Omega \langle a, \nabla\varphi\rangle u dx. \tag{29}$$

On the other hand

$$\int_\Omega (\mathrm{div}\, a_\epsilon) u_\epsilon \varphi dx = \int_\Omega f u_\epsilon \varphi dx \to \int_\Omega f u \varphi dx \tag{30}$$

since $f\varphi \in L^\infty(\Omega)$ and $u_\epsilon \to u$ in $L'(\Omega)$. Thus we get

$$\langle a_\epsilon, \nabla u_\epsilon\rangle \to \langle a, \nabla u\rangle \tag{31}$$

in $\mathcal{D}'(\Omega)$, as required.

4 Convergence to the homogenized problem

To prove convergence to the homogenized problem we need the following results:

Theorem 7 *Suppose λ satisfies hypothesis* I) *in the Introduction. Then there exist two positive constants $\delta = \delta(n,p,K)$ and $C = C(n,p,K)$ such that*

$$\begin{aligned} \left(\fint_Q \lambda^{1+\delta} dx \right)^{1/(1+\delta)} &\le C \fint_Q \lambda dx, \\ \left(\fint_Q \lambda^{-(1+\delta)/(p-1)} dx \right)^{1/(1+\delta)} &\le C \fint_Q \lambda^{-1/(p-1)} dx \end{aligned}$$

for every cube Q with faces parallel to the coordinate planes.

Proof. See [3], [8] and [13]. □

Lemma 3 *Let $f \in L^p_{loc}(R^n), 1 \leq p < +\infty$ be Y-periodic. Then, if $f_\epsilon(x) = f(\frac{x}{\epsilon})$,*

$$f_\epsilon \to \int_Y f dy \qquad in\ L^p_{loc}(\mathbb{R}^n) - weak. \tag{32}$$

Proof. See [14] page 5. □

Lemma 4 1) *The functions u_ϵ defined in Theorem* 5, *Problem (P_ϵ), satisfy* (a), (b), (c) *of Theorem* 5 *with $\nu \equiv 1$ and $u = u_0^*$ suitable. Moreover, the functions a_ϵ defined by $a_\epsilon(x) = a(\frac{x}{\epsilon}, \nabla u_\epsilon)$, where u_ϵ is defined in Theorem* 6, *Problem (P_ϵ), satisfy* (e), (f), (g) *of Theorem* 6 *with $\nu \equiv 1, f = f$ and $a = a_0$ suitable;*

2) *Analogously, the functions v_ϵ defined by $v_\epsilon = \epsilon v(\frac{x}{\epsilon}) := \langle \xi, x\rangle + \epsilon\tilde{v}(\frac{x}{\epsilon})$ (where v is defined in Theorem* 4 *as the solution of Problem* (11) *and $\tilde{v}$ is defined in the proof of Theorem* 4) *satisfy* (a), (b) *and* (c) *of Theorem* 6 *with $\nu \equiv 1$ and $u = \langle \xi, \cdot\rangle$. Moreover, the functions a_ϵ defined by $a_\epsilon(x) = a(\frac{x}{\epsilon}, \nabla v_\epsilon)$, where v_ϵ is defined in* 2) *above, satisfy* (e), (f), (g) *of Theorem* 6 *with $\nu \equiv 1, f \equiv 0$ and $a(x) = b(\xi)$;*

3) *If a_ϵ and u_ϵ are defined by one of the choices* 1) *or* 2) *independently (i.e. the choice for instance of u_ϵ as in* 1) *and a_ϵ as in* 2) *is allowed), then $\langle a_\epsilon, \nabla u_\epsilon\rangle \in L^1(\Omega)$ for all $\epsilon \in (0,1)$* ((d) *of Theorem* 6).

Remark 7 In fact, what will be proved below is not precisely 1), but, as for the convergence, a slightly weaker statement, i.e. the convergence of a subsequence to a suitable limit u_0^* that might depend on the subsequence itself. However, it will be proved at the end of the proof of Theorem 5 that u_0^* is a solution of Problem (P_0) that in turn admits a unique solution, by Remark 2 after Theorem 5. Thus, the statement 1) will be fully proved. However, we keep this formulation to avoid cumbersome arguments. Notice the same difficulty does not arise as for the statement 2).

Proof of Lemma 4. Let $K \geq 1$ be such that $\lambda \in A_p(K)$. For $\epsilon > 0$, we put $\lambda_\epsilon(x) = \lambda(\frac{x}{\epsilon})$, $x \in \mathbb{R}^n$; we have $\lambda_\epsilon \in A_p(K), \forall \epsilon > 0$. To prove 1), we want to verify that there exists a positive constant $C_1 = C_1(n, p, K, \Omega, f)$ such that

$$\int_\Omega (|u_\epsilon|^p + |\nabla u_\epsilon|^p)\lambda_\epsilon dx \leq C_1 \quad \forall \epsilon > 0 \tag{33}$$

First of all we notice that, thanks to a Poincaré type inequality (see [10]), we have, for $K \geq 1$ and Ω bounded

$$\int_\Omega |u_\epsilon|^p \lambda_\epsilon dx \leq C \int_\Omega |\nabla u_\epsilon|^p \lambda_\epsilon dx \tag{34}$$

where $C = C(n, p, K, \Omega)$. Hence

$$\int_\Omega (|u_\epsilon|^p + |\nabla u_\epsilon|^p)\lambda_\epsilon dx \le C \int_\Omega |\nabla u_\epsilon|^p \lambda_\epsilon dx. \tag{35}$$

The variational formulation of problem (P_ϵ) gives that

$$\int_\Omega \left\langle \alpha\left(\frac{x}{\epsilon}, \nabla u_\epsilon\right) A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon, \nabla u_\epsilon \right\rangle dx = \int_\Omega f u_\epsilon dx, \tag{36}$$

and hence, by assumption $(S_1), \mathrm{iii})$,

$$\int_\Omega \left\langle A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon, \nabla u_\epsilon \right\rangle^{p/2} dx \le C \int_\Omega |f u_\epsilon| dx. \tag{37}$$

By using (35) and (37) we then have:

$$\begin{aligned}
&\int_\Omega \left\langle A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon, \nabla u_\epsilon \right\rangle^{p/2} dx \le C(n,p,K,\Omega) \int_\Omega |f||u_\epsilon| dx \\
&= C(n,p,K,\Omega) \int_\Omega |f| \lambda_\epsilon^{-1/p} \lambda_\epsilon^{1/p} |u_\epsilon| dx \\
&\le C(n,p,K,\Omega) \left(\int_\Omega |u_\epsilon|^p \lambda_\epsilon dx\right)^{1/p} \left(\int_\Omega |f|^{p'} \lambda_\epsilon^{-1/(p-1)} dx\right)^{1/p'}.
\end{aligned}$$

Clearly, since $f \in L^\infty(\Omega)$,

$$\left(\int_\Omega |f|^{p'} \lambda_\epsilon^{-1/(p-1)} dx\right)^{1/p'} \le \|f\|_{L^\infty} \left(\int_\Omega \lambda_\epsilon^{-1/(p-1)} dx\right)^{1/p'}. \tag{38}$$

To estimate $\int_\Omega \lambda_\epsilon(x)^{-1/(p-1)} dx$ we put $\frac{x}{\epsilon} = t$ and we notice that, being Ω bounded, the dilated cube Ω/ϵ can be covered by a number of unit cubes Q proportional to $(\frac{1}{\epsilon})^n$. We then have

$$\begin{aligned}
\int_\Omega \lambda\left(\frac{x}{\epsilon}\right)^{-1/(p-1)} dx &= \int_{\frac{1}{\epsilon}\Omega} \lambda(t)^{-1/(p-1)} \epsilon^n dt \\
&\le C \left(\frac{1}{\epsilon}\right)^n \epsilon^n \int_Q \lambda(t)^{-1/(p-1)} dt \\
&\le C.
\end{aligned}$$

Thus, by Theorem 1, iv), we can conclude that

$$\int_\Omega \left\langle A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon, \nabla u_\epsilon \right\rangle^{p/2} dx \le C \tag{39}$$

and then (33) follows from (35), (39) and (S_2). This proves (a) and (b). To prove (e) holds we want to verify that there exists a positive constant C_2, independent of ϵ, such that:

$$\int_\Omega \left|\alpha\left(\frac{x}{\epsilon}, \nabla u_\epsilon\right) A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon\right|^{p'} \Lambda_\epsilon^{-1/(p-1)} dx \le C_2 \quad \forall \epsilon > 0. \tag{40}$$

We begin by estimating $|A(\frac{x}{\epsilon})\nabla u_\epsilon|$. By definition we have

$$\begin{aligned}
\left|A(\frac{x}{\epsilon})\nabla u_\epsilon\right| &= \left|\sqrt{A}\left(\frac{x}{\epsilon}\right)\sqrt{A}\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon\right| \\
&\le \left\|\sqrt{A}\left(\frac{x}{\epsilon}\right)\right\| \left\langle A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon, \nabla u_\epsilon\right\rangle^{1/2} \\
&= \sup_{|\xi|,|\eta|\le 1} \left|\left\langle \sqrt{A}\left(\frac{x}{\epsilon}\right)\xi, \eta\right\rangle\right| \left\langle A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon, \nabla u_\epsilon\right\rangle^{1/2} \\
&\le \sup_{|\xi|\le 1} \left|\sqrt{A}\left(\frac{x}{\epsilon}\right)\xi\right| \left\langle A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon, \nabla u_\epsilon\right\rangle^{1/2} \\
&= \sup_{|\xi|\le 1} \left\langle A\left(\frac{x}{\epsilon}\right)\xi, \xi\right\rangle^{1/2} \left\langle A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon, \nabla u_\epsilon\right\rangle^{1/2} \\
&\le \Lambda\left(\frac{x}{\epsilon}\right)^{1/p} \left\langle A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon, \nabla u_\epsilon\right\rangle^{1/2},
\end{aligned}$$

where in the last inequality we have used (S_2). Moreover, since by (S_1), iii) $\alpha\,(\frac{x}{\epsilon}, \nabla u_\epsilon) \le C\langle A(\frac{x}{\epsilon})\nabla u_\epsilon, \nabla u_\epsilon\rangle^{\frac{p}{2}-1}$, we have:

$$\begin{aligned}
&\int_\Omega \left|\alpha\left(\frac{x}{\epsilon}, \nabla u_\epsilon\right) A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon\right|^{p'} \Lambda_\epsilon^{-1/(p-1)} dx \\
&\qquad \le C \int_\Omega \left\langle A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon, \nabla u_\epsilon\right\rangle^{p'(p/2-1)+p'/2} \Lambda^{p'/p-1/(p-1)} dx \\
&\qquad \le \int_\Omega \left\langle A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon, \nabla u_\epsilon\right\rangle^{p/2} dx \\
&\qquad \le C \text{ by (39).}
\end{aligned}$$

To prove the convergence of $(u_\epsilon)_\epsilon$ and $(a_\epsilon)_\epsilon$ let us prove now that there exists $\sigma > 0$ such that $(u_\epsilon)_\epsilon$ is bounded in $\overset{\circ}{W}{}^{1,1+\sigma}(\Omega)$ and $(\alpha(\frac{x}{\epsilon}, \nabla u_\epsilon)A(\frac{x}{\epsilon})\nabla u_\epsilon)_\epsilon$ is bounded in $(L^{1+\sigma}(\Omega))^n$, uniformly with respect to $\epsilon \in (0,1)$. Indeed (keep in mind Theorem 1, ii)), by Hölder inequality we get

$$\begin{aligned}\int_\Omega (|u_\epsilon| + |\nabla u_\epsilon|)^{1+\sigma} dx \\ &\le \left(\int_\Omega (|u_\epsilon| + |\nabla u_\epsilon|)^p \lambda_\epsilon dx\right)^{\frac{1+\sigma}{p}} \left(\int_\Omega \lambda_\epsilon^{-\frac{1+\sigma}{p-(1+\sigma)}} dx\right)^{1-\frac{1+\sigma}{p}} \\ &\le C\left(\int_\Omega \lambda_\epsilon^{-\frac{1+\sigma}{p-(1+\sigma)}} dx\right)^{1-\frac{1+\sigma}{p}}\end{aligned}$$

by (33). We can choose $\sigma > 0$ such that $\frac{1+\sigma}{p-(1+\sigma)} = \frac{1+\delta}{p-1}$, where δ is defined in Theorem 7. Now, putting $\xi = \frac{x}{\epsilon}$,

$$\begin{aligned}\int_\Omega \lambda_\epsilon^{-\frac{1+\delta}{p-1}}(x) dx &= \epsilon^n \int_{\frac{1}{\epsilon}\Omega} \lambda^{-\frac{1+\delta}{p-1}}(\xi) d\xi \\ &\le \int_Y \lambda^{-\frac{1+\delta}{p-1}}(\xi) d\xi,\end{aligned}$$

by periodicity since $\frac{1}{\epsilon}\Omega$ can be covered by N copies of Y, with $N \le C(\frac{1}{\epsilon})^n$. Analogously, by (39)

$$\int_\Omega \left|\alpha\left(\frac{x}{\epsilon}, \nabla u_\epsilon\right) A\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon\right|^{1+\sigma} dx \le C\left(\int_\Omega \Lambda_\epsilon^{\frac{1+\sigma}{(p-1)(p'-1-\sigma)}} dx\right)^{1-\frac{1+\sigma}{p'}}.$$

Without loss of generality we can assume $\frac{1+\sigma}{(p'-1-\sigma)} = 1 + \mu$ (see condition II)) and then we can conclude as above. By reflexivity, it is then possible to find $u_0^* \in \overset{\circ}{W}{}^{1,1+\sigma}(\Omega)$ such that $u_\epsilon \to u_0^*$ in $\overset{\circ}{W}{}^{1,1+\sigma}(\Omega)$ – weak, and hence

$$u_\epsilon \to u_0^* \quad \text{in} \quad L^{1+\sigma}(\Omega), \tag{41}$$

and $a_0 \in (L^{1+\sigma}(\Omega))^n$ such that

$$\alpha\left(\frac{x}{\epsilon}, \nabla u_\epsilon\right) A\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon \to a_0 \text{ in } \in (L^{1+\sigma}(\Omega))^n - \text{weak}. \tag{42}$$

Thus u_ϵ satisfy (a), (b) (by (33) and (c) (by (41)). Moreover a_ϵ satisfy (e) (by 40), (f) by definition and (g) (by (42)). Let us prove now 2).

To prove that (a) holds, let us show for instance that $I = \int_\Omega |\sqrt{A}(\frac{x}{\epsilon}) \nabla v_\epsilon (x)|^p dx \le C < +\infty$ (remember $v_\epsilon \in W^{1,1}_{loc}(\mathbb{R}^n)$, and hence its gradient belongs to L^1_{loc}). Indeed, since $v(y) = \langle \xi, y\rangle + \tilde{v}(y)$, with $\tilde{v}(y) \in W^{1,p}_{A,\#}(Y)$, we have

$$\begin{aligned} I &= \epsilon^n \int_{\frac{1}{\epsilon}\Omega} |\sqrt{A}(y)\nabla v(y)|^p dy \\ &\le C \int_Y |\sqrt{A}(y)\nabla v(y)|^p dy \\ &\le C_p \left(\int_Y |\sqrt{A}(y)\xi|^p dy + \int_Y |\sqrt{A}(y)\nabla \tilde{v}(y)|^p dy \right) \\ &\le C_p|\xi|^p \int_Y \Lambda(y)dy + \|\tilde{v}\|_{W^{1,p}_A(Y)} = C_{\tilde{v}}, \end{aligned}$$

by definition of $W^{1,p}_{A,\#}$. An analogous estimate holds for $J = \int_\Omega |v_\epsilon(x)|^p \lambda_\epsilon(x)dx$. Since these estimates do not depend on ϵ, (b) follows. By Lemma 3, $v_\epsilon \to \langle \xi, \cdot\rangle$ weakly in $W^{1,1}(\Omega)$, and then, by Rellich's theorem, $v_\epsilon \to \langle \xi, \cdot\rangle$ in $L^1(\Omega)$. Thus, (c) is proved. To prove (e), it is enough to repeat the argument yielding (40) and replacing (39) by the estimates of I and J above. Assertion (f) (with $f \equiv 0$) can be proved arguing as in Lemma 1.6 of [9]. Eventually, (g) follows from the very definition of $b(\xi)$ and Lemma 3. As for 3), by (a) both $\langle a(\frac{x}{\epsilon}, \nabla u_\epsilon), \nabla u_\epsilon\rangle$ and $\langle a(\frac{x}{\epsilon}, \nabla v_\epsilon), \nabla v_\epsilon\rangle \in L^1(\Omega)$. Moreover (for instance)

$$\begin{aligned} &\int_\Omega \left\langle a\left(\frac{x}{\epsilon}, \nabla u_\epsilon\right), \nabla v_\epsilon \right\rangle dx \\ &\le C \int_\Omega |\sqrt{A}\nabla u_\epsilon|^{p-2} \langle A\nabla u_\epsilon, \nabla v_\epsilon\rangle dx \\ &\le C \int_\Omega |\sqrt{A}\nabla u_\epsilon|^{p-1} |\sqrt{A}\nabla v_\epsilon| dx \\ &\le C \left(\int_\Omega |\sqrt{A}\nabla u_\epsilon|^p dx \right)^{1/p} \left(\int_\Omega |\sqrt{A}\nabla v_\epsilon|^p dx \right)^{1/p} < +\infty \end{aligned}$$

Proof of Theorem 5. To get the thesis we need only, with the notations of Lemma 4, to prove that

$$u_0^* \in \overset{\circ}{W}{}^{1,p}(\Omega), \qquad a_0(x) = b(\nabla u_0^*(x)) \quad \text{a.e. in } \Omega. \tag{43}$$

Indeed, suppose (43) holds; then a limit argument shows that u_0^* is a variational solution of (P_0), and then, by uniqueness (Remark 2), it follows that

$$u = u_0^* \quad \text{a.e. in } \Omega.$$

Let us prove now that (43) holds. To prove that $u_0^* \in \overset{\circ}{W}{}^{1,p}(\Omega)$ it is sufficient to prove that

$$\nabla u_0^* \in (L^p(\Omega))^n \tag{44}$$

since Ω is a regular bounded open set. For any $\varphi \in C_0^0(\Omega)$, using Hölder inequality, for $j = 1, \ldots, n$ we have:

$$\begin{aligned} \left| \int_\Omega \partial_j u_\epsilon \varphi dx \right| &\leq \int_\Omega |\partial_j u_\epsilon| \lambda_\epsilon^{1/p} \lambda_\epsilon^{-1/p} |\varphi| dx \\ &\leq \left(\int_\Omega |\partial_j u_\epsilon|^p \lambda_\epsilon dx \right)^{1/p} \left(\int_\Omega \lambda_\epsilon^{-1/(p-1)} |\varphi|^{p'} dx \right)^{1/p'} \\ &\leq C_1^{1/p} \left(\int_\Omega \lambda_\epsilon^{-1/(p-1)} |\varphi|^{p'} dx \right)^{1/p'} \end{aligned} \tag{45}$$

where in the last inequality we have used (33). Since by Lemma 3 we have

$$\int_\Omega \lambda_\epsilon^{-1/(p-1)} |\varphi|^{p'} dx \to \int_Y \lambda^{-1/(p-1)} dy \int_\Omega |\varphi|^{p'} dx, \tag{46}$$

defining $C_3 = C_1^{1/p} (\int_\Omega \lambda^{-1/(p-1)} dy)^{1/p'}$ and taking the weak limit in (45) we obtain:

$$\left| \int_\Omega \partial_j u_0^* \varphi dx \right| \leq C_3 \|\varphi\|_{L^{p'}(\Omega)} \quad \forall \varphi \in C_0^0(\Omega), \tag{47}$$

that implies (44). We prove now that $a_0(x) = b(\nabla u_0^*(x))$ a.e. in Ω. Using inequality(40) we have, for any $\varphi \in (C_0^0(\Omega))^n$:

$$\left| \int_\Omega \alpha\left(\frac{x}{\epsilon}, \nabla u_\epsilon\right) \left\langle A\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon, \varphi \right\rangle dx \right| \tag{48}$$

$$\leq \int_\Omega \left| \alpha\left(\frac{x}{\epsilon}, \nabla u_\epsilon\right) A\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon \right| \Lambda_\epsilon^{-1/p} \Lambda_\epsilon^{1/p} |\varphi| dx$$

$$\leq \left(\int_\Omega \left| \alpha\left(\frac{x}{\epsilon}, \nabla u_\epsilon\right) A\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon \right|^{p'} \Lambda_\epsilon^{-1/(p-1)} dx \right)^{1/p'} \left(\int_\Omega \Lambda_\epsilon |\varphi|^p dx \right)^{1/p} \tag{49}$$

$$\leq \left(\int_\Omega \Lambda_\epsilon |\varphi|^p dx \right)^{1/p}, \tag{50}$$

and arguing as in (46) and (47) we obtain that

$$a_0 \in (L^{p'}(\Omega))^n. \tag{51}$$

Let $\xi \in \mathbb{R}^n$, and let v be the solution of problem (11) and set $v_\epsilon = \epsilon v(x/\epsilon)$. For every $\varphi \in \mathcal{D}(\Omega)$, with $\varphi \geq 0$ in Ω, by the monotonicity of $a(y, \xi) = \alpha(y, \xi) A(y) \xi$ (property (S_3)) it follows that

$$0 \leq \int_\Omega \langle a(x/\epsilon, \nabla u_\epsilon) - a(x/\epsilon, \nabla v_\epsilon), \nabla u_\epsilon - \nabla v_\epsilon \rangle \varphi dx. \tag{52}$$

Now by the compensated compactness theorem and Lemma 4 we can pass to the limit in inequality (52) and we finally obtain:

$$0 \leq \int_{\Omega} \langle a_0(x) - b(\xi), \nabla u_0^*(x) - \xi \rangle \varphi dx, \forall \xi \in \mathbb{R}^n, \forall \varphi \in \mathcal{D}(\Omega), \varphi \geq 0. \tag{53}$$

From the last inequality we deduce that there exists a subset N of Ω, of measure zero, such that

$$\langle a_0(x) - b(\xi), \nabla u_0^*(x) - \xi \rangle \geq 0 \quad \forall x \in \Omega \backslash N, \forall \xi \in Q^n. \tag{54}$$

Then by the continuity of b (Proposition 1, iv) we have that

$$\langle a_0(x) - b(\xi), \nabla u_0^*(x) - \xi \rangle \geq 0 \quad \forall x \in \Omega \backslash N, \forall \xi \in \mathbb{R}^n. \tag{55}$$

Now, for a.e. $x \in \Omega, \eta \in \mathbb{R}^n$ and $t > 0$ we set $\xi = \nabla u_0^*(x) - t\eta$. Writing (55) with this ξ an letting $t \to 0$ we obtain (using again the continuity of b)

$$\langle a_0(x) - b(\nabla u_0^*(x)), \eta \rangle \geq 0, \text{ for a.e. } x \in \Omega \backslash N, \forall \eta \in \mathbb{R}^n. \tag{56}$$

from which (43) follows.

Acknowledgements It is a pleasure to thank Matteo Giovanditti for many stimulating suggestions. We thank also Francesco Serra Cassano for many kind fruitful discussions. Finally, the authors thank the referee for his/her careful and helpful comments and suggestions.

References

[1] A. BENSOUSSAN, J.L. LIONS, G. PAPANICOLAOU, Asymptotic Analysis for Periodic Structures, North-Holland, Amsterdam, 1978.

[2] A. BRAIDES, A. DEFRANCESCHI, Homogenization of Multiple Integrals Clarendon Press, Oxford, 1998.

[3] R.R. COIFMAN, C. FEFFERMAN, Weighted Norm Inequalities for Maximal Function and Singular Integrals, *Studia Math.* **51** (1974), 241–250.

[4] S. CHANILLO, R.L. WHEEDEN, Weighted Poincaré and Sobolev inequalities and estimates for weighted Peano maximal functions, *Amer. J. Math.* **107** (1985), 1191–1226.

[5] S. CHANILLO, R.L. WHEEDEN, L^p estimates for fractional integrals and Sobolev inequalities with applications to Schrödinger operators, *Comm. Partial Differential Equations* **10** (1985), 1077–1116.

[6] V. CHIADÒ PIAT, F. SERRA CASSANO, Relaxation of degenerate variational integrals, *Nonlinear Anal.* **22** (1994), 409–424.

[7] D. CIORANESCU, P. DONATO, An Introduction to Homogenization, Oxford Univerty Press, Oxford, 1999.

[8] R. DE ARCANGELIS, Compactness and Convergence of Minimum Points for a Class of non Linear non Equicoercive Functionals, *Nonlinear Anal.* **15** (1990), 363–380.

[9] R. DE ARCANGELIS, F. SERRA CASSANO, On the homogenization of degenerate elliptic equations in divergence form, *J. Math. Pures Appl.* **71** (1992), 119–138.

[10] E.B. FABES, C.E. KENIG, R. SERAPIONI, The local regularity of solutions of degenerate elliptic equations, *Comm. Partial Differential Equations* **7** (1982), 77–116.

[11] B. FRANCHI, G. LU, R.L. WHEEDEN, Representation formulas and weighted Poincaré inequalities for Hörmander vector fields, *Ann. Inst. Fourier (Grenoble)* **45** (1995), 577–604.

[12] B. FRANCHI, R. SERAPIONI, F. SERRA CASSANO, Approximation and Imbedding Theorems for Weighted Sobolev Spaces Associated with Lipschitz Continuous Vector Fields, *Boll. Un. Mat. Ital.* (7) **11–B** (1997), 83–117.

[13] J. GARCIA CUERVA, J.L. RUBIO DE FRANCIA, Weighted norm inequalities and related topics, *North-Holland Math. Studies*, no. 116, North-Holland, 1985.

[14] V.V. JIKOV, S.M. KOZLOV, O.A. OLEINIK, Homogenization of Differential Operators and Integral Functionals, Springer-Verlag, 1991.

[15] D. KINDERLEHRER, G. STAMPACCHIA, An Introduction to Variational Inequalities and their Applications, Academic Press, 1980.

[16] P. KOSKELA, P. MACMANUS, Quasiconformal mappings and Sobolev spaces, *Studia Math.* **131** (1998), 1–17.

[17] F. MURAT, Compacité par compensation, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **5** (1978), 489–507.

[18] E. SANCHEZ-PALENCIA, Nonhomogeneous Media and Vibration Theory, *Lecture Notes in Physics 127*, Springer-Verlag, New York, 1980.

[19] S. SEMMES, Finding curves on general spaces through quantitative topology with applications to Sobolev and Poincaré inequalities, *Selecta Math. (N.S.)* **2** (1996), 155–295.

[20] L. TARTAR, Compensated Compactness and Applications to Partial Differential Equations, *Nonlinear Analysis and Mechanics*, Heriott-Watt Symposium, Pitman, London, 1979.

[21] R.L. WHEEDEN, A characterization of some weighted norm inequalities for the fractional maximal function, *Studia Math.* **107** (1993), 257–272.

Received December 1999