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Two-scale homogenization in the Heisenberg group ☆

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Abstract

In this paper, we consider a problem of periodic homogenization in the context of the Heisenberg group $\mathbb{H}^n$ that is the simplest noncommutative example of nilpotent stratified connected and simply connected Lie group, when the periodicity is defined through group translations and intrinsic anistropic dilations. In particular, we consider a Dirichlet problem for a generalized Kohn Laplacian operator with strongly oscillating (Heisenberg-)periodic coefficients in a domain that is perforated by interconnected (Heisenberg-)periodic pipes. Convergence to the homogenized problem is obtained by a two-scale method adapted to the geometry of the group with dilations.

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1. Introduction

The two-scale convergence method of Nguetseng [18] and Allaire [1,2] has proved to be very powerful in the framework of periodic homogenization. The method introduces a new notion of convergence, the "two-scale" convergence, which in particular implies weak convergence. The method has been successfully applied in several situations, always in a (periodic) Euclidean setting, including the homogenization of linear and nonlinear second order elliptic equations and the homogenization of nonlinear operators (see [1]).

It could be interesting to investigate the applicability of this method to a more general periodic context, where the periodicity has to be meant with respect to a class of non-Euclidean translations and nonisotropic dilations. Indeed, the notion of two-scale convergence relies basically on the fact that derivatives commute with translations and scale appropriately with respect to Euclidean homotheties. Thus, it is natural to imagine that such a procedure can be implemented in the more general setting of stratified nilpotent Lie groups with dilations (the so-called Carnot groups, see, e.g., [19]). The simplest but nevertheless significant example of noncommutative Carnot group is provided by the Heisenberg group $\mathbb{H}^n \simeq \mathbb{C}^n \times \mathbb{R} \simeq \mathbb{R}^{2n+1}$ endowed with the group multiplication $[z,t]\cdot[\zeta,\tau] = [z+\zeta, t+\tau+2\Im m(z\bar{\zeta})]$, $z,\zeta \in \mathbb{C}$, $t,\tau \in \mathbb{R}$ and the family of anisotropic dilations δ_λ, $\lambda \in \mathbb{R}$, given by $\delta_\lambda[z,t] = [\lambda z, \lambda^2 t]$. In this case the left invariant operators are $X_j = \partial_{x_j} + 2y_j\partial_t$ and $Y_j = \partial_{y_j} - 2x_j\partial_t$, $j = 1,\dots,n$, and the associated second order model operator is the Kohn Laplacian $\Delta_{\mathbb{H}} = \sum_{j=1}^n (X_j^2 + Y_j^2)$. This operator, that will play the role for our geometry of the Laplace operator, is not elliptic at any point, since the lowest eigenvalue of its principal quadratic form vanishes identically in $\mathbb{R}^{2n+1}$. However, we want to stress that we are not dealing here with a Riemannian geometry and associated Laplace–Beltrami operator. In fact, because of the noncommutativity of the group multiplication, the geometry of the Heisenberg group is not Euclidean and not even Riemannian at any scale (see, e.g., [10,20]).

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We point out that the two-scale procedure in the context of the Heisenberg group is not reduced to the n-scale version of the usual two-scale method (see [1], Corollary 1.16), since such a method fits periodicity with respect to Euclidean translations, whereas we are here dealing with periodicity with respect to the group translations.

Deeply related to the structure of $\mathbb{H}^n$ is the possibility to construct a periodic pavage in $\mathbb{H}^n$ by using group translations and homotheties (see [3]). In this way one is enabled to develop a periodic homogenization theory, analogously to the classical Euclidean setting.

There are several papers on homogenization in the Heisenberg group, but as far as we know this is the first time the two-scale method is generalized to the Heisenberg context.

The first paper concerning homogenization in the Heisenberg setting is due to Biroli, Mosco and Tchou [4]. In this paper, the author construct explicitly a periodic pavage associated with the operator $\Delta_{\mathbb{H}}$ and they study the asymptotic behaviour of its eigenfunctions in a domain with isolated Heisenberg periodic holes with Dirichlet boundary conditions on their boundaries. To show convergence to the homogenized problem they use Tartar's energy method.

In a subsequent paper, see [5], Biroli, Tchou and Zhikov studied the same problem, i.e., homogenization in a domain with holes periodically distributed with respect to the group, with Neumann boundary conditions on the holes. In this case, the method used in [4], essentially based on an extension of the solutions of approximating problems in the holes, does not work due to a lack of regularity on the boundary of the holes. In [5], to treat the problem the authors generalized to the Heisenberg group a method which is independent from the extension property, introduced by Zhikov [23] in an Euclidean setting to deal with an homogenization problem for periodic measures.

None of the methods mentioned so far seems to work when one is interested in treating the case of not necessarily isolated holes. Indeed an interesting case to examine is the one of periodic holes which may be not isolated (for example in $\mathbb{R}^3$ one can think of a domain perforated by interconnected pipes). This type of problems is quite common in physics or mechanics, think for example to the convection–diffusion of a liquid in a porous medium or to the viscoplasticity problem for a perforated material.

A successful method to prove convergence to the homogenized problem in this context, in an Euclidean framework, has turned out to be the two-scale method. The aim of the present paper is to verify the validity of the method in the same context in the framework of the Heisenberg group.

Let us conclude by mentioning few difficulties encountered in the present paper due to the noncommutative group structure.

The first one appears when dealing with interconnected holes (think for instance of a net of pipes): this problem is discussed in detail at the beginning of Section 3. Basically, because of the distortion of the microscopic cells generated by the group action, the pipes must be adequately positioned and "twisted" in order to produce a periodic net.

A second technical difficulty arises when we want to generalize the well known result holding that a function orthogonal to all divergence-free vector fields is a gradient. The classical proof [12,22] relies on continuation properties of distributions in a bounded regular domain. Since continuation properties associated with noncommutative vector fields are quite different from the Euclidean case (for instance, even smoothness does not guarantee a positive result because of the presence of characteristic points on the boundary), we use a different approach based on Poincaré inequality, that is a more geometrical condition.

2. Notations and preliminaries

Few notations and geometric preliminaries are in order to state in a simpler way our results. We follow the notations of [20], as well as the ones of [10].

In this paper, we indicate by $\mathbb{H}^n$ the n-dimensional Heisenberg group $\mathbb{H}^n \simeq \mathbb{C}^n \times \mathbb{R} \simeq \mathbb{R}^{2n+1}$. The points in $\mathbb{H}^n$ will be denoted as $p = [z,t] = [x + \mathrm{i}y, t] \simeq [p_1, p_2, \ldots, p_{2n}, p_{2n+1}]$. If $p = [z,t]$, $q = [\zeta, \tau] \in \mathbb{H}^n$ and $r > 0$, following the notations of [21], we define the group operation

$$p \cdot q := \left[z + \zeta, t + \tau + 2\Im m\left(z\bar{\zeta}\right)\right]$$

and the family of nonisotropic dilations δ_r,

$$\delta_r(p) := \left[rz, r^2 t\right].$$

It is also useful to consider the group translations $\tau_p : \mathbb{H}^n \to \mathbb{H}^n$ defined as $q \mapsto \tau_p(q) := p \cdot q$ for any fixed $p \in \mathbb{H}^n$. We denote as $p^{-1} := [-z, -t]$ the inverse of p and as 0 the origin of $\mathbb{R}^{2n+1}$. We shall endow $\mathbb{H}^n$ with the homogeneous (with respect to nonisotropic dilations) norm $\|p\|_\infty := \max\{|z|, |t|^{1/2}\}$ and with the distance, associated with the norm,

$$d(p,q) := \left\|p^{-1} \cdot q\right\|_\infty. \tag{1}$$

We explicitly observe that d is a distance in $\mathbb{H}^n$ (see [10]).

Proposition 2.1. *The function d defined by* (1) *is a distance in $\mathbb{H}^n$ and the usual properties related to translations and dilations hold, i.e., for any $p, q, q' \in \mathbb{H}^n$ and for any $r > 0$*

$$d(\tau_p q, \tau_p q') = d(q, q') \quad \text{and} \quad d(\delta_r q, \delta_r q') = r d(q, q'). \tag{2}$$

In addition, for any bounded subset Ω of $\mathbb{H}^n$ there exist positive constants $c_1(\Omega)$, $c_2(\Omega)$ such that

$$c_1(\Omega)|p-q|_{\mathbb{R}^{2n+1}} \leqslant d(p,q) \leqslant c_2(\Omega)|p-q|^{1/2}_{\mathbb{R}^{2n+1}} \quad \text{for } p, q \in \Omega.$$

In particular, the topologies defined by d and by the Euclidean distance coincide on $\mathbb{H}^n$.

From now on, $U(p,r)$ will be the open ball with center p and radius r with respect to the distance d. We notice explicitly that $U(p,r)$ is an Euclidean Lipschitz domain in $\mathbb{R}^{2n+1}$ (see [10]).

There is a natural measure $\mathrm{d}h$ on $\mathbb{H}^n$ which is given by the Lebesgue measure $\mathrm{d}\mathcal{L}^{2n+1} = \mathrm{d}z\,\mathrm{d}t$ on $\mathbb{C}^n \times \mathbb{R}$. The measure $\mathrm{d}h$ is left (and right) invariant and it is the Haar measure of the group. If $E \subset \mathbb{H}^n$ then $|E|$ is its Lebesgue measure and when $|E| < \infty$, $f \in L^1(E)$, $⨍_E f\,\mathrm{d}h$ will denote the average of f over E, i.e., $⨍_E f\,\mathrm{d}h = (1/|E|)\int_E f\,\mathrm{d}h$. We stress that $|\delta_\lambda(E)| = \lambda^{2n+2}|E|$ for any measurable set $E \subseteq \mathbb{H}^n$. In addition, if $s \geqslant 0$, we shall denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measure obtained starting from Euclidean balls, whereas $\mathcal{H}^s_{\mathbb{H}}$ will stand for the Hausdorff measure obtained from the distance d in (1).

The Lie algebra of the left invariant vector fields of $\mathbb{H}^n$ is given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}, \qquad j = 1, \dots, n, \qquad T = \frac{\partial}{\partial t},$$

and the only nontrivial commutator relations are

$$[X_j, Y_j] = -4T, \quad j = 1, \dots, n.$$

In the following, we shall identify vector fields and the associated first order differential operators. Notice that, if $\lambda > 0$, then $X_j(u(\delta_\lambda(p))) = \lambda(X_j u)(\delta_\lambda(p))$, $j = 1, \dots, n$, and the analogous statement holds for Y_j. The vector fields $X_1, \dots, X_n$, $Y_1, \dots, Y_n$ define a vector bundle over $\mathbb{H}^n$ (the *horizontal vector bundle* $H\mathbb{H}^n$) that can be canonically identified with a vector subbundle of the tangent vector bundle of $\mathbb{R}^{2n+1}$. Since each fiber of $H\mathbb{H}^n$ can be canonically identified with a vector subspace of $\mathbb{R}^{2n+1}$, each section ϕ of $H\mathbb{H}^n$ can be identified with a map $\phi : \mathbb{H}^n \to \mathbb{R}^{2n+1}$. At each point $p \in \mathbb{H}^n$ the horizontal fiber is indicated as $H\mathbb{H}^n_p$ and each fiber can be endowed with the scalar product $\langle \cdot, \cdot \rangle_p$ and the norm $|\cdot|_p$ that make the vector fields $X_1, \dots, X_n, Y_1, \dots, Y_n$ orthonormal. Hence we shall also identify a section of $H\mathbb{H}^n$ with its canonical coordinates with respect to this moving frame. In this way, a section ϕ will be identified with a function $\phi = (\phi_1, \dots, \phi_{2n}) : \mathbb{H}^n \to \mathbb{R}^{2n}$. As it is common in Riemannian geometry, when dealing with two sections ϕ and ψ whose argument is not explicitly written, we shall drop the index p in the scalar product writing $\langle \psi, \phi \rangle$ for $\langle \psi(p), \phi(p) \rangle_p$. The same convention shall be adopted for the norm.

If Ω is an open subset of $\mathbb{H}^n$ and $k \geqslant 0$ is a nonnegative integer, the symbols $\mathbb{C}^k(\Omega)$, $\mathbb{C}^\infty(\Omega)$ indicate the usual (Euclidean) spaces of real valued continuously differentiable functions. We denote by

$$\mathbb{C}^k(\Omega; H\mathbb{H}^n), \quad k \geqslant 0,$$

the set of all $\mathbb{C}^k$-sections of $H\mathbb{H}^n$ where the $\mathbb{C}^k$ regularity is understood as regularity between smooth manifolds. The notions of $\mathbb{C}^k_0(\Omega; H\mathbb{H}^n)$, $\mathbb{C}^\infty(\Omega; H\mathbb{H}^n)$ and $\mathbb{C}^\infty_0(\Omega; H\mathbb{H}^n)$ are defined analogously.

To stress the similarity among some statements in $\mathbb{H}^n$ with others in $\mathbb{R}^{2n+1}$ it is useful to use intrinsic notions of gradient for functions $\mathbb{H}^n \to \mathbb{R}$ and of divergence for sections of $H\mathbb{H}^n$.

Definition 2.2. If Ω is an open subset of $\mathbb{H}^n$, $f \in \mathbb{C}^1(\Omega)$ and $\phi = (\phi_1, \dots, \phi_{2n}) \in \mathbb{C}^1(\Omega; H\mathbb{H}^n)$ is a continuously differentiable section of $H\mathbb{H}^n$, we define

$$\nabla_{\mathbb{H}} f := (X_1 f, \dots, X_n f, Y_1 f, \dots, Y_n f) \tag{3}$$

and

$$\operatorname{div}_{\mathbb{H}} \phi = \sum_{j=1}^n (X_j\phi_j + Y_j\phi_{n+j}) \tag{4}$$

(notice that $X_j^* = -X_j$, $Y_j^* = -Y_j$, $j = 1, \dots, n$).

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Notice that both $\nabla_{\mathbb{H}}$ and $\mathrm{div}_{\mathbb{H}}$ are left invariant differential operators. Alternatively, $\nabla_{\mathbb{H}} f$ can be defined as a section of $\mathrm{H}\mathbb{H}^n$ as

$$\nabla_{\mathbb{H}} f = \sum_{j=1}^{n} \big((X_j f)X_j + (Y_j f)Y_j\big),$$

where the canonical coordinates of this section are $(X_1 f, \ldots, X_n f, Y_1 f, \ldots, Y_n f)$. This is consistent with the identification we mentioned of sections and their coordinates.

Finally, through this paper we shall use the following notation: $C = C(p)$ is the $2n \times (2n+1)$ matrix whose rows are the components of the vectors $X_1, \ldots, X_n, Y_1, \ldots, Y_n$, that is

$$C(p) := \begin{bmatrix} X_1(p) \\ \vdots \\ X_n(p) \\ Y_1(p) \\ \vdots \\ Y_n(p) \end{bmatrix} = \begin{bmatrix} 1 & \ldots & 0 & 0 & \ldots & 0 & 2y_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 1 & 0 & \ldots & 0 & 2y_n \\ 0 & \ldots & 0 & 1 & \ldots & 0 & -2x_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 1 & -2x_n \end{bmatrix}. \tag{5}$$

We point out that $\nabla_{\mathbb{H}} = C\nabla$, where ∇ is the Euclidean gradient in $\mathbb{R}^{2n+1}$, and that the identification of a vector $\xi = (\xi_1, \ldots, \xi_{2n}) \in \mathrm{H}\mathbb{H}^n_p$ with a vector $\tilde{\xi} \in \mathbb{R}^{2n+1}$ through the corresponding embedding of the horizontal fibers can be expressed by using C since $\tilde{\xi} = {}^t C(p)\xi$. Analogously, if $\varphi = (\varphi_1, \ldots, \varphi_{2n}) \in \mathbb{C}^1(\Omega; \mathrm{H}\mathbb{H}^n)$, then $\mathrm{div}_{\mathbb{H}}\, \varphi = \mathrm{div}({}^t C\varphi)$, where div is the Euclidean divergence in $\mathbb{R}^{2n+1}$.

If $\Sigma \subseteq \mathbb{H}^n$ is a regular $2n$-dimensional manifold, then we can define an intrinsic normal vector field $n_{\mathbb{H}}$ that is the section of $\mathrm{H}\mathbb{H}^n$ defined as follows: if n is an Euclidean normal unit vector field to Σ, we define $n_{\mathbb{H}}$ by means of its canonical coordinates as $n_{\mathbb{H}} = Cn$, or, alternatively, by means of the identification of $\mathbb{H}^n$ with $\mathbb{R}^{2n+1}$, as

$$n_{\mathbb{H}} = \sum_{k=1}^{2n} \langle Z_k, n\rangle_{\mathbb{R}^{2n+1}} Z_k,$$

where $Z_1 = X_1, \ldots, Z_n = X_n$, $Z_{n+1} = Y_1, \ldots, Z_{2n} = Y_n$. In fact this notion arises in much more general situations (see, e.g., [10]) and we stress here it is coherent with the divergence theorem, in the sense that, if Ω is a regular bounded open set in $\mathbb{R}^{2n+1}$ and $\phi \in \mathbb{C}^1(\overline{\Omega}; \mathrm{H}\mathbb{H}^n)$, then

$$\int_{\Omega} \mathrm{div}_{\mathbb{H}}\, \phi \,\mathrm{d}h = \int_{\partial\Omega} \langle \phi, n_{\mathbb{H}}\rangle_p \,\mathrm{d}\sigma := \int_{\partial\Omega} \big\langle \phi(p), n_{\mathbb{H}}(p)\big\rangle_p \,\mathrm{d}\sigma$$

(again, see [10]).

Through this paper, we shall denote by Y the cube $[-1,1)^{2n+1}$, by Y_0 the interior of Y. We shall say that a set $A \subseteq \mathbb{H}^n$ is $(\mathbb{H}, Y)$-periodic if for all $p \in A$ we have $\tau_{2k}(p) \in A$ for any $k = (k_1, \ldots, k_{2n+1}) \in \mathbb{Z}^{2n+1}$. Notice that for any set $B \subseteq Y$ we can define a $(\mathbb{H}, Y)$-periodic set $B_\#$ such that $B_\# \cap Y = B$. Finally, if $A \subseteq \mathbb{H}^n$ is $(\mathbb{H}, Y)$-periodic, and f is a function defined in A, we shall say that f is $(\mathbb{H}, Y)$-periodic if, for any $k \in \mathbb{Z}^{2n+1}$,

$$f\big(\tau_{2k}(p)\big) = f(p), \quad \forall p \in A. \tag{6}$$

Moreover, we shall say that a section φ of $\mathrm{H}\mathbb{H}^n$ defined on a $(\mathbb{H}, Y)$-periodic set A is $(\mathbb{H}, Y)$-periodic if for any $k \in \mathbb{Z}^{2n+1}$ and any $p \in A$, we have

$$d\tau_{2k}(p)\big(\varphi(p)\big) = \varphi\big(\tau_{2k}(p)\big), \tag{7}$$

i.e., if the canonical coordinates of φ are $(\mathbb{H}, Y)$-periodic, since $d\tau_{2k}(p)$ maps $\mathrm{H}\mathbb{H}^n_p$ onto $\mathrm{H}\mathbb{H}^n_{\tau_{2k}(p)}$ preserving the canonical coordinates.

As it is proved in [3,4], there is a canonical $(\mathbb{H}, Y)$-periodic *pavage* of $\mathbb{H}^n$ associated with the structure of $\mathbb{H}^n$ as a group with dilations, defined as follows.

Definition 2.3. Let $\epsilon > 0$ be fixed. Then the family of subsets of $\mathbb{H}^n$ obtained by taking

$$\delta_\epsilon(2k \cdot Y) = \delta_\epsilon(2k) \cdot \delta_\epsilon(Y), \quad k \in \mathbb{Z}^{2n+1} \tag{8}$$

is a *pavage* of $\mathbb{H}^n$, i.e.,

(i) $\delta_\epsilon(2k \cdot Y) \cap \delta_\epsilon(2h \cdot Y) = \emptyset$ if $k \neq h$;
(ii) $\mathbb{H}^n = \bigcup_k \delta_\epsilon(2k \cdot Y)$.

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In the rest of this paper we shall use several functions spaces. Let us list their definitions to avoid misunderstandings.

Definition 2.4. We define the following (nonperiodic) function spaces: let Ω be an open subset of $\mathbb{H}^n$, then:

- $W^{1,2}_{\mathbb{H}}(\Omega)$ denotes the set of functions $f \in L^2(\Omega)$ such that $X_j f$, $Y_j f$ belong to $L^2(\Omega)$ for $j = 1, \dots, n$, endowed with its natural norm. The space $W^{1,2}_{\mathbb{H},\mathrm{loc}}(\Omega)$ is defined in the standard way.
- $\mathring{W}^{1,2}_{\mathbb{H}}(\Omega)$ is the closure of $\mathbb{C}^\infty_0(\Omega)$ in $W^{1,2}_{\mathbb{H}}(\Omega)$. Such a definition is natural, since $\mathbb{C}^\infty(\Omega) \cap W^{1,2}_{\mathbb{H}}(\Omega)$ is dense in $W^{1,2}_{\mathbb{H}}(\Omega)$.
- $L^2(\Omega; H\mathbb{H}^n)$ denotes the space of all measurable sections $\phi = (\phi_1, \dots, \phi_{2n})$ of $H\mathbb{H}^n$ such that $\phi \in (L^2(\Omega))^{2n}$.

We then define the following periodic real-valued function spaces: let $A \subseteq \mathbb{H}^n$ be a $(\mathbb{H}, Y)$-periodic open subset of $\mathbb{H}^n$, then:

- $\mathbb{C}^\infty_{\#,\mathbb{H}}(Y, A)$ denotes the space of all $(\mathbb{H}, Y)$-periodic smooth real functions f defined on A. The spaces $\mathbb{C}^k_{\#,\mathbb{H}}(Y, A)$, $k \geqslant 0$, are defined analogously. When the domain of definition of the functions is the whole $\mathbb{H}^n$, the space will be denoted simply by $\mathbb{C}^\infty_{\#,\mathbb{H}}(Y)$; $(\mathbb{C}^k_{\#,\mathbb{H}}(Y))$ (dropping the A), and this notation will be adopted also for the spaces defined in the sequel.
- Let $1 \leqslant p < \infty$, we denote by $L^p_{\#,\mathbb{H}}(Y, A)$ the space of all $(\mathbb{H}, Y)$-periodic functions f on A such that $f|_Y \in L^p(Y \cap A)$, endowed with the norm $\|f\|_{L^p(Y\cap A)}$.
- $W^{1,2}_{\#,\mathbb{H}}(Y, A)$ denotes the space of all $(\mathbb{H}, Y)$-periodic real functions f on A such that $f \in W^{1,2}_{\mathbb{H}}(\delta_\lambda(Y_0) \cap A)$ for all $\lambda > 0$, endowed with the norm $\|f\|_{W^{1,2}_{\mathbb{H}}(Y_0\cap A)}$. We prove below (Proposition 2.9) that $W^{1,2}_{\#,\mathbb{H}}(Y, A)$ is a Hilbert space.
- $\mathring{W}^{1,2}_{\#,\mathbb{H}}(Y, A)$ denotes the closure of the set $\{u \in \mathbb{C}^\infty_{\#,\mathbb{H}}(Y, A)\colon \operatorname{supp} u \subset A\}$ in $W^{1,2}_{\#,\mathbb{H}}(Y, A)$.

In addition we define the following periodic vector-valued function spaces (Ω and A are as in the previous definitions):

- $\mathcal{D}(\Omega; \mathbb{C}^\infty_{\#,\mathbb{H}}(Y, A))$ denotes the space of all smooth functions on $\Omega \times \mathbb{H}^n$ such that $f(p, \cdot) \in \mathbb{C}^\infty_{\#,\mathbb{H}}(Y, A)$ for any $p \in \Omega$, and the map $p \in \Omega \to f(p, \cdot) \in \mathbb{C}^\infty_{\#,\mathbb{H}}(Y, A)$ is compactly supported in Ω.
- $L^2(\Omega; \mathbb{C}_{\#,\mathbb{H}}(Y, A))$ denotes the set of measurable functions f on $\Omega \times \mathbb{H}^n$ such that $f(p, \cdot) \in \mathbb{C}_{\#,\mathbb{H}}(Y, A)$ and $\max_{q\in\overline{Y}} |f(\cdot, q)| \in L^2(\Omega)$.

Finally we define the following spaces of periodic sections (A is as in the previous definitions):

- $\mathbb{C}^\infty_{\#,\mathbb{H}}(Y, A; H\mathbb{H}^n)$ denotes the space of all smooth $(\mathbb{H}, Y)$-periodic sections of $H\mathbb{H}^n$ defined on A. The spaces $\mathbb{C}^k_{\#,\mathbb{H}}(Y, A; H\mathbb{H}^n)$, $k \geqslant 0$, are defined analogously. By means of the canonical coordinates of $H\mathbb{H}^n$, $\mathbb{C}^\infty_{\#,\mathbb{H}}(Y, A; H\mathbb{H}^n)$ can be identified with $(\mathbb{C}^\infty_{\#,\mathbb{H}}(Y, A))^{2n}$, as well as $\mathbb{C}^k_{\#,\mathbb{H}}(Y, A; H\mathbb{H}^n)$ can be identified with $(\mathbb{C}^k_{\#,\mathbb{H}}(Y, A))^{2n}$.
- $W^{1,2}_{\#,\mathbb{H}}(Y, A; H\mathbb{H}^n)$ (respectively $L^2_{\#,\mathbb{H}}(Y, A; H\mathbb{H}^n)$) can be defined as above as the set $\phi = (\phi_1, \dots, \phi_{2n})$ of all measurable sections of $H\mathbb{H}^n$ such that $\phi \in (W^{1,2}_{\#,\mathbb{H}}(Y, A))^{2n}$ (respectively $\phi \in (L^2_{\#,\mathbb{H}}(Y, A))^{2n}$).
- $V(Y, A)$ is the closure in $W^{1,2}_{\#,\mathbb{H}}(Y, A; H\mathbb{H}^n)$ of the set
$$\mathcal{V}(Y, A) = \big\{u \in \mathbb{C}^\infty_{\#,\mathbb{H}}\big(Y, A; H\mathbb{H}^n\big)\colon \operatorname{supp} u \subset A, \operatorname{div}_{\mathbb{H}} u = 0\big\}.$$
- $\mathring{E}_{\#,\mathbb{H}}(Y, A; H\mathbb{H}^n)$ is the completion of $\{u \in \mathbb{C}^\infty_{\#,\mathbb{H}}(Y, A; H\mathbb{H}^n)\colon \operatorname{supp} u \subset A\}$ with respect to the norm
$$\|u\|^2_{\mathring{E}_{\#,\mathbb{H}}(Y,A;H\mathbb{H}^n)} = \|u\|^2_{L^2_{\#,\mathbb{H}}(Y,A;H\mathbb{H}^n)} + \|\operatorname{div}_{\mathbb{H}} u\|^2_{L^2_{\#,\mathbb{H}}(Y,A)}.$$

Lemma 2.5. *If $\varphi \in \mathbb{C}^\infty_{\#,\mathbb{H}}(Y; H\mathbb{H}^n)$, then*
$$\int_Y \operatorname{div}_{\mathbb{H}} \varphi \, dh = 0. \tag{9}$$
In particular, if $\varphi \in \mathbb{C}^\infty_{\#,\mathbb{H}}(Y, A; H\mathbb{H}^n)$, $\operatorname{supp} \varphi \subset A$, then
$$\int_{A\cap Y} \operatorname{div}_{\mathbb{H}} \varphi \, dh = 0. \tag{10}$$

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Proof. To prove the assertion, take $R > 0$ and set $\mathcal{F} = \{k \in \mathbb{Z}^{2n+1}\colon 2k \cdot \overline{Y} \cap \overline{U(0,R)} \neq \emptyset\}$, $\mathcal{F}_0 = \{k \in \mathcal{F}\colon 2k \cdot \overline{Y} \cap \partial U(0,R) \neq \emptyset\}$, $Y_R = \bigcup_{k\in\mathcal{F}}\colon 2k \cdot \overline{Y} \supseteq \overline{U(0,R)}$. Taking into account that $2k \cdot Y \cap 2h \cdot Y = \emptyset$ for $h \neq k$ and that $|2k \cdot Y| = |Y| = 2^{2n+1}$, we have

$$c_n R^{2n+1} = \left|U(0,R)\right| \leqslant |Y_R| = 2^{2n+1}(\operatorname{card}\mathcal{F}).$$

Moreover, $\partial Y_R \subseteq \bigcup_{k\in\mathcal{F}_0} 2k \cdot \partial Y = \bigcup_{k\in\mathcal{F}_0} \partial(2k \cdot Y)$. Indeed, if $p \in \partial Y_R$, then $p \in 2k_0 \cdot \overline{Y}$ for some $k_0 \in \mathcal{F}_0$; suppose by contradiction $k_0 \notin \mathcal{F}_0$, since $\bigcup_{k\in\mathcal{F}\setminus F_0} 2k \cdot \overline{Y} \subseteq U(0,R)$ and it is a closed set, then there exists $\delta > 0$ such that $\delta \leqslant d(p, \partial U(0,R))$. On the other hand, $p \in \partial Y_R$ and hence there exists $q \in U(p, \delta/2) \cap Y_R^c$, yielding a contradiction, since this would imply $q \in U(0,R) \cap Y_R^c \subseteq Y_R \cap Y_R^c = \emptyset$. Thus, by divergence theorem

$$(\operatorname{card}\mathcal{F})\left|\int_Y \operatorname{div}_{\mathbb{H}}\varphi\,\mathrm{d}h\right| \leqslant \left|\int_{Y_R} \operatorname{div}_{\mathbb{H}}\varphi\,\mathrm{d}h\right| \leqslant \int_{\partial Y_R} \left|\langle\varphi, n_{\mathbb{H}}\rangle\right| \mathrm{d}\mathcal{H}^{2n} \leqslant \max_{\partial Y}|\varphi| \sum_{k\in\mathcal{F}_0} \int_{2k\cdot\partial Y} |n_{\mathbb{H}}|\,\mathrm{d}\mathcal{H}^{2n}$$
$$= \max_{\partial Y}|\varphi|(\operatorname{card}\mathcal{F}_0)|\partial Y|_{\mathbb{H}}(\mathbb{H}^n) = C_\varphi(\operatorname{card}\mathcal{F}_0).$$

Now notice that, if $p \in \Delta_R := \bigcup_{k\in\mathcal{F}_0} 2k \cdot \overline{Y}$, then $d(p, \partial U(0,R)) \leqslant \operatorname{diam} Y = c_0$, and hence

$$\delta_{1/R}(\Delta_R) \subseteq \left\{p \in \mathbb{H}^n\colon d\big(p, \partial U(0,1)\big) < c_0/R\right\},$$

so that

$$c(\operatorname{card}\mathcal{F}_0)R^{-2n-2} \leqslant \left|\left\{p \in \mathbb{H}^n\colon d\big(p, \partial U(0,1)\big) < c_0/R\right\}\right|$$

and hence

$$\left|\int_Y \operatorname{div}_{\mathbb{H}}\varphi\,\mathrm{d}h\right| \leqslant C_\varphi \lim_{R\to\infty}(\operatorname{card}\mathcal{F}_0)R^{-2n-2} = C_\varphi \lim_{R\to\infty}\frac{1}{R}\limsup_{R\to\infty}\frac{R}{2c_0}\left|\left\{p \in \mathbb{H}^n\colon d\big(p, \partial U(0,1)\big) < \frac{c_0}{R}\right\}\right|$$
$$= 0,$$

since $R/(2c_0)|\{p \in \mathbb{H}^n\colon d(p, \partial U(0,1)) < c_0/R\}|$ is bounded for $R \to \infty$. This follows basically the result for the Minkowski content proved in [17]; however, since $U(0,1)$ is not smooth as required in [17], let us give a simple proof in this very particular situation. Indeed $\{p \in \mathbb{H}^n\colon d(p, \partial U(0,1)) < c_0/R\} \subseteq \{p \in \mathbb{H}^n\colon 1 - c_0/R < d(p,0) < 1 + c_0/R\}$, so that $|\{p \in \mathbb{H}^n\colon d(p, \partial U(0,1)) < c_0/R\}| \approx (1 + c_0/R)^{2n+2} - (1 - c_0/R)^{2n+2} \approx 1/R$, and we are done. □

Proposition 2.6. *Let A be a $(\mathbb{H}, Y)$-periodic open set. Then*

$$\left\{u \in \mathbb{C}^\infty_{\#,\mathbb{H}}(Y, A)\colon \operatorname{supp} u \subset A\right\} \textit{ is dense in } L^2_{\#,\mathbb{H}}(Y, A).$$

Analogously, $\{u \in \mathbb{C}^\infty_{\#,\mathbb{H}}(Y, A; H\mathbb{H}^n)\colon \operatorname{supp} u \subset A\}$ is dense in $L^2_{\#,\mathbb{H}}(Y, A; H\mathbb{H}^n)$.

Proof. If $f \in L^2_{\#,\mathbb{H}}(Y, A)$, then f can be approximated by smooth functions by means of the group convolution (see, e.g., [8] Proposition 1.20); since this convolution preserves the $(\mathbb{H}, Y)$-periodicity, then the proof can be carried out as in the Euclidean case. The second assertion follows arguing on canonical coordinates. □

Remark 2.7. If A is a $(\mathbb{H}, Y)$-periodic open set, we shall use the quotient space $L^2_{\#,\mathbb{H}}(Y, A)/\mathbb{R}$, endowed with the quotient norm. Since $\inf_t \|u - t\|_{L^2_{\#,\mathbb{H}}(Y,A)}$ is attained at $t = \fint_{A\cap Y} u\,\mathrm{d}h$, then $L^2_{\#,\mathbb{H}}(Y, A)/\mathbb{R}$ can be identified with $\{u \in L^2_{\#,\mathbb{H}}(Y, A)\colon \fint_{A\cap Y} u\,\mathrm{d}h = 0\}$.

In addition, notice that $\{u \in \mathbb{C}^\infty_{\#,\mathbb{H}}(Y, A)\colon u = v + \lambda,\ \lambda \in \mathbb{R},\ \operatorname{supp} v \subseteq A,\ \int_{A\cap Y} u\,\mathrm{d}h = 0\}$ is dense in $L^2_{\#,\mathbb{H}}(Y, A)/\mathbb{R}$. Indeed, if $u \in L^2_{\#,\mathbb{H}}(Y, A)$, $\int_{A\cap Y} u\,\mathrm{d}h = 0$, take $v_n \in \mathbb{C}^\infty_{\#,\mathbb{H}}(Y, A)$, $\operatorname{supp} v_n \subseteq A$, $v_n \to u$ in $L^2_{\#,\mathbb{H}}(Y, A)$, and notice that

$$\left|\int_{A\cap Y} v_n\,\mathrm{d}h\right| = \left|\int_{A\cap Y} (v_n - u)\,\mathrm{d}h\right| \leqslant |A \cap Y|^{1/2}\|v_n - u\|_{L^2(A\cap Y)} = |A \cap Y|^{1/2}\|v_n - u\|_{L^2_{\#,\mathbb{H}}(Y,A)} \to 0,$$

as $n \to \infty$. Thus we can take $u_n = v_n - \fint_{A\cap Y} v_n\,\mathrm{d}h$ and the assertion is proved.

Proposition 2.8. *As in the elliptic case, $\mathring{W}^{1,2}_{\mathbb{H}}(\Omega)^* = \{\operatorname{div}_{\mathbb{H}} f\colon f \in L^2(\Omega; H\mathbb{H}^n)\}$ is endowed with the norm*

$$\|\operatorname{div}_{\mathbb{H}} f\|_{\mathring{W}^{1,2}_{\mathbb{H}}(\Omega)^*} = \|f\|_{L^2(\Omega; H\mathbb{H}^n)}.$$

In addition, $\{\operatorname{div}_{\mathbb{H}} f\colon f \in \mathcal{D}(\Omega; H\mathbb{H}^n)\}$ is dense in $\mathring{W}^{1,1}_{\mathbb{H}}(\Omega)^*$.

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Proposition 2.9. *Let $A \subseteq \mathbb{H}^n$ be a $(\mathbb{H}, Y)$-periodic open set. Then $W^{1,2}_{\#,\mathbb{H}}(Y, A)$ is a Hilbert space.*

Proof. We want to show that $W^{1,2}_{\#,\mathbb{H}}(Y, A)$ is a complete metric space. Thus, let $(u_h)_{h\in\mathbb{N}}$ be a Cauchy sequence in $W^{1,2}_{\#,\mathbb{H}}(Y, A)$; by definition of norm in $W^{1,2}_{\#,\mathbb{H}}(Y, A)$ there exists $u \in W^{1,2}_{\mathbb{H}}(Y_0 \cap A)$ such that $u_h \to u$ in $W^{1,2}_{\mathbb{H}}(Y_0 \cap A)$. We need to prove that u is the restriction of a $(\mathbb{H}, Y)$-periodic function belonging to $W^{1,2}_{\mathbb{H}}(A \cap \delta_\lambda(Y_0))$. Let now Π be the operator of continuation by $(\mathbb{H}, Y)$-periodicity to all of A; we want to show that $Z_j(\Pi(u)) = \Pi(Z_j u)$, $j = 1, \ldots, 2n$, in the sense of distributions. If $\varphi \in \mathbb{C}^\infty_0(A; \mathbb{R})$, set $K_\varphi = \{k \in \mathbb{Z}^{2n+1}\colon \operatorname{supp}\varphi \cap \tau_{2k}(Y) \neq \emptyset\}$; then, keeping in mind that Z_j are left-invariant with respect to group translations and that $u_h \cdot \tau_{2k} = u_h$, we have

$$
\begin{aligned}
\int_A \Pi(u)(p) Z_j\varphi(p)\,\mathrm{d}p &= \sum_{k\in K_\varphi} \int_{\tau_{2k}(Y)\cap A} \Pi(u)(p) Z_j\varphi(p)\,\mathrm{d}p = \sum_{k\in K_\varphi} \int_{Y\cap A} u(p)(Z_j\varphi)\big(\tau_{-2k}(p)\big)\,\mathrm{d}p \\
&= \lim_{h\to\infty} \sum_{k\in K_\varphi} \int_{Y\cap A} u_h(p)(Z_j\varphi)\big(\tau_{-2k}(p)\big)\,\mathrm{d}p = \lim_{h\to\infty} \sum_{k\in K_\varphi} \int_{\tau_{2k}(Y)\cap A} u_h(p)(Z_j\varphi)(p)\,\mathrm{d}p \\
&= \lim_{h\to\infty} \int_A u_h(p)(Z_j\varphi)(p)\,\mathrm{d}p = -\lim_{h\to\infty} \int_A (Z_j u_h)(p)\varphi(p)\,\mathrm{d}p \\
&= -\lim_{h\to\infty} \sum_{k\in K_\varphi} \int_{\tau_{2k}(Y)\cap A} (Z_j u_h)(p)\varphi(p)\,\mathrm{d}p \\
&= -\lim_{h\to\infty} \sum_{k\in K_\varphi} \int_{Y\cap A} \big((Z_j u_h)\cdot\tau_{-2k}\big)(p)\varphi\big(\tau_{-2k}(p)\big)\,\mathrm{d}p \\
&= -\lim_{h\to\infty} \sum_{k\in K_\varphi} \int_{Y\cap A} \big(Z_j(u_h\cdot\tau_{-2k})\big)(p)\varphi\big(\tau_{-2k}(p)\big)\,\mathrm{d}p \\
&= -\lim_{h\to\infty} \sum_{k\in K_\varphi} \int_{Y\cap A} Z_j u_h(p)\varphi\big(\tau_{-2k}(p)\big)\,\mathrm{d}p = -\sum_{k\in K_\varphi} \int_{Y\cap A} Z_j u(p)\varphi\big(\tau_{-2k}(p)\big)\,\mathrm{d}p,
\end{aligned}
$$

since, by definition $Z_j u_h \to Z_j u$ in $L^2(Y \cap A)$ and in addition $p \to \varphi(\tau_{-2k}(p))$ again belongs to $L^2(Y \cap A)$. Thus

$$
\int_A \Pi(u)(p) Z_j\varphi(p)\,\mathrm{d}p = -\sum_{k\in K_\varphi} \int_{\tau_{2k}(Y)\cap A} \Pi(Z_j u)(p)\varphi(p)\,\mathrm{d}p = -\int_A \Pi(Z_j u)(p)\varphi(p)\,\mathrm{d}p
$$

and the assertion is proved. Thus, if $\lambda > 0$,

$$
\begin{aligned}
\int_{A\cap\delta_\lambda(Y_0)} \big|Z_j\big(\Pi(u)\big)\big|^2\,\mathrm{d}p &= \int_{A\cap\delta_\lambda(Y_0)} \big|\Pi(Z_j u)\big|^2\,\mathrm{d}p \leqslant \sum_{\tau_{2k}(Y)\cap\delta_\lambda(Y)\neq\emptyset} \int_{\tau_{2k}(Y)\cap A} \big|\Pi(Z_j u)\big|^2\,\mathrm{d}p \\
&= \operatorname{card}\big\{k \in \mathbb{Z}^{2n+1}\colon \tau_{2k}(Y) \cap \delta_\lambda(Y) \neq \emptyset\big\} \cdot \int_{Y\cap A} |Z_j u|^2\,\mathrm{d}p < \infty,
\end{aligned}
$$

and hence $\Pi(u)$ belongs to $W^{1,2}_{\mathbb{H}}(A \cap \delta_\lambda(Y_0))$. Since $\Pi(u)|_{Y_0} = u$, the proof is complete. □

Proposition 2.10. *Let $A \subseteq \mathbb{H}^n$ be a $(\mathbb{H}, Y)$-periodic open set. Then the set*

$$
\big\{u \in \mathbb{C}^\infty_{\#,\mathbb{H}}(Y, A)\colon u \in W^{1,2}_{\mathbb{H}}\big(A \cap \delta_\lambda(Y_0)\big) \text{ for any } \lambda > 0\big\}
$$

is dense in $W^{1,2}_{\#,\mathbb{H}}(Y, A)$.

Proof. First we notice that any function $u \in W^{1,2}_{\#,\mathbb{H}}(Y, A)$ such that $\operatorname{supp} u \Subset A$ can be approximated by smooth $(\mathbb{H}, Y)$-periodic functions supported in an arbitrary given neighborhood of $\operatorname{supp} u$ by means of the group convolution with suitable Friedrichs' mollifiers for the group (see [8] Proposition 1.20). Indeed, the group convolution preserves the $(\mathbb{H}, Y)$-periodicity, since group translations and dilations do. Thus, to prove the assertion we can repeat verbatim the classical Meyers–Serrin's proof (see,

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e.g., [13], Theorem 7.9), provided we can find a family of $(\mathbb{H}, Y)$-periodic open sets $\{A_j\colon j \in \mathbb{N}\}$, such that $A_j \Subset A_{j+1}$ and $\bigcup_{j\in\mathbb{N}} A_j = A$, and a partition of unity $\{\psi_j\colon j \in \mathbb{N}\}$ subordinated to the covering $\{A_{j+1}\setminus\bar{A}_{j-1}\colon j \in \mathbb{N}\}$ ($A_0 = A_{-1} = \emptyset$) such that $\psi_j \in \mathbb{C}^\infty_{\#,\mathbb{H}}(Y, A)$. Again by the group convolution, it is enough to prove the existence of the A_j, since we can obtain the function ψ_j by a regularization of the characteristic function of, say $A_{j+2}\setminus A_{j-2}$. To this end, notice that the function $p \to d(p, \partial A)$ is $(\mathbb{H}, Y)$-periodic. Indeed, if $p \in \mathbb{H}^n$,

$$d(2k\cdot p, \partial A) = \inf\bigl\{d(2k\cdot p, q)\colon q \in \partial A\bigr\} = \inf\bigl\{d(2k\cdot p, 2k\cdot(-2k)\cdot q)\colon q \in \partial A\bigr\} = \inf\bigl\{d(p, (-2k)\cdot q)\colon q \in \partial A\bigr\}.$$

Thus, to prove that $d(2k\cdot p, \partial A) = d(p, \partial A)$, we need only to prove that any $\tilde{q} \in \partial A$ can be written as $\tilde{q} = (-2k)\cdot q$, for q suitable in ∂A. But this is straightforward, since $q := 2k\cdot\tilde{q} \in \partial A$, by $(\mathbb{H}, Y)$-periodicity.

Then we can take $A_j = \{p \in A\colon d(p, \partial A) > 1/(j+m)\}$, where $m \in \mathbb{N}$ has to be chosen in such a way that $A_j \neq \emptyset$ for $j \geqslant 1$. □

Theorem 2.11. *Let $A \subseteq \mathbb{H}^n$ be a $(\mathbb{H}, Y)$-periodic open set. We have*

$$\mathring{E}_{\#,\mathbb{H}}(Y, A)^* = \bigl\{f + \nabla_{\mathbb{H}} g\colon f \in L^2_{\#,\mathbb{H}}\bigl(Y, A; \mathrm{H}\mathbb{H}^n\bigr),\ g \in L^2_{\#,\mathbb{H}}(Y, A)\bigr\}$$

and the action of $f + \nabla_{\mathbb{H}} g$ on $\varphi \in \mathring{E}_{\#,\mathbb{H}}(Y, A)$ is given by the expression

$$\langle f + \nabla_{\mathbb{H}} g, \varphi\rangle = \langle f, \varphi\rangle_{L^2(Y\cap A;\mathrm{H}\mathbb{H}^n)} - \langle g, \operatorname{div}_{\mathbb{H}}\varphi\rangle_{L^2(Y\cap A)}.$$

In particular, if $\varphi \in \mathbb{C}^\infty_{\#,\mathbb{H}}(Y, A; \mathrm{H}\mathbb{H}^n)$ and $\operatorname{supp}\varphi \subset A$, then the action of $f + \nabla_{\mathbb{H}} g$ on φ coincides with its action as a distribution. Moreover,

$$\|F\|^2_{\mathring{E}_{\#,\mathbb{H}}(Y,A)^*} = \inf\Bigl\{\|f\|^2_{L^2_{\#,\mathbb{H}}(Y,A;\mathrm{H}\mathbb{H}^n)} + \|g\|^2_{L^2_{\#,\mathbb{H}}(Y,A)}\colon F = f + \nabla_{\mathbb{H}} g\Bigr\}. \tag{11}$$

Proof. Let us identify a section of the horizontal fibre bundle with its canonical $2n$ coordinates. Consider the map $S\colon \mathring{E}_{\#,\mathbb{H}}(Y, A) \to L^2_{\#,\mathbb{H}}(Y, A)^{2n+1}$ given by $S(u) = (u, \operatorname{div}_{\mathbb{H}} u)$. By definition, S is an isometry on the closed subspace $R(S)$.

If $F \in \mathring{E}_{\#,\mathbb{H}}(Y, A)^*$, it defines a functional G on $R(S)$, $G = F\circ S^{-1}$ such that $\|G\| = \|F\|$. By Hahn–Banach theorem, such a functional can be continued on all of $L^2_{\#,\mathbb{H}}(Y, A)^{2n+1}$ preserving its norm. Then we can conclude by Riesz' representation theorem. □

Proposition 2.12. *Let $A \subseteq \mathbb{H}^n$ be a $(\mathbb{H}, Y)$-periodic open set. The set*

$$\bigl\{f + \nabla_{\mathbb{H}} g\colon f \in \mathbb{C}^\infty_{\#,\mathbb{H}}\bigl(Y, A; \mathrm{H}\mathbb{H}^n\bigr),\ g \in \mathbb{C}^\infty_{\#,\mathbb{H}}(Y, A),\ \operatorname{supp} f, \operatorname{supp} g \subset A\bigr\}$$

is dense in $\mathring{E}_{\#,\mathbb{H}}(Y, A)^*$.

Proof. If $f + \nabla_{\mathbb{H}} g \in \mathring{E}_{\#,\mathbb{H}}(Y, A)^*$, take $f_k \in \mathbb{C}^\infty_{\#,\mathbb{H}}(Y, A; \mathrm{H}\mathbb{H}^n)$, $g_k \in \mathbb{C}^\infty_{\#,\mathbb{H}}(Y, A)$, $\operatorname{supp} f_k, \operatorname{supp} g_k \subset A$ such that $f_k \to f$ in $L^2_{\#,\mathbb{H}}(Y, A; \mathrm{H}\mathbb{H}^n)$ and $g_k \to g$ in $L^2_{\#,\mathbb{H}}(Y, A)$ (see Proposition 2.6). Then the assertion follows by (11). □

Definition 2.13. Let A be a $(\mathbb{H}, Y)$-periodic connected open set, we shall say that A is of 2-Poincaré type if there exists $c > 0$ such that

$$\int_{A\cap Y} \Bigl|u - \fint_{A\cap Y} u\,\mathrm{d}h\Bigr|^2 \mathrm{d}h \leqslant c \int_{A\cap Y} |\nabla_{\mathbb{H}} u|^2\,\mathrm{d}h \tag{12}$$

for any $u \in W^{1,2}_{\#,\mathbb{H}}(Y, A)$. For sake of simplicity, from now on we shall write $\fint_{A\cap Y} u\,\mathrm{d}h = u_{A\cap Y}$.

Remark 2.14. We stress the fact that, since periodic functions are involved, the above property does not depend on the boundary of the unit cell Y, but only on ∂A. Indeed, since $\int_{A\cap Y} |u - t|^2\,\mathrm{d}h$ attains its minimum for $t = \fint_{A\cap Y} u\,\mathrm{d}h$, an easy periodicity argument shows that (12) follows provided there exists a bounded set B, with $Y \subseteq B$, such that the Poincaré inequality

$$\int_{A\cap B} \Bigl|u - \fint_{A\cap B} u\,\mathrm{d}h\Bigr|^2 \mathrm{d}h \leqslant c \int_{A\cap B} |\nabla_{\mathbb{H}} u|^2\,\mathrm{d}h$$

holds for any $u \in W^{1,2}_{\mathbb{H}}(A\cap B)$. For conditions implying (12), see [9,11].

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Sets of 2-Poincaré type will play a major role in our results, so that it is natural to look for simple assumptions insuring a Poincaré type inequality. In general Carnot groups, this is a very difficult problem far from being well understood, but luckily in the setting of the Heisenberg group, a neat and very deep result has been proved recently by R. Monti and D. Morbidelli [16] for Step 2 Carnot groups. In our setting, their result reads as follows.

Theorem 2.15. *Let A be a $(\mathbb{H}, Y)$-periodic connected open set with $\mathbb{C}^{1,1}$-boundary (in the usual sense). Then A is of 2-Poincaré type.*

Theorem 2.16. *Let $A \subseteq \mathbb{H}^n$ be a $(\mathbb{H}, Y)$-periodic connected open set of 2-Poincaré type, and let $F \in L^2(\Omega; \mathring{E}_{\#,\mathbb{H}}(Y, A)^*)$ be such that*

$$\int_{\Omega} F(p)\big(\varphi(p,\cdot)\big)\, \mathrm{d}p = 0 \tag{13}$$

for any $\varphi = \varphi(p,q) \in L^2(\Omega; \mathring{E}_{\#,\mathbb{H}}(Y, A))$ with $\operatorname{div}_{\mathbb{H},q} \varphi = 0$ for a.e. $p \in \Omega$. Then $F = \nabla_{\mathbb{H},q}\, g$, with $g \in L^2(\Omega; L^2_{\#,\mathbb{H}}(Y, A)/\mathbb{R})$.

Proof. Consider the map

$$T : L^2\big(\Omega; L^2_{\#,\mathbb{H}}(Y, A)/\mathbb{R}\big) \to L^2\big(\Omega; \mathring{E}_{\#,\mathbb{H}}(Y, A)^*\big)$$

given by $Tu = \nabla_{\mathbb{H},q} u$. We want to show that

$$\|Tu\|_{L^2(\Omega; \mathring{E}_{\#,\mathbb{H}}(Y,A)^*)} \geqslant c\|u\|_{L^2(\Omega; L^2_{\#,\mathbb{H}}(Y,A)/\mathbb{R})}. \tag{14}$$

Clearly, T is continuous, by (11). Arguing as in Remark 2.7 and again by continuity, we can assume

$$u = v + \lambda, \quad v \in L^2\big(\Omega; \mathbb{C}^{\infty}_{\#,\mathbb{H}}(Y, A)\big), \qquad \operatorname{supp} v(p,\cdot) \subset A \quad \text{for a.e. } p \in \Omega, \qquad \lambda \in L^2(\Omega, \mathbb{R})$$

and

$$\int_{A\cap Y} u(p,q)\, \mathrm{d}q = 0 \quad \text{for a.e. } p \in \Omega.$$

Let us fix the function u and notice that

$$T(u)(\varphi) := -\int_{\Omega} \langle u, \operatorname{div}_{\mathbb{H},q} \varphi\rangle_{L^2(A\cap Y)}\, \mathrm{d}p = \int_{\Omega} \langle \nabla_{\mathbb{H},q} u, \varphi\rangle_{(L^2(A\cap Y))^{2n}}\, \mathrm{d}p \quad \text{for any } \varphi \in L^2\big(\Omega; \mathring{E}_{\#,\mathbb{H}}(Y, A)\big). \tag{15}$$

To prove (14), consider now the bilinear form on $L^2(\Omega; W^{1,2}_{\#,\mathbb{H}}(Y, A)/\mathbb{R})$ defined by

$$Q(\varphi, \psi) = \int_{\Omega} \Bigg(\int_{A\cap Y} \langle \nabla_{\mathbb{H},q} \varphi, \nabla_{\mathbb{H},q} \psi\rangle\, \mathrm{d}q \Bigg)\, \mathrm{d}p. \tag{16}$$

Clearly, Q is continuous. To prove that Q is coercive it is enough to notice that, by Poincaré inequality (12),

$$Q(\varphi, \varphi) \geqslant c \int_{\Omega} \Bigg(\int_{A\cap Y} |\varphi - \varphi_{A\cap Y}|^2\, \mathrm{d}q \Bigg)\, \mathrm{d}p = c\|\varphi\|^2_{L^2(\Omega; L^2_{\#,\mathbb{H}}(Y;A)/\mathbb{R})}.$$

Hence, by Lax–Milgram theorem, there exists a (unique) $\varphi \in L^2(\Omega; W^{1,2}_{\#,\mathbb{H}}(Y, A)/\mathbb{R})$ such that

$$Q(\varphi, \psi) = \int_{\Omega} \Bigg(\int_{A\cap Y} u\psi\, \mathrm{d}q \Bigg)\, \mathrm{d}p \tag{17}$$

for all $\psi \in L^2(\Omega; W^{1,2}_{\#,\mathbb{H}}(Y, A)/\mathbb{R})$. If we choose $\psi = \varphi$, then, keeping in mind Poincaré inequality (12), in particular, we get

$$\|\nabla_{\mathbb{H},q} \varphi\|^2_{L^2(\Omega; L^2(Y\cap A))} \leqslant \frac{1}{c}\|u\|_{L^2(\Omega; L^2(Y\cap A))}. \tag{18}$$

In addition, since we know $\int_{A\cap Y} u(p,q)\, \mathrm{d}q = 0$ for a.e. $p \in \Omega$, identity (17) in particular holds when we choose $\psi = u$.

Suppose now for a while we know

$$\nabla_{\mathbb{H},q}\varphi \in L^2\big(\Omega;\mathring{E}_{\#,\mathbb{H}}(Y,A)\big) \quad \text{with} \tag{19}$$

$$\|\nabla_{\mathbb{H},q}\varphi\|_{L^2(\Omega;\mathring{E}_{\#,\mathbb{H}}(Y,A))} \leqslant \frac{1}{c}\|u\|_{L^2(\Omega;L^2_{\#,\mathbb{H}}(Y,A)/\mathbb{R})}; \tag{20}$$

then we would have (by (15), (19), and (20))

$$\|Tu\|_{L^2(\Omega;\mathring{E}_{\#,\mathbb{H}}(Y,A)^*)} = \sup_{\psi\in L^2(\Omega;\mathring{E}_{\#,\mathbb{H}}(Y,A))} \frac{T(u)(\psi)}{\|\psi\|_{L^2(\Omega;\mathring{E}_{\#,\mathbb{H}}(Y,A))}} \geqslant \frac{T(u)(\nabla_{\mathbb{H},q}\varphi)}{\|\nabla_{\mathbb{H},q}\varphi\|_{L^2(\Omega;\mathring{E}_{\#,\mathbb{H}}(Y,A))}}$$
$$\geqslant c\frac{Q(u,\varphi)}{\|u\|_{L^2(\Omega;L^2_{\#,\mathbb{H}}(Y,A)/\mathbb{R})}} = (\text{by (17) with } \psi = u)\ c\|u\|_{L^2(\Omega;L^2_{\#,\mathbb{H}}(Y,A)/\mathbb{R})}$$

and (14) would be proved.

To prove (19) and (20), consider now the functional L_φ on $L^2(\mathring{E}_{\#,\mathbb{H}}(Y,A)^*;\mathbb{R})$ defined by

$$L_\varphi(f+\nabla_{\mathbb{H},q}g) := \langle f,\nabla_{\mathbb{H},q}\varphi\rangle_{L^2(\Omega;L^2(Y\cap A;H\mathbb{H}^n))} + \langle\nabla_{\mathbb{H},q}g,\nabla_{\mathbb{H},q}\varphi\rangle_{L^2(\Omega;L^2(Y\cap A;H\mathbb{H}^n))},$$

with $f\in L^2(\Omega;\mathbb{C}^\infty_{\#,\mathbb{H}}(Y,A;H\mathbb{H}^n))$, $g\in L^2(\Omega;\mathbb{C}^\infty_{\#,\mathbb{H}}(Y,A))$, $\operatorname{supp} f\subset A$, $\operatorname{supp} g\subset A$ (as we showed in Proposition 2.12, such a set is dense in $L^2(\Omega;\mathring{E}_{\#,\mathbb{H}}(Y,A)^*)$). Notice also that the definition of L_φ is well posed, in the sense that it does not depend on the choice of f,g associated with a given functional in $\mathring{E}_{\#,\mathbb{H}}(Y,A)^*$.

Keeping in mind (18), we have:

$$\begin{aligned}
&\big|L_\varphi(f+\nabla_{\mathbb{H},q}g)\big|\\
&\quad=\big|\langle f,\nabla_{\mathbb{H},q}\varphi\rangle_{L^2(\Omega;L^2(Y\cap A;H\mathbb{H}^n))}+\langle g,u\rangle_{L^2(\Omega;L^2(Y\cap A))}\big|\\
&\quad\leqslant\|f\|_{L^2(\Omega;L^2(Y\cap A;H\mathbb{H}^n))}\|\nabla_{\mathbb{H},q}\varphi\|_{L^2(\Omega;L^2(Y\cap A;H\mathbb{H}^n))}+\|g\|_{L^2(\Omega;L^2(Y\cap A))}\|u\|_{L^2(\Omega;L^2(Y\cap A))}\\
&\quad\leqslant\big(\|f\|^2_{L^2(\Omega;L^2(Y\cap A;H\mathbb{H}^n))}+\|g\|^2_{L^2(\Omega;L^2(Y\cap A))}\big)^{1/2}\big(\|\nabla_{\mathbb{H},q}\varphi\|^2_{L^2(\Omega;L^2(Y\cap A;H\mathbb{H}^n))}+\|u\|^2_{L^2(\Omega;L^2(Y\cap A))}\big)^{1/2}\\
&\quad\leqslant\frac{1}{c}\big(\|f\|^2_{L^2(\Omega;L^2(Y\cap A;H\mathbb{H}^n))}+\|g\|^2_{L^2(\Omega;L^2(Y\cap A))}\big)^{1/2}\|u\|_{L^2(\Omega;L^2(Y\cap A))}.
\end{aligned}$$

Taking the infimum with respect to all pairs f,g representing the same functional, we obtain that

$$L_\varphi\in\mathring{E}_{\#,\mathbb{H}}(Y,A)^{**}=\mathring{E}_{\#,\mathbb{H}}(Y,A)\quad\text{and}\quad\|L_\varphi\|\leqslant\|u\|_{L^2(\Omega;L^2(Y\cap A))}=\|u\|_{L^2(\Omega;L^2_{\#,\mathbb{H}}(A))}.$$

Thus (19) and (20) are proved and hence (14) follows.

Hence, there exists $\psi\in\mathring{E}_{\#,\mathbb{H}}(Y,A)$ such that

$$L_\varphi(f+\nabla_{\mathbb{H},q}g)=\langle f,\psi\rangle_{L^2(\Omega;L^2(Y\cap A;H\mathbb{H}^n))}-\langle g,\operatorname{div}\psi\rangle_{L^2(\Omega;L^2(Y\cap A))}$$

for all $f\in L^2(\Omega;L^2_{\#,\mathbb{H}}(Y,A;H\mathbb{H}^n))$, $g\in L^2(\Omega;L^2_{\#,\mathbb{H}}(Y,A))$.

Taking $g\equiv 0$, we get $\nabla_{\mathbb{H},q}\varphi=\psi$ and statements (19), (20) together with (14) are proved, so that $R(T)$, the range of T, is closed in $L^2(\Omega;\mathring{E}_{\#,\mathbb{H}}(Y,A)^*)$. This implies that $R(T)=R(T)^{\perp\perp}=(\ker T^*)^\perp$. Let us show now that

$$\ker T^*=\big\{\varphi\in L^2\big(\Omega;\mathring{E}_{\#,\mathbb{H}}(Y,A)\big)\colon\ \operatorname{div}_{\mathbb{H},q}\varphi\equiv 0 \text{ for a.e. } p\in\Omega\big\}. \tag{21}$$

Indeed, consider the map

$$S: L^2\big(\Omega;\mathring{E}_{\#,\mathbb{H}}(Y,A)\big)\to L^2\big(\Omega;L^2_{\#,\mathbb{H}}(Y,A)/\mathbb{R}\big)$$

where $S(\varphi)=\operatorname{div}_{\mathbb{H},q}\varphi$ (remember that $\int_{A\cap Y}\operatorname{div}_{\mathbb{H},q}\varphi\,\mathrm{d}h(q)=0$ for a.e. $p\in\Omega$, and Remark 2.7). If $u\in L^2(\Omega;L^2_{\#,\mathbb{H}}(Y,A))$, then by (15)

$$\int_\Omega T(u)(\varphi)\,\mathrm{d}p := -\big\langle u,S(\varphi)\big\rangle_{L^2(\Omega;L^2_{\#,\mathbb{H}}(Y,A))},$$

so that $T^*=-S$ and (21) follows. Thus the functional F in (13) belongs to $(\ker T^*)^\perp=R(T)$ and hence the assertion follows. □

The proof of Theorem 2.16 can be adapted (more precisely, simplified) in order to prove the following result:

Corollary 2.17. *Let $A\subseteq\mathbb{H}^n$ be a $(\mathbb{H},Y)$-periodic connected open set of 2-Poincaré type, and let $F\in\mathring{E}_{\#,\mathbb{H}}(Y,A)^*$ be such that $F(\varphi)=0$ for any $\varphi\in\mathring{E}_{\#,\mathbb{H}}(Y,A)$ with $\operatorname{div}_{\mathbb{H}}\varphi=0$. Then $F=\nabla_{\mathbb{H}}g$, with $g\in L^2_{\#,\mathbb{H}}(Y,A)/\mathbb{R}$.*

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Remark 2.18. It follows from the proof of Theorem 2.16 that if we drop the assumption A is of 2-Poincaré type we still get $F=\nabla_{\mathbb{H}}g$, but $g\in L^2(\Omega; L^2_{\#,\mathbb{H},\mathrm{loc}}(Y,A)/\mathbb{R})$. Indeed it is enough to replace $W^{1,2}_{\#,\mathbb{H}}(Y,A)$ by the completion of $\mathbb{C}^\infty_{\#,\mathbb{H}}(Y,A)$ with respect to the norm given by (16), that is a true norm, by local Poincaré inequality [15] and by the connectedness of A.

We will need later the two following results:

Proposition 2.19. *Let A be an open $(\mathbb{H},Y)$-periodic set in $\mathbb{H}^n$. If $\beta\in \mathring{E}_{\#,\mathbb{H}}(Y,A)$, then the function $\tilde\beta$ obtained by continuing β with zero outside A still belongs to $\mathring{E}_{\#,\mathbb{H}}(Y,\mathbb{H}^n)$ and $\operatorname{div}_{\mathbb{H}}\tilde\beta$ is $\widetilde{\operatorname{div}_{\mathbb{H}}\beta}$, i.e., the continuation of $\operatorname{div}_{\mathbb{H}}\beta$ by zero outside A.*

Proof. Let $\beta\in \mathring{E}_{\#,\mathbb{H}}(Y,A)$; then β is the limit in $\mathring{E}_{\#,\mathbb{H}}(Y,A)$ of the sequence $\beta_n\in\mathbb{C}^\infty_{\#,\mathbb{H}}(Y,A;H\mathbb{H}^n)$, $\operatorname{supp}\beta_n\subseteq A$. Clearly, the functions $\tilde\beta_n$ obtained by continuing β_n by zero outside of A belong to $\mathbb{C}^\infty_{\#,\mathbb{H}}(Y,\mathbb{H}^n;H\mathbb{H}^n)$, and $\tilde\beta_n\to\tilde\beta$ in $L^2_{\#,\mathbb{H}}(Y,\mathbb{H}^n;H\mathbb{H}^n)$. Notice that $(\tilde\beta_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathring{E}_{\#,\mathbb{H}}(Y,\mathbb{H}^n)$, since $\|\tilde\beta_n-\tilde\beta_m\|_{\mathring{E}_{\#,\mathbb{H}}(Y,\mathbb{H}^n)}=\|\beta_n-\beta_m\|_{\mathring{E}_{\#,\mathbb{H}}(Y,A)}$, so that $\tilde\beta\in\mathring{E}_{\#,\mathbb{H}}(Y,\mathbb{H}^n)$. Thus to accomplish the proof we have to show that $\operatorname{div}\tilde\beta=\widetilde{\operatorname{div}\beta}$.

If $\varphi\in\mathcal{D}(\mathbb{H}^n)$, then

$$\begin{aligned}\int_{\mathbb{H}^n}\langle\tilde\beta,\nabla_{\mathbb{H}}\varphi\rangle\,\mathrm{d}h &= \int_A\langle\beta,\nabla_{\mathbb{H}}\varphi\rangle\,\mathrm{d}h=\lim_{n\to\infty}\int_A\langle\beta_n,\nabla_{\mathbb{H}}\varphi\rangle\,\mathrm{d}h=-\lim_{n\to\infty}\int_A(\operatorname{div}_{\mathbb{H}}\beta_n)\varphi\,\mathrm{d}h=-\int_A(\operatorname{div}_{\mathbb{H}}\beta)\varphi\,\mathrm{d}h\\ &=-\int_{\mathbb{H}^n}\big(\widetilde{\operatorname{div}_{\mathbb{H}}\beta}\big)\varphi\,\mathrm{d}h,\end{aligned}$$

and we are done. □

Lemma 2.20. *Let $A\subseteq\mathbb{H}^n$ be a $(\mathbb{H},Y)$-periodic connected open set. Let $\xi\in\mathbb{R}^{2n}$ be identified with a smooth section of $H\mathbb{H}^n$. If*

$$\int_{A\cap Y}\langle\xi,\varphi\rangle\,\mathrm{d}h=0 \tag{22}$$

for any $\varphi\in\mathcal{V}(A)$, then $\xi=0$.

Proof. Let $B\subset\subset A$ be any open $(\mathbb{H},Y)$-periodic connected set and let $\varphi\in\mathring{E}_{\#,\mathbb{H}}(Y,B)$, with $\operatorname{div}_{\mathbb{H}}\varphi=0$, be fixed. Denote by $\tilde\varphi$ the continuation of φ by zero outside B. By Proposition 2.19 $\tilde\varphi\in\mathring{E}_{\#,\mathbb{H}}(Y,\mathbb{H}^n)$. Arguing as in Proposition 2.6, using the group convolution we can approximate $\tilde\varphi$ in $\mathring{E}_{\#,\mathbb{H}}(Y,\mathbb{H}^n)$ by means of smooth functions φ_n supported in A and such that $\operatorname{div}_{\mathbb{H}}\varphi_n=0$ (all this because the left invariant vector fields commute with the group convolution). In particular, $\varphi_n\in\mathcal{V}(A)$ and hence

$$\int_{B\cap Y}\langle\xi,\varphi\rangle\,\mathrm{d}h=\int_Y\langle\xi,\tilde\varphi\rangle\,\mathrm{d}h=\lim_{n\to\infty}\int_{A\cap Y}\langle\xi,\varphi_n\rangle\,\mathrm{d}h=0$$

by (22). Thus, we can apply Corollary 2.17 and Remark 2.18 to conclude that there exists $w\in L^2_{\#,\mathbb{H},\mathrm{loc}}(Y,A)/\mathbb{R}$ such that $\xi=\nabla_{\mathbb{H}}w$. In particular $0=\operatorname{div}_{\mathbb{H}}(\nabla_{\mathbb{H}}w)=\Delta_{\mathbb{H}}w$ in B, and hence $w\in\mathbb{C}^\infty_{\#,\mathbb{H}}(Y,B)$, since the Khon Laplacian $\Delta_{\mathbb{H}}$ is hypoelliptic ([14]). Moreover, $\partial_t w=-(1/4)(X_1Y_1-Y_1X_1)w=0$ in B and hence w is independent of t, since B is connected. Take now a point $p\in A$; by connectedness, there exists a connected open $(\mathbb{H},Y)$-periodic set B containing p and $\tau_{2e_k}(p)$, $k=1,\dots,2n$. Let $w\in\mathbb{C}^\infty_{\#,\mathbb{H},\mathrm{loc}}(Y,B)$ be such that $\xi=\nabla_{\mathbb{H}}w$ in B. As we saw above w is independent of t in B and $X_jw=\partial w/\partial x_j$, $Y_jw=\partial w/\partial y_j$, $j=1,\dots,n$ so that $w=\sum_{j=1}^{2n}\xi_jp_j+c$ in B. On the other hand, by periodicity $0=w(p+\tau_{2e_k}(p))-w(p)=2\xi_k$, and hence $\xi=0$. □

The following result is completely analogous to the corresponding result for periodic functions with respect to the standard structure of $\mathbb{R}^n$ (see, e.g., [6], Theorem 2.6). As for the Heisenberg group, see also [5], Lemma 3.4.

Lemma 2.21. *Let $1<p<+\infty$ and $f\in L^p_{\#,\mathbb{H}}(Y)$, then, as $\epsilon\to0$,*

$$f\circ\delta_{1/\epsilon}\rightharpoonup\frac{1}{|Y|}\int_Y f\,\mathrm{d}h \tag{23}$$

weakly in $L^p(\Omega)$, for any bounded open subset Ω of $\mathbb{H}^n$.

Proof. Let Ω be a bounded open subset of $\mathbb{H}^n$ and, for sake of simplicity, let $\{Y_k\colon\ k\in\mathbb{N}\}=\{\tau_{2\ell}(Y)\colon\ \ell\in\mathbb{Z}^{2n+1}\}$ be the intrinsic pavage by unitary cubes in $\mathbb{H}^n$, and let $N(\epsilon)=\{\delta_\epsilon(Y_k)\colon\ \delta_\epsilon(Y_k)\subset\Omega\}$, $N'(\epsilon)=\{\delta_\epsilon(Y_k)\colon\ \delta_\epsilon(Y_k)\cap\partial\Omega\neq\emptyset\}$. We have

(1) $\lim_{\epsilon\to 0}\epsilon^{2n+2}\operatorname{card}N(\epsilon)=|\Omega|/|Y|$,
(2) $\limsup_{\epsilon\to 0}\epsilon^{2n+1}\operatorname{card}N'(\epsilon)\leqslant C|\partial\Omega|_{\mathbb{H}}$.

We begin by proving (1). Since $|Y|\epsilon^{2n+2}=|\delta_\epsilon(Y_k)|$, then

$$\operatorname{card}N(\epsilon)\cdot|Y|\epsilon^{2n+2}=\Bigl|\bigcup_{\delta_\epsilon(Y_k)\in N(\epsilon)}\delta_\epsilon(Y_k)\Bigr|=\int\limits_{\mathbb{R}^3}\chi_{\bigcup_{\delta_\epsilon(Y_k)\in N(\epsilon)}\delta_\epsilon(Y_k)}(p)\,\mathrm{d}p.$$

If we prove that $\chi_{\bigcup_{\delta_\epsilon(Y_k)\in N(\epsilon)}\delta_\epsilon(Y_k)}\to\chi_\Omega$, then (1) follows from the dominated convergence theorem. If $p\notin\Omega$ we have $\chi(p)=0$ for all ϵ, which implies $\chi_{\bigcup_{\delta_\epsilon(Y_k)\in N(\epsilon)}\delta_\epsilon(Y_k)}(p)\to\chi_\Omega(p)$ for $\epsilon\to 0$. Let $p\in\Omega$, then there exists $\delta_p>0$ such that $U(p,\delta_p)\subseteq\Omega$; on the other hand, if $p\in\delta_\epsilon(Y_k)$ for a given $k\in\mathbb{N}$, then for all $\xi\in\delta_\epsilon(Y_k)$, $d(p,\xi)<\operatorname{diam}_d(\delta_\epsilon(Y_k))=\epsilon\operatorname{diam}(Y)<\delta_p$ for small ϵ, and then for small ϵ we have $\delta_\epsilon(Y_k)\subseteq\Omega$ which implies $\delta_\epsilon(Y_k)\in N(\epsilon)$ leading to $\chi_{\bigcup_{\delta_\epsilon(Y_k)\in N(\epsilon)}\delta_\epsilon(Y_k)}(p)=1$.

We now prove (2). If $p\in\bigcup_{\delta_\epsilon(Y_k)\in N'(\epsilon)}$ then $d(p,\partial\Omega)\leqslant C\epsilon$. Indeed there exists $\xi\in\delta_\epsilon(Y_k)\cap\partial\Omega$, and so $d(p,\partial\Omega)\leqslant d(p,\xi)\leqslant 2\operatorname{diam}_d(\delta_\epsilon(Y_k))=2\epsilon\operatorname{diam}_d(Y)=c_1\epsilon$. This implies that $\bigcup_{\delta_{\epsilon/c_1}(Y_k)\in N'(\epsilon/c_1)}\delta_{\epsilon/c_1}(Y_k)$ is contained in a ϵ-neighborhood of $\partial\Omega$. But this implies

$$\Bigl|\bigcup\delta_{\epsilon/c_1}(Y_k)\Bigr|=|Y|\left(\frac{\epsilon}{c_1}\right)^{2n+2}\operatorname{card}N'\left(\frac{\epsilon}{c_1}\right)\leqslant|\epsilon\text{-neighborhood of }\partial\Omega|$$

which in turn implies $\limsup\epsilon^{2n+1}|Y|\operatorname{card}N'(\epsilon)\leqslant c_1|\partial\Omega|_{\mathbb{H}}$ by [17]. $\square$

Definition 2.22. A family of functions $\{u_\epsilon\}\in L^2(\Omega)$ is said to converge two-scale in $\mathbb{H}^n$ to $u_0\in L^2(\Omega\times Y)$, if for any $\psi\in\mathcal{D}(\Omega;\mathbb{C}^\infty_{\#,\mathbb{H}}(Y))$ we have

$$\lim_{\epsilon\to 0}\int\limits_\Omega u_\epsilon(p)\psi\bigl(p,\delta_{1/\epsilon}(p)\bigr)\,\mathrm{d}p=\frac{1}{|Y|}\int\limits_\Omega\int\limits_Y u_0(p,q)\psi(p,q)\,\mathrm{d}q\,\mathrm{d}p.\tag{24}$$

As in [1], Proposition 1.6, we have:

Proposition 2.23. *Let u_ϵ be a sequence of functions in $L^2(\Omega)$ which converges two-scales to a limit $u_0\in L^2(\Omega\times Y)$. Then u_ϵ converges weakly in $L^2(\Omega)$ to $u_0(p)=\frac{1}{|Y|}\int_Y u_0(p,q)\,\mathrm{d}q$.*

Proposition 2.24. *The following results hold*:

(i) *if $\psi(p,q)=\alpha(p)\beta(q)$, $\alpha\in\mathcal{D}(\Omega)$, $\beta\in L^2_{\#,\mathbb{H}}(Y)$, then $\psi_\epsilon(p):=\psi(p,\delta_{1/\epsilon}(p))$ two-scale converges to $\psi(p,q)$ as $\epsilon\to 0$, and*

$$\lim_{\epsilon\to 0}\int\limits_\Omega\alpha^2(p)\beta^2\bigl(\delta_{1/\epsilon}(p)\bigr)\,\mathrm{d}p=\frac{1}{|Y|}\int\limits_\Omega\int\limits_Y\alpha^2(p)\beta^2(q)\,\mathrm{d}q\,\mathrm{d}p;\tag{25}$$

(ii) *if $\psi(p,q)\in L^2(\Omega;\mathbb{C}_{\#,\mathbb{H}}(Y))$, then $\psi_\epsilon(p):=\psi(p,\delta_{1/\epsilon}(p))$ is a measurable function on Ω that two-scale converges to $\psi(p,q)$ as $\epsilon\to 0$, and, moreover,*

$$\lim_{\epsilon\to 0}\int\limits_\Omega\psi^2\bigl(p,\delta_{1/\epsilon}(p)\bigr)\,\mathrm{d}p=\frac{1}{|Y|}\int\limits_\Omega\int\limits_Y\psi^2(p,q)\,\mathrm{d}q\,\mathrm{d}p\tag{26}$$

$$\bigl\|\psi\bigl(p,\delta_{1/\epsilon}(p)\bigr)\bigr\|_{L^2(\Omega)}\leqslant\bigl\|\psi(p,q)\bigr\|_{L^2(\Omega;\mathbb{C}_{\#,\mathbb{H}}(Y))};\tag{27}$$

(iii) *if $\psi(p,q)\in L^2(\Omega;\mathbb{C}_{\#,\mathbb{H}}(Y))$ and $\varphi(p,q)\in\mathcal{D}(\Omega;\mathbb{C}^\infty_{\#,\mathbb{H}}(Y))$, then $\varphi\psi\in L^2(\Omega;\mathbb{C}_{\#,\mathbb{H}}(Y))$ and hence the conclusions of* (ii) *still hold when ψ is replaced by $\varphi\psi$.*

Corollary 2.25. *If ψ satisfies the structure assumptions of* (i) *or* (ii) *in Proposition* 2.24*, then for any sequence $(v_\epsilon)_{\epsilon>0} \in L^2(\Omega)$ such that $v_\epsilon \to v_0 \in L^2(\Omega \times Y)$ two-scale, we have*

$$\int_\Omega v_\epsilon(p)\psi\big(p, \delta_{1/\epsilon}(p)\big)\,\mathrm{d}p \to \frac{1}{|Y|}\int_\Omega\int_Y v_0(p,q)\psi(p,q)\,\mathrm{d}q\,\mathrm{d}p,$$

i.e., we can take in (24) *ψ as a test function.*

Proof of Corollary 2.25. The statement follows from Proposition 2.24 and the following lemma, that can be proved by repeating verbatim the argument of Theorem 1.8 in [1]:

Lemma 2.26. *Let $(u_\epsilon)_{\epsilon>0}$ be a sequence in $L^2(\Omega)$ that two-scale converges to $u_0 \in L^2(\Omega \times Y)$ and such that*

$$\int_\Omega u_\epsilon^2(p)\,\mathrm{d}p \to \frac{1}{|Y|}\int_\Omega\int_Y u_0^2(p,q)\,\mathrm{d}q\,\mathrm{d}p.$$

Then, if $(v_\epsilon)_{\epsilon>0}$ is a sequence in $L^2(\Omega)$ such that $v_\epsilon \to v_0 \in L^2(\Omega \times Y)$ two-scale, we have

$$u_\epsilon v_\epsilon \to \frac{1}{|Y|}\int_Y u_0(\cdot,q)v_0(\cdot,q)\,\mathrm{d}q \quad \text{in } \mathcal{D}'(\Omega) \quad \text{as } \epsilon \to 0.$$

In fact, to prove the corollary, let us consider first the case with ψ as in Proposition 2.24(ii). If $\eta > 0$, take $\psi \in \mathcal{D}(\Omega; \mathbb{C}_{\#,\mathbb{H}}(Y))$ such that

$$\int_\Omega \big\|\psi(p,\cdot) - \tilde\psi(p,\cdot)\big\|^2_{\mathbb{C}_{\#,\mathbb{H}}(Y)}\,\mathrm{d}p < \eta^2.$$

By Proposition 2.24(ii) the limit (26) holds for $\tilde\psi$ and hence we can apply Lemma 2.26 with $u_\epsilon(p) = \tilde\psi(p, \delta_{1/\epsilon}(p))$, to conclude that

$$\tilde\psi\big(p, \delta_{1/\epsilon}(p)\big)v_\epsilon(p) \to \frac{1}{|Y|}\int_Y \tilde\psi(p,q)v_0(p,q)\,\mathrm{d}q \tag{28}$$

in $\mathcal{D}'(\Omega)$. Now

$$\begin{aligned}
&\left|\int_\Omega \psi\big(p, \delta_{1/\epsilon}(p)\big)v_\epsilon(p)\,\mathrm{d}p - \frac{1}{|Y|}\int_\Omega\int_Y \psi(p,q)v_0(p,q)\,\mathrm{d}q\,\mathrm{d}p\right| \\
&\quad \leqslant \int_\Omega \big|\psi\big(p, \delta_{1/\epsilon}(p)\big) - \tilde\psi\big(p, \delta_{1/\epsilon}(p)\big)\big|\big|v_\epsilon(p)\big|\,\mathrm{d}p + \left|\int_\Omega \tilde\psi\big(p, \delta_{1/\epsilon}(p)\big)v_\epsilon(p)\,\mathrm{d}p - \frac{1}{|Y|}\int_\Omega\int_Y \tilde\psi(p,q)v_0(p,q)\,\mathrm{d}q\,\mathrm{d}p\right| \\
&\qquad + \frac{1}{|Y|}\int_\Omega\int_Y \big|\psi(p,q) - \tilde\psi(p,q)\big|\big|v_0(p,q)\big|\,\mathrm{d}q\,\mathrm{d}p \\
&\quad = I_1 + I_2 + I_3.
\end{aligned}$$

We now have the following estimate:

$$I_1 \leqslant \int_\Omega \big\|\psi(p,\cdot) - \tilde\psi(p,\cdot)\big\|_{\mathbb{C}_{\#,\mathbb{H}}(Y)}\big|v_\epsilon(p)\big|\,\mathrm{d}p \leqslant \bigg(\int_\Omega |v_\epsilon(p)|^2\,\mathrm{d}p\bigg)^{1/2}\eta \leqslant C\eta,$$

since $(v_\epsilon)_{\epsilon>0}$ weakly converges in $L^2(\Omega)$ (Proposition 2.23).

Analogously:

$$I_3 \leqslant \frac{1}{|Y|}\bigg(\int_\Omega\int_Y \big|\psi(p,q) - \tilde\psi(p,q)\big|^2\,\mathrm{d}q\,\mathrm{d}p\bigg)^{1/2}\bigg(\int_\Omega\int_Y \big|v_0(p,q)\big|^2\,\mathrm{d}q\,\mathrm{d}p\bigg)^{1/2} \leqslant C\eta.$$

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Take now $\varphi \in \mathcal{D}(\Omega)$, $\varphi \equiv 1$ on supp $\tilde{\psi}(\cdot, q)$, then

$$I_2 = \left| \int_\Omega \tilde{\psi}\big(p, \delta_{1/\epsilon}(p)\big) v_\epsilon(p)\varphi(p)\,\mathrm{d}p - \frac{1}{|Y|}\int_\Omega \int_Y \tilde{\psi}(p,q) v_0(p,q)\,\mathrm{d}q\varphi(p)\,\mathrm{d}p \right| < \eta$$

if $\epsilon < \epsilon(\eta)$, by (28), and we are done.

It remains the case with ψ as in Proposition 2.24(i). In this case, by Proposition 2.24(i) and Lemma 2.26, we have

$$\psi\big(p, \delta_{1/\epsilon}(p)\big) v_\epsilon(p) \to \frac{1}{|Y|}\int_Y \psi(p,q) v_0(p,q)\,\mathrm{d}q$$

in $\mathcal{D}'(\Omega)$. Taking now $\varphi \equiv 1$ on supp α we can conclude. □

Proof of Proposition 2.24. Let us prove (i). We want to show that

$$\int_\Omega \alpha(p)\beta\big(\delta_{1/\epsilon}(p)\big)\varphi\big(p, \delta_{1/\epsilon}(p)\big)\,\mathrm{d}p \to \frac{1}{|Y|}\int_\Omega\int_Y \alpha(p)\beta(q)\varphi(p,q)\,\mathrm{d}q\,\mathrm{d}p \tag{29}$$

as $\epsilon \to 0$ for any $\varphi \in \mathcal{D}(\Omega; \mathbb{C}^\infty_{\#,\mathbb{H}}(Y))$. To this end, notice first that

$$\big\|\beta\big(\delta_{1/\epsilon}(\cdot)\big)\big\|_{L^2(\Omega)} \leqslant C\|\beta\|_{L^2(Y)}$$

since, arguing as in Lemma 2.21,

$$\int_\Omega \beta^2\big(\delta_{1/\epsilon}(p)\big)\,\mathrm{d}p = \epsilon^{2n+2}\int_{\delta_{1/\epsilon}(\Omega)} \beta^2(q)\,\mathrm{d}q,$$

and $\delta_{1/\epsilon}(\Omega)$ can be covered by $N_\Omega(\epsilon)$ copies of the unit cube Y, with $N_\Omega(\epsilon) \leqslant C_\Omega \epsilon^{-2n-2}$.

Let now η be given, and let $p_\eta(p,q) = \sum_{|\gamma|,|\delta| \leqslant M_\eta} c_{\gamma,\delta} p^\gamma q^\delta$ be a polynomial such that

$$\max_{\Omega\times\bar{Y}} \big|\varphi(p,q) - p_\eta(p,q)\big| < \eta.$$

Then set

$$\varphi_\eta(p,q) = \sum_{|\gamma|,|\delta|\leqslant M_\eta} c_{\gamma,\delta} p^\gamma g_\delta(q),$$

where $g_\delta(q) \in L^\infty(\mathbb{H}^n)$ is obtained by continuing the function $c_{\gamma,\delta}q^\delta$ by $(\mathbb{H}, Y)$-periodicity on all of $\mathbb{H}^n$. Clearly

$$\sup_{\Omega\times\mathbb{H}^n} \big|\varphi(p,q) - \varphi_\eta(p,q)\big| < \eta.$$

Thus

$$\begin{aligned}
&\left|\int_\Omega \alpha(p)\beta\big(\delta_{1/\epsilon}(p)\big)\varphi\big(p,\delta_{1/\epsilon}(p)\big)\,\mathrm{d}p - \frac{1}{|Y|}\int_\Omega\int_Y \alpha(p)\beta(q)\varphi(p,q)\,\mathrm{d}q\,\mathrm{d}p\right| \\
&\quad\leqslant \int_\Omega |\alpha(p)|\big|\beta\big(\delta_{1/\epsilon}(p)\big)\big|\big|\varphi\big(p,\delta_{1/\epsilon}(p)\big) - \varphi_\eta\big(p,\delta_{1/\epsilon}(p)\big)\big|\,\mathrm{d}p \\
&\qquad + \left|\int_\Omega \alpha(p)\beta\big(\delta_{1/\epsilon}(p)\big)\varphi_\eta\big(p,\delta_{1/\epsilon}(p)\big)\,\mathrm{d}p - \frac{1}{|Y|}\int_\Omega\int_Y \alpha(p)\beta(q)\varphi_\eta(p,q)\,\mathrm{d}q\,\mathrm{d}p\right| \\
&\qquad + \frac{1}{|Y|}\int_\Omega\int_Y |\alpha(p)||\beta(q)|\big|\varphi_\eta(p,q) - \varphi(p,q)\big|\,\mathrm{d}q\,\mathrm{d}p \\
&\quad= I_1 + I_2 + I_3.
\end{aligned}$$

We have now the following inequalities:

$$I_1 \leqslant \eta\int_\Omega |\alpha(p)|\big|\beta\big(\delta_{1/\epsilon}(p)\big)\big|\,\mathrm{d}p \leqslant \eta\|\alpha\|_{L^2(\Omega)}\|\beta\|_{L^2(Y)},$$

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and analogously

$$I_3 \leqslant \eta\|\alpha\|_{L^1(\Omega)}\|\beta\|_{L^2(Y)}|Y|^{-1}.$$

On the other hand, the function $g_\delta(q)\beta(q)$ belongs to $L^2_{\#,\mathbb{H}}(Y)$, and hence $(g_\delta\beta)\circ\delta_{1/\epsilon}$ converges weakly in $L^2(\Omega)$ to its average on Y for any δ, $|\delta|\leqslant M_\eta$. Since $\alpha(p)p^\gamma$ belongs to $L^2(\Omega)$, we conclude that

$$\int_\Omega \alpha(p)\beta\big(\delta_{1/\epsilon}(p)\big)p^\gamma g_\delta\big(\delta_{1/\epsilon}(p)\big)\,\mathrm{d}p \to \frac{1}{|Y|}\int_\Omega \alpha(p)p^\gamma\int_Y \beta(q)g_\delta(q)\,\mathrm{d}q\,\mathrm{d}p,$$

as $\epsilon\to 0$. Summing up for $\gamma,\delta,|\gamma|,|\delta|\leqslant M_\eta$ we conclude that $I_2<\eta$ for $\epsilon<\epsilon(\eta,\varphi_\eta)=\epsilon(\eta)$ and then (29) is proved.

We want to prove now that (25) holds. If $N>0$, put $\beta_N=\min\{|\beta|,N\}$. Since $\beta_N\in L^\infty_{\#,\mathbb{H}}(Y)$, by Lemma 2.21,

$$\begin{aligned}\liminf_{\epsilon\to 0}\int_\Omega \alpha^2(p)\beta^2\big(\delta_{1/\epsilon}(p)\big)\,\mathrm{d}p &\geqslant \lim_{\epsilon\to 0}\int_\Omega \alpha^2(p)\beta_N^2\big(\delta_{1/\epsilon}(p)\big)\,\mathrm{d}p = \frac{1}{|Y|}\int_\Omega\int_Y \alpha^2(p)\beta_N^2(q)\,\mathrm{d}q\,\mathrm{d}p\\ &\geqslant \frac{1}{|Y|}\int_\Omega\int_Y \alpha^2(p)\beta^2(q)\,\mathrm{d}q\,\mathrm{d}p - \frac{\eta}{|Y|}\int_\Omega \alpha^2(p)\,\mathrm{d}p\end{aligned}$$

if $N>N(\eta)$, by dominated convergence theorem. On the other hand, arguing as in the first part of the proof, we have:

$$\begin{aligned}\int_\Omega \alpha^2(p)\big[\beta^2\big(\delta_{1/\epsilon}(p)\big)-\beta_N^2\big(\delta_{1/\epsilon}(p)\big)\big]\,\mathrm{d}p &\leqslant \max_{\overline{\Omega}}\alpha^2\epsilon^{2n+2}\int_{\delta_{1/\epsilon}(\Omega)}\big[\beta^2(q)-\beta_N^2(q)\big]\,\mathrm{d}q\\ &\leqslant C_\Omega\max_{\overline{\Omega}}\alpha^2\int_Y\big[\beta^2(q)-\beta_N^2(q)\big]\,\mathrm{d}q<\eta\end{aligned}$$

if $N>N(\eta)$, so that

$$\begin{aligned}\limsup_{\epsilon\to 0}\int_\Omega \alpha^2(p)\beta^2\big(\delta_{1/\epsilon}(p)\big)\,\mathrm{d}p &\leqslant \lim_{\epsilon\to 0}\int_\Omega \alpha^2(p)\beta_N^2\big(\delta_{1/\epsilon}(p)\big)\,\mathrm{d}p+\eta = \frac{1}{|Y|}\int_\Omega\int_Y \alpha^2(p)\beta_N^2(q)\,\mathrm{d}q\,\mathrm{d}p+\eta\\ &\leqslant \frac{1}{|Y|}\int_\Omega\int_Y \alpha^2(p)\beta^2(q)\,\mathrm{d}q\,\mathrm{d}p+\eta,\end{aligned}$$

and (25) follows since $\eta>0$ is arbitrary.

Coming to point (ii), for the measurability of $\psi(p,\delta_{1/\epsilon}(p))$ see [1, Lemma 1.3]. We want to show here that given $\psi\in L^2(\Omega;\mathbb{C}_{\#,\mathbb{H}}(Y))$ we have

$$\int_\Omega \psi\big(p,\delta_{1/\epsilon}(p)\big)\varphi\big(p,\delta_{1/\epsilon}(p)\big)\,\mathrm{d}p \to \frac{1}{|Y|}\int_\Omega\int_Y \psi(p,q)\varphi(p,q)\,\mathrm{d}q\,\mathrm{d}p$$

for any $\varphi\in\mathcal{D}(\Omega;\mathbb{C}^\infty_{\#,\mathbb{H}}(Y))$. By the very definition of the space $L^2(\Omega;\mathbb{C}_{\#,\mathbb{H}}(Y))$ we have the following inequality:

$$\int_\Omega \bigg\|\psi(p,\cdot)-\sum_{j=1}^m \alpha_j(p)\psi_j(\cdot)\bigg\|^2_{\mathbb{C}_{\#,\mathbb{H}}(Y)}\,\mathrm{d}p<\eta, \tag{30}$$

where $\alpha_j\in L^2(\Omega)$ and ψ_j are continuous Y-periodic functions in $\mathbb{H}^n$. Thus we have:

$$\begin{aligned}&\bigg|\int_\Omega \psi\big(p,\delta_{1/\epsilon}(p)\big)\varphi\big(p,\delta_{1/\epsilon}(p)\big)\,\mathrm{d}p-\frac{1}{|Y|}\int_\Omega\int_Y \psi(p,q)\varphi(p,q)\,\mathrm{d}q\,\mathrm{d}p\bigg|\\ &\quad\leqslant \int_\Omega\bigg|\psi\big(p,\delta_{1/\epsilon}(p)\big)-\sum_{j=1}^m\alpha_j(p)\psi_j\big(\delta_{1/\epsilon}(p)\big)\bigg|\big|\varphi\big(p,\delta_{1/\epsilon}(p)\big)\big|\,\mathrm{d}p\\ &\qquad+\bigg|\int_\Omega\sum_{j=1}^m\alpha_j(p)\psi_j\big(\delta_{1/\epsilon}(p)\big)\varphi\big(p,\delta_{1/\epsilon}(p)\big)\,\mathrm{d}p-\frac{1}{|Y|}\int_\Omega\int_Y\sum_{j=1}^m\alpha_j(p)\psi_j(q)\varphi(p,q)\,\mathrm{d}q\,\mathrm{d}p\bigg|\end{aligned}$$

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$$
\begin{aligned}
&+\frac{1}{|Y|}\int_\Omega\int_Y\left|\sum_{j=1}^m \alpha_j(p)\psi_j(q)-\psi(p,q)\right|\bigl|\varphi(p,q)\bigr|\,\mathrm{d}q\,\mathrm{d}p\\
&=I_1+I_2+I_3.
\end{aligned}
$$

Using (30) we immediately have

$$I_1+I_3<C\eta.$$

On the other hand, the function $\psi_j(\delta_{1/\epsilon}(\cdot))\varphi(\cdot,\delta_{1/\epsilon}(\cdot))$ converges weakly in $L^2(\Omega)$ to $1/|Y|\int_Y\psi_j(q)\varphi(\cdot,q)\,\mathrm{d}q$ as $\epsilon\to 0$ (see for instance [6, page 174]), and since $\alpha_j\in L^2(\Omega)$ we have

$$\int_\Omega\sum_{j=1}^m\alpha_j(p)\psi_j\bigl(\delta_{1/\epsilon}(p)\bigr)\varphi\bigl(p,\delta_{1/\epsilon}(p)\bigr)\,\mathrm{d}p\to\frac{1}{|Y|}\int_\Omega\int_Y\sum_{j=1}^m\alpha_j(p)\psi_j(q)\varphi(p,q)\,\mathrm{d}q\,\mathrm{d}p$$

as $\epsilon\to 0$, which implies $I_2<\eta$ for $\epsilon<\epsilon(\eta)$. This gives the desired result. The proof of (26) and (27) is completely analogous to the one in [1, Lemma 1.3]. In our case one still works with Euclidean cubes, the only difference being that the periodicity of the functions involved is respect to the Heisenberg group.

To prove (iii) notice that $q\to\psi(p,q)\varphi(p,q)$ is continuous and $(\mathbb{H},Y)$-periodic for every $p\in\Omega$, and that $\operatorname{supp}\psi\varphi\subseteq\varphi$. Moreover,

$$\int_\Omega\Bigl(\max_{q\in\bar Y}\bigl|\varphi(p,q)\psi(p,q)\bigr|\Bigr)^2\,\mathrm{d}p\leqslant\max_{\bar\Omega\times\bar Y}|\varphi|^2\int_\Omega\Bigl(\max_{q\in\bar Y}\bigl|\psi(p,q)\bigr|\Bigr)^2\,\mathrm{d}p<\infty$$

so that $\varphi\psi\in L^2(\Omega;\mathbb{C}_{\#,\mathbb{H}}(Y))$, and the proof is completed. □

We now state and prove a crucial compactness result concerning two-scale convergence. The proof is basically the one given in [6].

Theorem 2.27. *Let $\{u_\epsilon\}$ be a bounded sequence in $L^2(\Omega)$.Then there exists a subsequence $\{u_{\epsilon_n}\}$ and a function $u_0\in L^2(\Omega\times Y)$ such that $\{u_{\epsilon_n}\}$ two-scale converges in $\mathbb{H}^n$ to u_0.*

Proof. Let $\psi\in L^2(\Omega;\mathbb{C}_{\#,\mathbb{H}}(Y))$. Then from the boundedness of $\{u_\epsilon\}$, Hölder inequality and Proposition 2.24 we have

$$\left|\int_\Omega u_\epsilon(p)\psi\bigl(p,\delta_{1/\epsilon}(p)\bigr)\,\mathrm{d}h(p)\right|\leqslant C\|\psi\|_{L^2(\Omega;\mathbb{C}_{\#,\mathbb{H}}(Y))},\tag{31}$$

with C not depending on ϵ.

So we can consider u_ϵ as an element U_ϵ of the dual space of $L^2(\Omega;\mathbb{C}_{\#,\mathbb{H}}(Y))$. From (31) it follows

$$
\begin{aligned}
\|U_\epsilon\|_{(L^2(\Omega;\mathbb{C}_{\#,\mathbb{H}}(Y)))^*}&=\sup\bigl\{\langle U_\epsilon,\psi\rangle_{(L^2(\Omega;\mathbb{C}_{\#,\mathbb{H}}(Y)))^*,L^2(\Omega;\mathbb{C}_{\#,\mathbb{H}}(Y))}\colon\ \psi\in L^2\bigl(\Omega;\mathbb{C}_{\#,\mathbb{H}}(Y)\bigr),\ \|\psi\|_{L^2(\Omega;\mathbb{C}_{\#,\mathbb{H}}(Y))}\leqslant 1\bigr\}\\
&\leqslant C,
\end{aligned}\tag{32}
$$

and recalling that $L^2(\Omega;\mathbb{C}_{\#,\mathbb{H}}(Y))$ is separable, there exists a subsequence ϵ_n such that

$$U_{\epsilon_n}\rightharpoonup U_0\quad\text{weakly}^*\text{ in }\bigl(L^2\bigl(\Omega;\mathbb{C}_{\#,\mathbb{H}}(Y)\bigr)\bigr)^*.$$

So we have:

$$\langle U_0,\psi\rangle_{(L^2(\Omega;\mathbb{C}_{\#,\mathbb{H}}(Y)))^*,L^2(\Omega;\mathbb{C}_{\#,\mathbb{H}}(Y))}=\lim_{\epsilon_n\to 0}\int_\Omega u_{\epsilon_n}(p)\psi\bigl(p,\delta_{1/\epsilon_n}(p)\bigr)\,\mathrm{d}h(p).\tag{33}$$

On the other hand, from (31) we have

$$\lim_{\epsilon_n\to 0}\left|\int_\Omega u_{\epsilon_n}(p)\psi\bigl(p,\delta_{1/\epsilon_n}(p)\bigr)\,\mathrm{d}h(p)\right|\leqslant C\|\psi\|_{L^2(\Omega\times Y)}.\tag{34}$$

From (34) and (33) it follows the estimate

$$\bigl|\langle U_0,\psi\rangle_{(L^2(\Omega;\mathbb{C}_{\#,\mathbb{H}}(Y)))^*,L^2(\Omega;\mathbb{C}_{\#,\mathbb{H}}(Y))}\bigr|\leqslant C\|\psi\|_{L^2(\Omega\times Y)},\quad\forall\psi\in L^2\bigl(\Omega;\mathbb{C}_{\#,\mathbb{H}}(Y)\bigr).\tag{35}$$

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But the space $L^2(\Omega;\mathbb{C}_{\#,\mathbb{H}}(Y))$ is dense in $L^2(\Omega\times Y)$ (indeed $\mathcal{D}(Y)$ is dense in $L^2(Y)$ which implies $L^2(\Omega;\mathcal{D}(Y))$ dense in $L^2(\Omega;L^2(Y))$. Then the inclusion $L^2(\Omega;\mathcal{D}(Y))\subset L^2(\Omega;\mathbb{C}_{\#,\mathbb{H}}(Y))\subset L^2(\Omega;L^2(Y))$ gives the assertion, since $L^2(\Omega;L^2(Y))=L^2(\Omega\times Y)$). Thus inequality (35) holds for any function $\psi\in L^2(\Omega\times Y)$, and so U_0 can be extended continuously to $L^2(\Omega\times Y)$. But then from the Riesz representation theorem U_0 can be identified with an element $u\in L^2(\Omega\times Y)$ such that

$$\langle U_0,\psi\rangle_{(L^2(\Omega;\mathbb{C}_{\#,\mathbb{H}}(Y)))^*,L^2(\Omega;\mathbb{C}_{\#,\mathbb{H}}(Y))}=\int\limits_{\Omega\times Y} u(p,q)\psi(p,q)\,\mathrm{d}h(p)\,\mathrm{d}h(q). \tag{36}$$

But (36) with (33) gives

$$\lim_{\epsilon_n\to 0}\int\limits_{\Omega} u_{\epsilon_n}(p)\psi\big(p,\delta_{1/\epsilon_n}(p)\big)\,\mathrm{d}h(p)=\int\limits_{\Omega\times Y} u(p,q)\psi(p,q)\,\mathrm{d}h(p)\,\mathrm{d}h(q) \tag{37}$$

which implies the assertion, with $u_0=|Y|u$. □

3. Homogenization in perforated domains

As an example of the two-scale technique applied in the Heisenberg group we will consider homogenization in perforated domains. We will concentrate here on the case of domains perforated with nonisolated holes (the case of isolated holes has been treated in [3,4] essentially using the compensated compactness method).

We recall that Y is the cube $[-1,1[^{2n+1}$ in $\mathbb{R}^{2n+1}$. Let now $Y^*\subset Y$ be a relatively open set such that $\operatorname{int} Y^*$ is connected and the set E^* obtained by $(\mathbb{H},Y)$-periodicity from Y^* is a smooth connected open subset of $\mathbb{R}^{2n+1}$, with ∂E^* that coincides with the set we obtain if $\overline{\partial Y^*\backslash\partial Y}$ is continued by $(\mathbb{H},Y)$-periodicity. We can think of E^* as a perforated domain, usually called the material domain, and of $\mathbb{H}^n\backslash E^*$ as a family of periodic holes, the void domain. If $\Omega\subseteq\mathbb{H}^n$ is a bounded open set, then we define a sequence Ω_ϵ of periodically perforated subdomains as follows:

$$\Omega_\epsilon=\big\{p\in\Omega\colon\ \chi\big(\delta_{1/\epsilon}(p)\big)=1\big\},$$

where χ is the characteristic function of E^*. Clearly the sets Ω_ϵ are $(\mathbb{H},\epsilon Y^*)$-periodic.

We notice that we are not assuming that $Y\backslash Y^*\Subset Y$, so that the holes need not to be isolated.

For instance, consider $n=1$ and let $p=(x,y,t)$ be a point in $\mathbb{H}^1$. Let U_+ and U_-, $U_-\Subset Y\cap\{y=-1\}$, $U_+\Subset Y\cap\{y=1\}$ be open smooth sets such that

$$(0,2,0)\cdot U_-=U_+,$$

and let $Y\backslash Y^*$ be a “pipe" connecting $\overline{U}_-$ and $\overline{U}_+$ and not touching ∂Y outside of $\overline{U}_-$ and $\overline{U}_+$.

Obviously, we need further assumptions on the shape of the pipe near $y=\pm1$ in order to get the regularity of the global pipe obtained by periodicity. Notice that a certain degree of regularity guarantees that Y^* is a 2-Poincaré domain (Theorem 2.15), so that all our previous theory applies.

For instance, we might assume that there exist smooth functions $g=g(x,t)$ and $h=h(y)$ such that the pipe is the restriction to Y of a smooth manifold that can be written as $h(y)=g(x,t)$ when $|y+1|<\delta$, and as $h(y-2)=g(x,t-4x)$ when $|y-1|<\delta$.

The condition $U_-\Subset Y\cap\{y=-1\}$ requires that $g(x,t)=h(-1)$ implies $|x|<1$, $|t|<1$; analogously $U_+\Subset Y\cap\{y=1\}$ requires $g(x,t-4x)=h(-1)$ implies $|t|<1$. For instance, if we strength the first assumption (corresponding to $U_-\Subset Y\cap\{y=-1\}$) by requiring that $g(x,t)=h(-1)$ implies $|x|<\alpha$, $|t|<\beta$ with $\beta+4\alpha<1$, then the second property is automatically satisfied. Indeed if the above condition is satisfied, then in the set $1-\delta<y<1$ the pipe continued by periodicity has the equation $h(y-2)=g(x,t-4x)$, whereas in the set $1<y<1+\delta$, it is given by the family of points $\{(\xi,\eta,\tau)\colon\ \xi=x,\ \eta=y+2,\ \tau=t+4x,\ h(y)=g(x,t)\}=\{h(y-2)=g(x,t-4x)\}$, and clearly, by our choice, the two pieces well fit across $y=1$.

We stress the fact that the above example is only the most elementary we can produce. For instance, we can replace the assumption $(0,2,0)\cdot U_-=U_+$ by $(0,2,2k)\cdot U_-=U_+$ for some $k\in Z$. This means roughly speaking that the second end of the pipe in Y well fits the first end of the pipe we obtain by translating first in the t-direction and then in the y-direction.

Again, more generally, we can imagine a network of pipes originated by periodicity by a system of 3 pipes smoothly connecting opposite regions U_-^x, U_+^x, U_-^y, U_+^y, U_-^t, U_+^t where $(2,0,0)\cdot U_-^x=U_+^x$, $(0,2,0)\cdot U_-^y=U_+^y$, $(0,0,2)\cdot U_-^t=U_+^t$.

To produce a mind picture of the situation we are considering, suppose $U_-=\{(x,-1,t)\colon\ x^2+t^2\leqslant r^2\}$, where $r<1/5$; then a pipe satisfying our assumptions can be obtained by taking a smooth function $\sigma:[-1,1]\to[0,4]$ such that $\sigma\equiv0$ in $[-1,-1+\eta]$, $\sigma\equiv4$ in $[1-\eta,1]$ and the pipe being defined by $x^2+(t-\sigma(y)x)^2\leqslant r^2$, $-1\leqslant y\leqslant1$.

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Consider now a $2n \times 2n$ symmetric matrix $A = A(p,q)$, $p \in \Omega$, $q \in \mathbb{H}^n$, A being $(\mathbb{H}, Y)$-periodic in q for a.e. $p \in \Omega$ and such that the entries of $A = (A)_{i,j}$ are admissible functions in the sense of Proposition 2.24 and

$$\lambda|\xi|^2 \leqslant \langle A(p,q)\xi, \xi\rangle \leqslant \Lambda|\xi|^2 \tag{38}$$

with $\Lambda \geqslant \lambda > 0$, for $\xi \in \mathbb{R}^{2n}$. The map $p \to A(p, \delta_{1/\epsilon}(p))$ can be identified through the canonical moving frame with a section of the vector bundle of symmetric linear endomorphisms of the horizontal fibers.

We put

$$L_\epsilon u = -\operatorname{div}_{\mathbb{H}}(A\nabla_{\mathbb{H}} u) + u \tag{39}$$

or, in coordinates,

$$L_\epsilon u = -\sum_{i,j} Z_j\big(A_{i,j}\big(p, \delta_{1/\epsilon}(p)\big) Z_i u\big) + u.$$

Consider now the following boundary value problem:

$$(P_\epsilon)\begin{cases} L_\epsilon u_\epsilon = f \in L^2(\Omega) & \text{in } \ \Omega_\epsilon, \\ \big\langle A\big(p, \delta_{1/\epsilon}(p)\big)\nabla_{\mathbb{H}} u_\epsilon, n_{\mathbb{H}}(p)\big\rangle_p = 0 & \text{on } \ \partial\Omega_\epsilon \backslash \partial\Omega, \\ u_\epsilon = 0 & \text{on } \ \partial\Omega_\epsilon \cap \partial\Omega\ , \end{cases}$$

or, more precisely, its weak formulation: we look for a function $u_\epsilon \in V_\epsilon$, where V_ϵ is the completion in $W^{1,2}_{\mathbb{H}}(\Omega_\epsilon)$ of the space $\{u \in \mathbb{C}^\infty(\Omega_\epsilon) \cap \mathbb{C}(\overline{\Omega}_\epsilon) \cap W^{1,2}_{\mathbb{H}}(\Omega_\epsilon)\colon\ u \equiv 0 \text{ on } \partial\Omega_\epsilon \cap \partial\Omega\}$, such that for all $\phi \in \mathbb{C}^\infty(\Omega_\epsilon) \cap \mathbb{C}(\overline{\Omega}_\epsilon)$, $\phi \equiv 0$ on $\partial\Omega_\epsilon \cap \partial\Omega$,

$$(PV_\epsilon)\ \int_{\Omega_\epsilon} \langle A_\epsilon \nabla_{\mathbb{H}} u_\epsilon, \nabla_{\mathbb{H}}\phi\rangle\,\mathrm{d}h + \int_{\Omega_\epsilon} u_\epsilon\phi\,\mathrm{d}h = \int_{\Omega_\epsilon} f\phi\,\mathrm{d}h;$$

or, in coordinates,

$$\sum_{i,j}\int_{\Omega_\epsilon} A_{i,j}\big(p, \delta_{1/\epsilon}(p)\big) Z_i u_\epsilon Z_j\phi\,\mathrm{d}h + \int_{\Omega_\epsilon} u_\epsilon\phi\,\mathrm{d}h = \int_{\Omega_\epsilon} f\phi\,\mathrm{d}h.$$

Clearly, problem (PV_ϵ) has a unique solution, by Lax–Milgram theorem. Moreover,

$$\|u_\epsilon\|_{W^{1,2}_{\mathbb{H}}(\Omega_\epsilon)} \leqslant C\|f\|_{L^2(\Omega_\epsilon)} \leqslant C\|f\|_{L^2(\Omega)} \tag{40}$$

for any $\epsilon \in (0,1)$. Following [1, Theorem 2.9], put $\tilde{u}_\epsilon$ and $\widetilde{\nabla_{\mathbb{H}} u_\epsilon}$ the continuation respectively of u_ϵ and $\nabla_{\mathbb{H}} u_\epsilon$ by zero in $\Omega\backslash\Omega_\epsilon$. We have:

Theorem 3.1. *Suppose E^* is a connected open set of 2-Poincaré type. The following two-scale convergences hold*:

(1) $\tilde{u}_\epsilon \to u(p)\chi(q)$;
(2) $\widetilde{\nabla_{\mathbb{H}} u_\epsilon} \to \chi(q)\big(\nabla_{\mathbb{H}} u(p) + \nabla_{\mathbb{H},q} u_1(p,q)\big)$,

where $\chi = 1_{E^*}$, and (u, u_1) is the unique variational solution in the space

$$\mathring{W}^{1,2}_{\mathbb{H}}(\Omega) \times L^2\big(\Omega; W^{1,2}_{\#,\mathbb{H}}(Y, E^*)/\mathbb{R}\big)$$

of the following two-scale homogenized problem:

$$\begin{cases} -\operatorname{div}_{\mathbb{H},p} \displaystyle\int_{Y^*} A(p,q)\big(\nabla_{\mathbb{H}} u(p) + \nabla_{\mathbb{H},q} u_1(p,q)\big)\,\mathrm{d}h(q) + |Y^*|u(p) = |Y^*|f(p) & \textit{a.e. in } \ \Omega, \\ -\operatorname{div}_{\mathbb{H},q}\big[A(p,q)\big(\nabla_{\mathbb{H}} u(p) + \nabla_{\mathbb{H},q} u_1(p,q)\big)\big] = 0 & \textit{a.e. in } \ \Omega \times Y^*, \\ \big\langle A(p,q)\big(\nabla_{\mathbb{H}} u(p) + \nabla_{\mathbb{H},q} u_1(p,q)\big), n_{\mathbb{H}}\big\rangle = 0 & \textit{on } \ \partial Y^* \backslash \partial Y. \end{cases}$$

Remark 3.2. If we drop the assumption E^* is a set of 2-Poincaré type, then the assertion still holds with

$$u_1 \in L^2\big(\Omega; L^2_{\#,\mathbb{H},\mathrm{loc}}(Y, E^*)/\mathbb{R}\big) \quad \text{and} \quad \nabla_{\mathbb{H},q} u_1 \in L^2\big(\Omega; L^2_{\#,\mathbb{H}}(Y, E^*)\big).$$

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Proof of Theorem 3.1. By (40) and Theorem 2.27, there exists a subsequence $\{\epsilon_j\}_{j\in\mathbb{N}}$ such that, as $j\to\infty$,

$$\tilde{u}_{\epsilon_j}\to u_0(p,q)\in L^2(\Omega\times Y)\quad\text{and}\quad \widetilde{\nabla_{\mathbb{H}}u_{\epsilon_j}}\to \xi_0(p,q)\in L^2\big(\Omega;L^2(Y,\mathrm{H}\mathbb{H}^n)\big)$$

two scales. We claim that $u_0(p,\cdot)\equiv 0$, $\xi_0(p,\cdot)\equiv 0$ in $Y_0\backslash\overline{Y^*}$ for a.e. $p\in\Omega$. To prove this, it is enough to choose in Definition 2.22 $\psi(p,q)=g(p)h(q)$, where $g\in\mathcal{D}(\Omega)$ and $h\in\mathbb{C}_0^\infty(Y_0\backslash\overline{Y^*})$; without loss of generality we can think of h as continued by $(\mathbb{H},Y)$-periodicity to all of $\mathbb{H}^n$ so that $\operatorname{supp} h\subseteq(\overline{E^*})^c$. Then, keeping into account that $\tilde{u}_\epsilon\equiv 0$ and $\widetilde{\nabla_{\mathbb{H}}u_\epsilon}\equiv 0$ in $\Omega\backslash\Omega_\epsilon$, and then for instance $\tilde{u}_\epsilon(p)=\tilde{u}_\epsilon(p)\chi(\delta_{1/\epsilon}(p))$, where χ is the characteristic function of E^*, we have

$$0=\int_\Omega u_\epsilon(p)g(p)h\big(\delta_{1/\epsilon}(p)\big)\,\mathrm{d}p\to\frac{1}{|Y|}\int_\Omega\bigg(\int_Y u_0(p,q)h(q)\,\mathrm{d}q\bigg)g(p)\,\mathrm{d}p.$$

Since the set $\{g\in\mathbb{C}_0^\infty(Y_0\backslash\overline{Y^*})\}$ is dense in $L^2(Y_0\backslash\overline{Y^*})$, which is separable, we can extract a countable subfamily $\mathcal{C}\subseteq C_0^\infty(Y_0\backslash\overline{Y^*})$ that is dense in $L^2(Y_0\backslash\overline{Y^*})$. Since $\int_\Omega(\int_Y u_0(p,q)h(q)\,\mathrm{d}q)g(p)\,\mathrm{d}p=0$ for any $g\in\mathcal{D}(\Omega)$ and for any $h\in\mathcal{C}$, we can conclude that $u_0(p,\cdot)\equiv 0$ on $Y_0\backslash\overline{Y^*}$ for a.e. $p\in\Omega$. An analogous argument works for ξ_0.

We want to show now that u_0 and ξ_0 are as in (1) and (2), respectively. To this end, it turns out simpler to work in coordinates, i.e., to identify sections of $\mathrm{H}\mathbb{H}^n$ with their canonical $2n$-dimensional coordinates. Thus, take

$$\psi=\psi(p,q)\in\big(\mathcal{D}\big(\Omega;\mathbb{C}^\infty_{\#,\mathbb{H}}(Y)\big)\big)^{2n},\qquad \operatorname{supp}\psi(p,\cdot)\subseteq E^*\quad\text{for } p\in\Omega.$$

By definition, $\operatorname{supp}\psi(\cdot,\delta_{1/\epsilon}(\cdot))\subseteq\Omega_\epsilon$, and we can write

$$\int_{\Omega_\epsilon}\big\langle\nabla_{\mathbb{H}}u_\epsilon(p),\psi\big(p,\delta_{1/\epsilon}(p)\big)\big\rangle_{\mathbb{R}^{2n}}\,\mathrm{d}p=-\int_{\Omega_\epsilon}u_\epsilon(p)\bigg[(\operatorname{div}_{\mathbb{H},p}\psi)\big(p,\delta_{1/\epsilon}(p)\big)+\frac{1}{\epsilon}(\operatorname{div}_{\mathbb{H},q}\psi)\big(p,\delta_{1/\epsilon}(p)\big)\bigg]\,\mathrm{d}p.\tag{41}$$

Since $\operatorname{div}_{\mathbb{H},p}\psi$ and $\operatorname{div}_{\mathbb{H},q}\psi$ are admissible test functions, taking the limit as $\epsilon\to 0$ we get

$$\int_\Omega\mathrm{d}p\int_{Y^*}\mathrm{d}q\,u_0(p,q)\operatorname{div}_{\mathbb{H},q}\psi(p,q)=0.$$

Choose now $\psi=g(p)h(q)$, $h=(h_1,\dots,h_{2n})$ where g is arbitrary in $\mathcal{D}(\Omega)$ and $h\in(\mathbb{C}^\infty_{\#,\mathbb{H}}(Y))^{2n}$, $\operatorname{supp} h\cap Y\subseteq\operatorname{int}(Y^*)$. The map $p\to\int_{Y^*}u_0(p,q)\operatorname{div}_{\mathbb{H},q}\psi(p,q)\,\mathrm{d}q$ belongs to $L^2(\Omega)$, and then, arguing as above it follows that

$$\int_{Y^*}u_0(p,q)\operatorname{div}_{\mathbb{H}}h(q)\,\mathrm{d}q=0\tag{42}$$

for a.e. $p\in\Omega$.

In turn, this implies that $u_0(p,\cdot)$ is constant on Y^* for a.e. $p\in\Omega$. This statement is well known in the Euclidean setting, but it deserves few further words in our case. Indeed (42) implies that $X_ju_0(p,\cdot)\equiv 0$ and $Y_ju_0(p,\cdot)\equiv 0$ in $\operatorname{int}(Y^*)$, $j=1,\dots,n$, in the sense of distributions. By Theorem 6.4 in [10] this implies $u_0(p,\cdot)\in W^{1,2}_{\mathbb{H}}(\operatorname{int}(Y^*))$, and then that $u_0(p,\cdot)$ is locally constant, by Poincaré inequality. Since $\operatorname{int}(Y^*)$ is connected, the assertion follows. Thus, we can assume $u_0(p,q)=u(p)\chi(q)$, where χ is the characteristic function of Y^*. By the way, $u\in L^2(\Omega)$.

The following lemma will provide the tool to achieve the proof.

Lemma 3.3. *For any $\theta\in(L^2(\Omega))^{2n}$ there exists $\psi\in L^2(\Omega,V)$ such that*

$$\int_{Y^*}\psi(p,q)\,\mathrm{d}h(q)=\theta(p),\quad p\in\Omega,\qquad\|\psi\|_{L^2(\Omega,V)}\leqslant C\|\theta\|_{(L^2(\Omega))^{2n}}.$$

More precisely, $\psi(p,q)=\sum_{j=1}^{2n}\langle B^{-1}\theta(p),e_j\rangle v_j(q)$, where B is a positive definite constant matrix and $v_j\in V$, $j=1,\dots,2n$. Hence, if $\theta\in\mathcal{D}(\Omega)^{2n}$ then $\psi\in\mathcal{D}(\Omega;V)^{2n}$.

Proof. We consider the bilinear form on $V=V(Y,E^*)$:

$$Q(v,\varphi)=\sum_{j=1}^{2n}\int_{Y^*}\big\langle\nabla_{\mathbb{H}}v^j,\nabla_{\mathbb{H}}\varphi^j\big\rangle\,\mathrm{d}h,$$

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where $v=(v^1,\dots,v^{2n})$, $\varphi=(\varphi^1,\dots,\varphi^{2n})$, and the linear functionals on V:

$$L_i(\varphi)=\int_{Y^*}\langle e_i,\varphi\rangle\,\mathrm{d}h,\quad i=1,\dots,2n,$$

where e_i is the ith vector of the canonical orthonormal basis of $\mathbb{R}^{2n}$. We can prove that each of the problems

$$Q(v,\varphi)=L_i(\varphi),\quad i=1,\dots,2n,\tag{43}$$

has a unique solution $v_i=(v_i^1,\dots,v_i^{2n})$ in V by the Lax–Milgram lemma. To this end, let us prove that the form Q is coercive on V. For this it will be enough to show that

$$\int_{Y^*}|\nabla_{\mathbb{H}}v|^2\,\mathrm{d}h\geqslant c\int_{Y^*}|v|^2\,\mathrm{d}h\quad\text{for all } v\in\mathbb{C}^\infty_{\#,\mathbb{H}}(Y),\ \operatorname{supp}v\subseteq E^*.\tag{44}$$

By Proposition 2.1 we have $Y\subseteq U(0,\sqrt{n}\,)\subseteq U_C(0,\sqrt{n}/c)$. Let us prove preliminarily that $U_C(0,\sqrt{n}/c)$ overlaps only a finite number of the tiles on the $(\mathbb{H},Y)$-periodic pavage. Indeed, if $p\in 2k\cdot Y$, then $p=2k\cdot q$, with $q\in Y$, and we have

$$\begin{aligned}d_C(p,0)\ &\geqslant\ d_C(2k,0)-d_C(2k\cdot q,2k)=d_C(2k,0)-d_C(q,0)\geqslant\frac{1}{c}d(2k,0)-\frac{\sqrt{n}}{c}\\&=\ \frac{1}{c}\max\bigl\{2\bigl|(k_1,\dots,k_{2n})\bigr|,\sqrt{2}\sqrt{k_{2n+1}}\bigr\}-\frac{\sqrt{n}}{c}\geqslant\frac{\sqrt{2}}{c}\max\bigl\{\sqrt{k_1},\dots,\sqrt{k_{2n}},\sqrt{k_{2n+1}}\bigr\}-\frac{\sqrt{n}}{c}>\frac{\sqrt{n}}{c}\end{aligned}$$

provided $\max\{\sqrt{k_1},\dots,\sqrt{k_{2n}},\sqrt{k_{2n+1}}\}\geqslant 2\sqrt{n/2}$, so that $U_C(0,\sqrt{n}/c)\cap 2k\cdot Y$ is nonempty only for a finite family of k. Denote now by m the average of v on $U_C(0,\sqrt{n}/c)$; by [15], we have

$$\int_{U_C(0,\sqrt{n}/c)}|v-m|^2\,\mathrm{d}h\leqslant c_n^2\int_{U_C(0,\sqrt{n}/c)}|\nabla_{\mathbb{H}}v|^2\,\mathrm{d}h,$$

which implies

$$c_n^2\int_{U_C(0,\sqrt{n}/c)}|\nabla_{\mathbb{H}}v|^2\,\mathrm{d}h\geqslant\int_Y|v-m|^2\,\mathrm{d}h\geqslant\int_{Y^*}|v-m|^2\,\mathrm{d}h.$$

On the other hand, keeping in mind that $E^*\cap Y=Y^*$ and $v\equiv 0$ on $Y\setminus Y^*$, we have,

$$c_n^2\int_{U_C(0,\sqrt{n}/c)}|\nabla_{\mathbb{H}}v|^2\,\mathrm{d}h\geqslant\int_{Y\setminus Y^*}|v-m|^2\,\mathrm{d}h=m^2\bigl(|Y|-|Y^*|\bigr),$$

so that, by $(Y,\mathbb{H})$-periodicity, we have

$$\begin{aligned}\Bigl(\int_{Y^*}|v|^2\,\mathrm{d}h\Bigr)^{1/2}\ &\leqslant\ \Bigl(\int_{Y^*}|v-m|^2\,\mathrm{d}h\Bigr)^{1/2}+|Y^*|^{1/2}m\leqslant c_n\Biggl(1+\Bigl(\frac{|Y^*|}{|Y|-|Y^*|}\Bigr)^{1/2}\Biggr)\Bigl(\int_{U_C(0,\sqrt{n}/c)}|\nabla_{\mathbb{H}}v|^2\,\mathrm{d}h\Bigr)^{1/2}\\&\leqslant\ c\sum_{(2k\cdot Y)\cap U_C(0,\sqrt{n}/c)\neq\emptyset}\Bigl(\int_{2k\cdot Y}|\nabla_{\mathbb{H}}v|^2\,\mathrm{d}h\Bigr)^{1/2}=c\Bigl(\int_Y|\nabla_{\mathbb{H}}v|^2\,\mathrm{d}h\Bigr)^{1/2}=c\Bigl(\int_{Y^*}|\nabla_{\mathbb{H}}v|^2\,\mathrm{d}h\Bigr)^{1/2}.\end{aligned}$$

Consider now the constant $2n\times 2n$ matrix B defined by

$$B_{ij}=\int_{Y^*}\sum_{k=1}^{2n}\bigl\langle\nabla_{\mathbb{H}}v_i^k,\nabla_{\mathbb{H}}v_j^k\bigr\rangle\,\mathrm{d}h,$$

where v_i, $i=1,\dots,2n$, are the solutions of the above mentioned problems. Let us prove that $B_{ij}>0$. To this end take $\xi=(\xi_1,\dots,\xi_n)\in\mathbb{R}^n$. We have

$$\begin{aligned}\sum_{ij}B_{ij}\xi_i\xi_j\ &=\ \int_{Y^*}\sum_{k=1}^{2n}\sum_{ij}\bigl\langle\nabla_{\mathbb{H}}\xi_iv_i^k,\nabla_{\mathbb{H}}\xi_jv_j^k\bigr\rangle\,\mathrm{d}h=\int_{Y^*}\sum_{k=1}^{2n}\Bigl\langle\nabla_{\mathbb{H}}\Bigl(\sum_i\xi_iv_i\Bigr)^k,\nabla_{\mathbb{H}}\Bigl(\sum_j\xi_jv_j\Bigr)^k\Bigr\rangle\,\mathrm{d}h\\&=\ Q\Bigl(\sum_i\xi_iv_i,\sum_j\xi_jv_j\Bigr)\geqslant 0,\end{aligned}$$

and $\sum_{ij} B_{ij}\xi_i\xi_j = 0$ if and only if $\sum_i \xi_i v_i \equiv 0$. Suppose now $\sum_i \xi_i v_i \equiv 0$, and take $\varphi \in V$. Then

$$\int_{Y^*} \left\langle \sum_i \xi_i e_i, \varphi \right\rangle \mathrm{d}h = \sum_i \xi_i \int_{Y^*} \langle e_i, \varphi\rangle \,\mathrm{d}h = \sum_i \xi_i Q(v_i, \varphi) = Q\left(\sum_i \xi_i v_i, \varphi\right) = 0.$$

Hence, by Lemma 2.20, $\sum_i \xi_i e_i = 0$, that in turn implies $\xi = 0$. Thus $\sum_{ij} B_{ij}\xi_i\xi_j > 0$, when $\xi \neq 0$; in particular the matrix B is invertible and $\|B^{-1}\| < \infty$.

Then, for given $\theta \in (L^2(\Omega))^{2n}$, the function

$$\psi(p,q) = \sum_{j=1}^{2n} \langle B^{-1}\theta(p), e_j\rangle v_j(q) \tag{45}$$

satisfies all the required properties. Indeed, concerning the first property we have:

$$\begin{aligned} \left\langle \int_{Y^*} \psi(p,q)\,\mathrm{d}q, e_i \right\rangle &= \sum_{j=1}^{2n} \langle B^{-1}\theta(p), e_j\rangle \left\langle \int_{Y^*} v_j(q)\,\mathrm{d}q, e_i \right\rangle \quad \left(\text{noticing that } B_{ij} = \left\langle e_i, \int_{Y^*} v_j(q)\,\mathrm{d}q \right\rangle\right) \\ &= \sum_{j=1}^{2n} \langle B^{-1}\theta(p), e_j\rangle B_{ji} = \sum_{l,j,k} \left(B^{-1}\right)_{kl} \theta_l(p) e_j^k B_{ji} = \sum_{l,j} \left(B^{-1}\right)_{jl} B_{ji}\theta_l = \sum_{l,j} \left(B^{-1}\right)_{lj} B_{ji}\theta_l \\ &= \sum_l \theta_l \delta_{li} = \theta_i. \end{aligned}$$

Concerning the second property, notice that $c\|v_j\|_V^2 \leqslant Q(v_j, v_j) = L_j(v_j) \leqslant \|v_j\|_V$, so that $\|v_j\|_V \leqslant C$, and hence we have:

$$\begin{aligned} \left(\int_\Omega \|\psi(p,\cdot)\|_V^2 \,\mathrm{d}p\right)^{1/2} &= \left(\int_\Omega \left\| \sum_j^{2n} \langle B^{-1}\theta(p), e_j\rangle v_j(\cdot) \right\|_V^2 \mathrm{d}p\right)^{1/2} \leqslant C\left(\int_\Omega \sum_j^{2n} \left|\langle B^{-1}\theta(p), e_j\rangle\right|^2 \|v_j(\cdot)\|_V^2 \,\mathrm{d}p\right)^{1/2} \\ &\leqslant C\left(\int_\Omega \left\|B^{-1}\right\|^2 \left|\theta(p)\right|^2 \mathrm{d}p\right)^{1/2} \leqslant C\|\theta\|_{(L^2(\Omega))^{2n}}. \qquad \square \end{aligned}$$

Take now $\psi \in \mathcal{D}(\Omega; \mathcal{V})$, so that $\operatorname{supp}\psi(p,\cdot) \subseteq E^*$ for $p \in \Omega$, $\operatorname{div}_{\mathbb{H},q}\psi(p,\cdot) \equiv 0$. From (41) we get

$$\int_{\Omega_\epsilon} \left\langle \nabla_{\mathbb{H}} u_\epsilon(p), \psi\left(p, \delta_{1/\epsilon}(p)\right)\right\rangle_{2n} \mathrm{d}p = -\int_{\Omega_\epsilon} u_\epsilon(p) \operatorname{div}_{\mathbb{H},p} \psi\left(p, \delta_{1/\epsilon}(p)\right) \mathrm{d}p.$$

Taking the two-scale limit, and taking into account what we proved about ξ_0 and u_0, we obtain

$$\int_\Omega \int_{Y^*} \left\langle \xi_0(p,q), \psi(p,q)\right\rangle_{2n} \mathrm{d}q\,\mathrm{d}p = -\int_\Omega u(p) \operatorname{div}_{\mathbb{H},p}\left(\int_{Y^*} \psi(p,q)\,\mathrm{d}q\right) \mathrm{d}p. \tag{46}$$

In particular, if $\psi = \sum_{i=1}^m \alpha_i(p)\beta_i(q)$, $m \in \mathbb{N}$, with $\alpha_i \in \mathcal{D}(\Omega)$ and $\beta_i \in \mathcal{V}$ for $i = 1, \ldots m$, (46) takes the form

$$\int_\Omega \int_{Y^*} \left\langle \xi_0(p,q), \psi(p,q)\right\rangle_{2n} \mathrm{d}q\,\mathrm{d}p = -\sum_i \int_\Omega u(p) \left\langle \nabla_{\mathbb{H}} \alpha_i(p), \int_{Y^*} \beta_i(q)\,\mathrm{d}q \right\rangle_{2n} \mathrm{d}p. \tag{47}$$

Clearly, by density, (47) still holds for $\beta_i \in V$, $i = 1, \ldots, m$. Thus, by Lemma 3.3 (45), if $\theta \in \mathcal{D}(\Omega)^{2n}$ identity (46) still holds for the function ψ associated with θ as in (45). Since $\theta = \int_{Y^*} \psi(p,q)\,\mathrm{d}q$, we get

$$\int_\Omega \int_{Y^*} \left\langle \xi_0(p,q), \psi(p,q)\right\rangle_{2n} \mathrm{d}q\,\mathrm{d}p = -\int_\Omega u(p) \operatorname{div}_{\mathbb{H}} \theta(p)\,\mathrm{d}p. \tag{48}$$

Hence, since for all $p \in \Omega$, $\psi(p,\cdot) \in V$, then

$$\begin{aligned} \left|\int_\Omega u(p) \operatorname{div}_{\mathbb{H}} \theta(p)\,\mathrm{d}p\right| &\leqslant \|\xi_0\|_{(L^2(\Omega\times Y^*))^{2n}} \|\psi\|_{(L^2(\Omega\times Y^*))^{2n}} \leqslant \|\xi_0\|_{(L^2(\Omega\times Y^*))^{2n}} \|\psi\|_{L^2(\Omega,V)} \\ &\leqslant C\|\xi_0\|_{(L^2(\Omega\times Y^*))^{2n}} \|\theta\|_{(L^2(\Omega))^{2n}}. \end{aligned}$$

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Thus, u can be identified with a linear continuous map S_u from $\operatorname{div}(\mathcal{D}(\Omega)^{2n}) \subseteq (\mathring{W}^{1,2}_{\mathbb{H}}(\Omega))^*$ to $\mathbb{R}$. Since $\operatorname{div}(\mathcal{D}(\Omega)^{2n})$ is dense in $(\mathring{W}^{1,2}_{\mathbb{H}}(\Omega))^*$, then $S_u \in (\mathring{W}^{1,2}_{\mathbb{H}}(\Omega))^{**}$, and hence $u \in \mathring{W}^{1,2}_{\mathbb{H}}(\Omega)$.

Take now $\beta \in \mathring{E}_{\#,\mathbb{H}}(Y, A)$, with $\operatorname{div}\beta = 0$ and let $\tilde{\beta}$ be as in Proposition 2.19. Obviously $\operatorname{div}\tilde{\beta} = 0$. If $\alpha \in \mathcal{D}(\Omega)$, set $\psi(p,q) = \alpha(p)\tilde{\beta}(q)$, and let $\beta_n \in (\mathbb{C}^{\infty}_{\#,\mathbb{H}}(Y))^{2n}$ be such that $\beta_n \to \tilde{\beta}$ in $\mathring{E}_{\#,\mathbb{H}}(Y)$. In (41) take

$$\psi(p,q) = \psi_n(p,q) = \alpha(p)\,\beta_n(q)$$

and get

$$\begin{aligned}\int_{\Omega_\epsilon} \alpha(p)\big\langle \nabla_{\mathbb{H}} u_\epsilon(p), \beta_n\big(\delta_{1/\epsilon}(p)\big)\big\rangle\,\mathrm{d}p = {} & -\int_{\Omega_\epsilon} u_\epsilon(p)\big\langle \nabla_{\mathbb{H}}\alpha(p), \beta_n\big(\delta_{1/\epsilon}(p)\big)\big\rangle\,\mathrm{d}p \\ & + \frac{1}{\epsilon}\int_{\Omega_\epsilon} u_\epsilon(p)\alpha(p)\big(\operatorname{div}_{\mathbb{H}}\beta_n\big)\big(\delta_{1/\epsilon}(p)\big)\,\mathrm{d}p.\end{aligned} \tag{49}$$

We want to show that the above identity still holds with β_n replaced by $\tilde{\beta}$. To this end remember that $\beta_n \to \tilde{\beta}$, $\operatorname{div}\beta_n \to \operatorname{div}\tilde{\beta}$ in $(L^2(\Omega))^{2n}$ and $L^2(\Omega)$, respectively, and notice that $\alpha|\nabla_{\mathbb{H}} u_\epsilon|$, $u_\epsilon|\nabla_{\mathbb{H}}\alpha|$, $u_\epsilon\alpha$ belong to $L^2(\Omega)$. Since $\operatorname{div}\tilde{\beta} = 0$, we get

$$\int_{\Omega_\epsilon} \alpha(p)\big\langle \nabla_{\mathbb{H}} u_\epsilon(p), \tilde{\beta}\big(\delta_{1/\epsilon}(p)\big)\big\rangle\,\mathrm{d}p = -\int_{\Omega_\epsilon} u_\epsilon(p)\big\langle \nabla_{\mathbb{H}}\alpha(p), \tilde{\beta}\big(\delta_{1/\epsilon}(p)\big)\big\rangle\,\mathrm{d}p.$$

By Proposition 2.19, we can take now the limit as $\epsilon \to 0^+$ and we get

$$\begin{aligned}\int_{\Omega}\int_{Y^*} \big\langle \xi_0(p,q), \alpha(p)\beta(q)\big\rangle\,\mathrm{d}q\,\mathrm{d}p &= -\int_{\Omega}\int_{Y^*} u(p)\big\langle \nabla_{\mathbb{H}}\alpha(p), \beta(q)\big\rangle\,\mathrm{d}q\,\mathrm{d}p \quad \big(\text{since } \operatorname{supp}\tilde{\beta} \subseteq Y^*\big) \\ &= -\int_{\Omega}\int_{Y^*} u(p)\operatorname{div}_{\mathbb{H},p}\big(\alpha(p)\beta(q)\big)\,\mathrm{d}q\,\mathrm{d}p = \int_{\Omega}\int_{Y^*} \big\langle \nabla_{\mathbb{H}} u(p), \alpha(p)\beta(q)\big\rangle\,\mathrm{d}q\,\mathrm{d}p.\end{aligned}$$

Hence,

$$\int_{\Omega} \alpha(p)\bigg(\int_{Y^*} \big\langle \xi_0(p,q) - \nabla_{\mathbb{H}} u(p), \beta(q)\big\rangle\,\mathrm{d}q\bigg)\,\mathrm{d}p = 0. \tag{50}$$

With $p \in \Omega$ we can associate $F(p) \in (\mathring{E}_{\#,\mathbb{H}}(Y, E^*))^*$ given by

$$F(p)(\varphi) = \int_{Y^*} \big\langle \xi_0(p,q) - \nabla_{\mathbb{H}} u(p), \varphi(q)\big\rangle\,\mathrm{d}q$$

for $\varphi \in \mathring{E}_{\#,\mathbb{H}}(Y, E^*)$. Indeed

$$\begin{aligned}\big|F(p)(\varphi)\big| &\leqslant \int_{Y^*} \big(\big|\xi_0(p,q)\big| + \big|\nabla_{\mathbb{H}} u(p)\big|\big)\big|\varphi(q)\big|\,\mathrm{d}q \leqslant \bigg(\bigg(\int_{Y^*} \big|\xi_0(p,q)\big|^2\,\mathrm{d}q\bigg)^{1/2} + |Y^*|^{1/2}\big|\nabla_{\mathbb{H}} u(p)\big|\bigg)\|\varphi\|_{L^2_{\#,\mathbb{H}}(Y,E^*)} \\ &\leqslant \bigg(\bigg(\int_{Y^*} \big|\xi_0(p,q)\big|^2\,\mathrm{d}q\bigg)^{1/2} + |Y^*|^{1/2}\big|\nabla_{\mathbb{H}} u(p)\big|\bigg)\|\varphi\|_{\mathring{E}_{\#,\mathbb{H}}(Y,E^*)},\end{aligned}$$

and

$$\big\|F(p)\big\|_{(\mathring{E}_{\#,\mathbb{H}}(Y,E^*))^*} \leqslant \bigg(\bigg(\int_{Y^*} \big|\xi_0(p,q)\big|^2\,\mathrm{d}q\bigg)^{1/2} + |Y^*|^{1/2}\big|\nabla_{\mathbb{H}} u(p)\big|\bigg) < \infty$$

for a.e. $p \in \Omega$. Moreover,

$$\int_{\Omega} \big\|F(p)\big\|^2_{(\mathring{E}_{\#,\mathbb{H}}(Y,E^*))^*}\,\mathrm{d}p \leqslant 2\big(\|\xi_0\|^2_{L^2(\Omega\times Y^*)} + |Y^*|\|u\|_{\mathring{W}^{1,2}_{\#,\mathbb{H}}(Y,E^*)}\big).$$

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Thus (50) reads as

$$\int_\Omega F(p)\big(\alpha(p)\beta\big)\,\mathrm{d}p = 0 \tag{51}$$

for any $\alpha \in \mathcal{D}(\Omega)$ and $\beta \in \mathring{E}_{\#,\mathbb{H}}(Y, E^*)$ with $\operatorname{div}\beta = 0$.

Since the map

$$\alpha \to \int_\Omega F(p)\big(\alpha(p)\beta\big)\,\mathrm{d}p$$

is continuous when $\alpha \in L^2(\Omega)$, then (51) still holds when α is a characteristic function, so that

$$\int_\Omega F(p)\big(\psi(p,q)\big)\,\mathrm{d}p = 0 \tag{52}$$

for any $\psi = \sum_{j=1}^N \chi_{E_j}(p)\beta_j(q)$, with $\beta_j \in \mathring{E}_{\#,\mathbb{H}}(Y, E^*)$, $\operatorname{div}\beta_j = 0$, $j = 1, \ldots, N$. On the other hand, these functions are dense in $L^2(\Omega; H)$, where $H = \{\beta \in \mathring{E}_{\#,\mathbb{H}}(Y, E^*)\colon \operatorname{div}\beta = 0\}$, and hence (52) holds with $\psi \in L^2(\Omega; H)$.

By Theorem 2.16, there exists $u_1 = u_1(p,q) \in L^2(\Omega; L^2_{\#,\mathbb{H}}(Y, E^*)/\mathbb{R})$ such that

$$\xi_0(p,q) - \nabla_{\mathbb{H}} u(p) = \nabla_{\mathbb{H},q} u_1(p,q) \text{ on } E^*.$$

Hence $u_1 \in L^2(\Omega; W^{1,2}_{\#,\mathbb{H}}(Y, E^*)/\mathbb{R})$ and since ξ_0 is supported in Y^*, then

$$\xi_0(p,q) = \chi(q)\big(\nabla_{\mathbb{H}} u(p) + \nabla_{\mathbb{H},q} u_1(p,q)\big).$$

(If E^* is not of 2-Poincaré type, then u_1 is only in $L^2(\Omega; L^2_{\#,\mathbb{H},\mathrm{loc}}(Y, E^*)/\mathbb{R})$.)

We can conclude now as in [1]: in (PV_ϵ), choose ϕ of the form $\phi(p) + \epsilon\phi_1(p, \delta_{1/\epsilon}(p))$, where $\phi \in \mathcal{D}(\Omega)$ and $\phi_1 \in \mathcal{D}(\Omega; \mathbb{C}^\infty_{\#,\mathbb{H}}(Y))$. Since the vector fields Z_j are homogeneous with respect to the dilations δ_λ, we can conclude that

$$\begin{aligned}&\sum_{i,j}\int_\Omega\int_{Y^*} A_{i,j}(p,q)\big(Z_i u(p) + Z_{i,q} u_1(p,q)\big)\big(Z_j\phi(p) + Z_{j,q}\phi_1(p,q)\big)\,\mathrm{d}p\,\mathrm{d}q + |Y^*|\int_\Omega u(p)\phi(p)\,\mathrm{d}p\\ &\quad = |Y^*|\int_\Omega f(p)\phi(p)\,\mathrm{d}p.\end{aligned}$$

Thus, the pair $(u, u_1) \in \mathring{W}^{1,2}_{\mathbb{H}}(\Omega) \times L^2(\Omega; W^{1,2}_{\#,\mathbb{H}}(Y, E^*)/\mathbb{R})$ satisfies

$$\sum_{i,j}\int_\Omega\left(\int_{Y^*} A_{i,j}(p,q)\big(Z_i u(p) + Z_{i,q} u_1(p,q)\big)\,\mathrm{d}q\right) Z_j\phi(p)\,\mathrm{d}p + |Y^*|\int_\Omega u(p)\phi(p)\,\mathrm{d}p = |Y^*|\int_\Omega f(p)\phi(p)\,\mathrm{d}p$$

and

$$\sum_{i,j}\int_\Omega\int_{Y^*} A_{i,j}(p,q)\big(Z_i u(p) + Z_{i,q} u_1(p,q)\big) Z_{j,q}\phi_1(p,q)\,\mathrm{d}q\,\mathrm{d}p = 0$$

so that (u, u_1) is the variational solution of the system

$$-\operatorname{div}_{\mathbb{H},p}\int_{Y^*} A(p,q)\big(\nabla_{\mathbb{H}} u(p) + \nabla_{\mathbb{H},q} u_1(p,q)\big)\,\mathrm{d}q + |Y^*|u(p) = |Y^*|f(p) \quad \text{a.e. in } \Omega$$

and

$$\begin{cases} -\operatorname{div}_{\mathbb{H},q}\big[A(p,q)\big(\nabla_{\mathbb{H}} u(p) + \nabla_{\mathbb{H},q} u_1(p,q)\big)\big] = 0 & \text{a.e. in } \Omega \times Y^*,\\ \big\langle A(p,q)\big(\nabla_{\mathbb{H}} u(p) + \nabla_{\mathbb{H},q} u_1(p,q)\big), n_{\mathbb{H}}\big\rangle = 0 & \text{on } \partial Y^* \setminus \partial Y. \end{cases} \qquad \square$$

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