On the Cahn-Hilliard equations with dynamic boundary conditions and irregular potentials

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Abstract

Several papers regard the Cahn-Hilliard equations

$$\partial_t u - \Delta w = 0 \quad \text{in } \Omega, \qquad \partial_n w = 0 \text{ on } \Gamma := \partial\Omega \tag{1}$$

$$w = -\Delta u + \mathcal{W}'(u) - f \quad \text{in } \Omega, \qquad \text{some B.C. and I.C. for } u \tag{2}$$

where $\mathcal{W}$ is a double-well potential, typically $\mathcal{W}(r) = (r^2-1)^2$ for $r \in \mathbf{R}$. Moreover, even the case of singular potentials (e.g., with constraints) has already been considered in the literature, as well as some systems obtained by coupling the above equations to other ones (e.g., generalized heat equations and elasticity systems). However, the results that are known generally deal with the classical boundary condition $\partial_n u = 0$ for u. Recently, the dynamic boundary condition

$$\partial_t u + \partial_n u - \Delta_\Gamma u + \mathcal{W}_\Gamma'(u) = f_\Gamma \tag{3}$$

has been studied, where Δ_Γ is the Laplace-Beltrami operator and $\mathcal{W}_\Gamma$ is another potential. However, just a regular framework has been considered so far.

The present talk regards equations (1-3) in the case of possibly singular potentials $\mathcal{W}$ and $\mathcal{W}_\Gamma$, like the so-called logarithmic potential (double well if $c > 1$)

$$\mathcal{W}(r) := \int_0^r \left(-2c + \ln \frac{1+s}{1-s} \right) ds, \quad \text{for } |r| < 1,$$

which is the thermodynamically relevant, and potentials related to constraints on the solution. Namely, the regularity required is the following: $\mathcal{W}' = \beta + \pi$ and $\mathcal{W}_\Gamma' = \beta_\Gamma + \pi_\Gamma$, where $\beta, \beta_\Gamma : \mathbf{R} \to 2^{\mathbf{R}}$ are just maximal monotone operators satisfying $\beta(0) \ni 0$ and $\beta_\Gamma(0) = \{0\}$ and $\pi, \pi_\Gamma : \mathbf{R} \to \mathbf{R}$ are just Lipschitz continuous functions (in such cases, some derivatives might be subdifferentials and equations (2-3) should be read as differential inclusions). Moreover, the so-called viscous Cahn-Hilliard equations (obtained by adding $\partial_t u$ to the right-hand side of (2)) and the less regular dynamic boundary condition obtained by cancelling the Δ_Γ-term in (3) are noteworthy and thus considered as well.

The results I present regard well-posedness in such a general framework and have been proved very recently in collaboration with A. Miranville (Poitiers) and G. Schimperna (Pavia) by using monotonicity and compactness methods. The existence of a solution is guaranteed under rather natural compatibility conditions on the structure of the system. The most relevant one is the following

$$D(\beta_\Gamma) \supseteq D(\beta) \quad \text{and} \quad \beta(r) \geq \alpha\, \beta_\Gamma(r) - C \quad \text{for every } r \in D(\beta)$$

for some positive constants α and C.