The caputo Fractional derivative and memory effects in systems with memory

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Fractional derivatives can be defined in several different ways. One of the possible definition, the *Caputo fractional derivative,* is as follows: given a function f which is $(n+1)$–times differentiable in the usual sense, $n \geq 0$, and a number $\gamma \in (0,1)$

$$D^{n+\gamma} f = J^{1-\gamma} D^{n+1} f$$

where D^{n+1} denotes the usual derivative of integer order while $J^{1-\gamma}$ is the usual Rieman-Liouville fractional integral,

$$\left(J^{1-\gamma} u\right)(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} u(s)\, \mathrm{d}s\,.$$

The Caputo fractional derivative is more suited then the usual Riemann-Liouville derivative for the applications in several engineering problems due to the fact that it has better relations with the Laplace transform and because the differentiation appears *inside* instead then outside, the integral, so to alleviate the effects of noise and numerical differentiation.

The use of the Caputo fractional derivative of order $\gamma \in (0,1)$, instead then the first order derivative, in the stabilizing feedback loop of a wave equation has been recently advocated. Let us consider the simplest case:

$$w_{tt} = w_{xx}\,, \qquad w(t,0) = u(t)\,, \quad w(t,1) = 0$$

and initial conditions $w(0,x) = a(x)$, $w_t(0,x) = b(x)$. It is known that this system is exponentially stable when $u(t) = -w_t(t,0)$. However, it is known that exponential stability is destroyed in the presence of time delays, as small as we wish.

In this talk we discuss the limitations of this approach, which can be interpreted as a new form of the lack of robustness under the effect of delays.

The Caputo fractional derivative introduces a memory in the feedback loop. We plan to discuss in general the limitations induced by memory and also issues related to the numerical computations of Caputo derivatives.