Discrete Carleman estimates for elliptic operators and uniform controllability of semi-discretized parabolic equations

Jérôme Le Rousseau*
Université de Provence†/ INRIA‡

In 1995 G. Lebeau and L. Robbiano proved a spectral inequality that led to the first proof of the null controllability of parabolic linear equations. Let ϕ_j and μ_j, $j \in \mathbf{N}^*$, denote an orthonormal basis of eigenfunctions and the associated eigenvalues for the elliptic operator $A = -\nabla \cdot (\gamma(x)\nabla)$, with γ smooth, on an open set $\Omega \subset \mathbf{R}^n$ with homogeneous Dirichlet boundary conditions. The spectral inequality reads

$$\sum_{\mu_j \leq \mu} |a_j|^2 \leq C e^{C\sqrt{\mu}} \int_\omega |\sum_{\mu_j \leq \mu} a_j \phi_j(x)|^2 \, dx, \qquad \forall (a_j)_{j \in \mathbf{N}^*} \subset \mathbf{C}, \tag{1}$$

where ω is any non-empty open subset of Ω. It thus measures the loss of orthogonality of the eigenfunctions when restricted to ω. This inequality can be obtained through the derivation of a global Carleman estimate for the elliptic operator $-\partial_{x_0}^2 + A$, where x_0 is an additional variable.

Here, we shall consider a discrete version A_h of A, using finite differences or finite volumes, for quasi-uniform meshes. We derive a global Carleman estimate for the semi-discrete operator $-\partial_{x_0}^2 + A_h$, in which the size of the Carleman large parameter is connected to the discretization step h. We can then prove an inequality of the form of (1) for the lower portion of the spectrum of A_h, $\mu \leq C/h^2$. This in turn leads to the null controllability of that part of the spectrum in the associated parabolic problem. The L^2 norm of the remaining of the spectrum can also be estimated by $e^{-C/h}\|u_0\|_{L^2}$, where u_0 is the initial condition used in the parabolic control problem. This result can be considered almost-sharp in view of the existing counterexamples that involve the higher end of the discrete spectrum.

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[2]Laboratoire d'Analyse Topologie Probabilités (LATP), CNRS UMR 6632, Universités d'Aix-Marseille, 39 rue F. Joliot-Curie, 13453 Marseille cedex 13, France.

[3]On a research leave at Laboratoire POEMS, INRIA Paris-Rocquencourt/ENSTA, CNRS UMR 2706, France.