Sturm-Liouville Problems for an Abstract Differential Equation of Elliptic Type in UMD Spaces

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In this talk we give some new results on Sturm-Liouville abstract problems for second order differential equations of elliptic type in UMD spaces. Existence, uniqueness and maximal regularity of the strict solution are proved using the celebrated Dore-Venni Theorem.

More precisely, we consider the second order abstract differential equation in X

$$u''(x) + Au(x) - \omega u(x) = f(x), \quad \text{a.e.} \quad x \in (0,1), \tag{1}$$

together with the abstract boundary conditions of Sturm-Liouville type

$$u'(0) - Hu(0) = d_0, \; u(1) = u_1. \tag{2}$$

where $f \in L^p(0,1;X), 1 < p < \infty$ (X being a complex X Banach space), $d_0, u_1 \in X$, A, H are closed linear operators in X and ω is some large positive number.

The particularity lies in the boundary condition containing operator H, which can be unbounded. This difficulty will require a precise study of a sum of two linear operators. To this end, we assume that:

1. X is a UMD space.
2. $[0,+\infty[\subset \rho(A_{\omega_0}), \mathbb{R}_- \subset \rho(H)$ and
$$\sup_{\lambda \geqslant 0} \left\| \lambda \left(A_{\omega_0} - \lambda I \right)^{-1} \right\|_{L(X)} < +\infty \text{ and } \sup_{\lambda \geqslant 0} \left\| \lambda (H + \lambda I)^{-1} \right\|_{L(X)} < +\infty,$$
(here ω_0 is some fixed positive number and $A_{\omega_0} = A - \omega_0 I$).
3. $\forall \, \lambda \geqslant \omega_0, \zeta \geqslant 0, \; (A - \lambda I)^{-1}(H + \zeta I)^{-1} = (H + \zeta I)^{-1}(A - \lambda I)^{-1}$.
4. $\exists C \geqslant 1, \exists \theta_A, \theta_H \in]0,\pi[$ (with $\dfrac{\theta_A}{2} + \theta_H \in]0,\pi[$) :
$$\begin{cases} \forall s \in \mathbb{R}, (H)^{is} \in L(X), (-A_{\omega_0})^{is} \in L(X) \text{ and} \\ \left\| (-A_{\omega_0})^{is} \right\|_{L(X)} \leqslant C e^{\theta_A |s|}, \left\| (H)^{is} \right\|_{L(X)} \leqslant C e^{\theta_H |s|}. \end{cases}$$

Then, our main result is the following:

Theorem *Under the previous assumptions and for $\omega > 0$ large enough, the following assertions are equivalent*

1. $d_0 \in \left(D\left(A\right), X\right)_{\frac{1}{2p}+\frac{1}{2},p}, \; u_1 \in \left(D\left(A\right), X\right)_{\frac{1}{2p},p}$.
2. *Problem (1) and (2) has a strict solution u, that is $u \in W^{2,p}(0,1;X) \cap L^p(0,1;D(A)), u(0) \in D(H)$ and u satisfies (1) and (2).*

Moreover, in this case, u is uniquely determined by an explicit representation formula.