An identification problem with evolution on the boundary of hyperbolic type

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We consider an equation of the type

$$A(u + k * u) = f, \tag{0.1}$$

where A is a linear second-order elliptic. operator, k is a scalar function depending on time only and $k * u$ denotes the standard time convolution of functions defined in $(-\infty, T)$ with their supports in $[0, T]$. The previous equation is endowed with a second-order dynamical boundary condition of the flux form,

$$D_t^2 u + bD_{\nu_A} u = g, \qquad \text{on} \quad (0, T) \times \partial\Omega,$$

where D_{ν_A} stands for the conormal derivative.
The identification problem we deal with in the present paper is concerned with identifying the unknown convolution *scalar* kernel k. To recover k we prescribe an additional condition of the form

$$\Phi[u(t, \cdot)] = l(t), \quad t \in [0, T],$$

Φ and l being a given functional and a given function. Global existence, uniqueness and continuous dependence results can be shown.
To our knowledge the problem of determining the memory kernel k is new under dynamical boundary conditions.
First we observe that our treatment to solve our identification problem leads to the following (equivalent) identification problem, for the new unknown v such that

$$v = u + k * u \quad \Longleftrightarrow \quad u = v + h * v, \tag{0.2}$$

$$Av = f \quad \text{in } (0, T) \times \Omega, \tag{0.3}$$

$$D_t^2 v + D_t h * D_t v + h(0) D_t v + u_0 D_t h + bD_{\nu_A} v + bh * D_{\nu_A} v = g \quad \text{on } (0, T) \times \partial\Omega, \tag{0.4}$$

$$v(0, \cdot) = u_0 \quad \text{on } (0, T) \times \partial\Omega, \tag{0.5}$$

$$D_t v(0, \cdot) = u_1 - h(0) u_0 \quad \text{on } (0, T) \times \partial\Omega, \tag{0.6}$$

$$\Phi[v(t, \cdot)] + h * \Phi[v(t, \cdot)] = l(t), \quad t \in [0, T]. \tag{0.7}$$

Note that the unknown kernel h convolves not only with the space operator $D_{\nu_A} v$, but also with the time derivative $D_t v$.
For such a kind of identification problems, to the authors' knowledge, the global in time existence of the solution (v, h) is not a standard result.

In this paper this difficulty is overcome by using a suitable estimate strategy and a fixed-point approach strictly related to the problem itself - we could say "suggested" by the problem itself. In particular we transform our problem into an equivalent problem related to the boundary of Ω.
Finally, we observe that a simple treatment of the elliptic problem (0.1) (endowed with auxiliary nonhomogeneous Dirichlet conditions) forces to choose for u - and consequently for v - an $L^2(\Omega)$-framework.
In contrast to this, we have chosen for the time dependence of v and h the Hölder spaces C^α, $\alpha \in (0,1)$, to ensure a good regularity for the unknown kernel h - and consequently for k.