LINEAR EVOLUTION EQUATION OF HYPERBOLIC TYPE WITH APPLICATION TO SCHRÖDINGER EQUATIONS

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Abstract

Let $\{A(t); 0 \le t \le T\}$ be a family of closed linear operators in a complex Hilbert space X. This talk is concerned with linear evolution equations of the form

$$\text{(E)} \qquad du(t)/dt + A(t)u(t) = f(t) \quad \text{on} \quad (0, T).$$

Let S be a selfadjoint operator in X, satisfying $(u, Su) \ge \|u\|^2$ for $u \in D(S)$. Assume that the following four conditions are satisfied:

(I) There is $\alpha \in L^1(0, T)$, $\alpha \ge 0$, such that

$$|\mathrm{Re}\,(A(t)v, v)| \le \alpha(t)\|v\|^2, \; v \in D(A(t)), \; \text{a.a. } t \in (0, T);$$

(II) $Y := D(S^{1/2}) \subset D(A(t)), \; 0 \le t \le T$;

(III) There is $\beta \in L^1(0, T)$, $\beta \ge \alpha$, such that

$$|\mathrm{Re}\,(A(t)u, Su)| \le \beta(t)\|S^{1/2}u\|^2, \; u \in D(S), \; \text{a.a. } t \in (0, T);$$

(IV) $A(\cdot) \in L^2\big(0, T; B(Y, X)\big)$.

Then we have

Theorem. Let $f(\cdot) \in L^2(0, T; X) \cap L^1(0, T; Y)$. Then there exists a unique strong solution $u(\cdot)$ of (E) with $u(0) = u_0 \in Y$ such that $u(\cdot) \in H^1(0, T; X) \cap C([0, T]; Y)$.

Now let $k \in \mathbb{N}$. Then, by introducing

$$S := 1 + \Delta^{2k} + |x|^{4k}, \quad D(S^{1/2}) = H^{2k}(\mathbb{R}^N) \cap D(|x|^{2k}),$$

the above-mentioned theorem can be applied to the Cauchy problem for Schrödinger evolution equations ($i := \sqrt{-1}$):

$$\text{(SE)} \qquad i\,du(t)/dt - \big(-\Delta + V(x, t)\big)u(t) = 0, \quad 0 < t < \infty,$$

in $L^2(\mathbb{R}^N)$. The assumption is satisfied under the following three conditions:

(V0) $V(\cdot, t) \in C^{2k}(\mathbb{R}^N)$ a.a. $t \in (0, \infty)$;

(V1) $(1 + |x|^{2k})^{-1}V \in L^2_{\mathrm{loc}}([0, \infty); L^\infty(\mathbb{R}^N))$;

(V2) $\exists\, g_j \in L^1_{\mathrm{loc}}[0, \infty)$; $\displaystyle\sum_{|\alpha|=j} |D^\alpha V(x, t)| \le g_j(t)(1 + |x|^j), \quad 1 \le j \le 2k.$