Time reversal and data assimilation

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Given a skew-adjoint operator $A : X_1 \longrightarrow X$ generating a C^0 group of isometries $\mathbb{T}$ on X, consider the system

$$\dot{z}(t) = Az(t), \quad z(0) = x \in X. \tag{0.1}$$

with the observation

$$y(t) = Cz(t)$$

where $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}$. Under the assumption that the pair (A, C) is exactly observable in some time $\tau > 0$ we

Let $\Psi_\tau \in \mathcal{L}(X, L^2([0, \tau]; Y))$ be defined by

$$(\Psi_\tau z_0)(t) = C\mathbb{T}_t z_0 \text{ for } \ t \in [0, \tau], \quad z_0 \in X.$$

We are interested in solving the inverse problem of determining the initial data x from the observation y (the data assimilation problem). In other words, we want to solve the equation

$$\Psi_\tau x = y, \tag{0.2}$$

where $y \in L^2([0, \tau]; Y)$ is supposed to be a given element of the range of Ψ_τ. This is done by using the version from [1] of the time reversal method of Fink. More precisely we consider the iterative method

$$\begin{cases} \dot{v}^j(t) = (A - C^*C)v^j(t) + 2C^*C(\boldsymbol{\mathcal{R}}_\tau e^{j-1})(t), \\ v^j(0) = 0, \end{cases} \qquad \forall j \geq 1 \tag{0.3}$$

$$e^j(t) = v^j(t) - \boldsymbol{\mathcal{R}}_\tau e^{j-1}(t), \tag{0.4}$$

where $\boldsymbol{\mathcal{R}}_\tau \in \mathcal{L}(L^2([0, \tau]; X))$ is the "time reversal operator". The choice of this operator depends on A and C. We have, for instance, for a Schrödinger equation with internal observation that

$$(\boldsymbol{\mathcal{R}}_\tau v)(t) = \overline{v(\tau - t)} \qquad (v \in L^2(0, \tau; L^2(\Omega)).$$

Our main abstract result asserts that, under the above assumptions, we have

$$\left\| x - \sum_{k=0}^{N} v^{2k}(\tau) \right\| = \|e^{2N}(\tau)\| \leq Me^{-2N\alpha\tau}\|x\|.$$

The above estimate provides an efficient method for reconstructiong initial data. We apply this approach, both theotericaly and numerically, to the wave and to the Schrödinger equations.

References

[1] K. D Phung, X. Zhang, *Time reversal foccussing of the initial state for the Kirhhoff plate*, to appear.