From exact observability to identification of singular sources

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We list some recent results which are relevant in the identification of point sources for the wave equation, using boundary measurements. The results are in a more general framework, dealing with an arbitrary operator semigroup and an admissible observation operator for it. Our standing assumptions are that $\mathbb{T}$ is a strongly continuous semigroup of operators on the Hilbert space X, with generator $A : \mathcal{D}(A) \to X$. For a Hilbert space V, $m \in \mathbb{N}$ and for $\tau > 0$ we set

$$\mathcal{H}_R^m(0,\tau;V) = \left\{ u \in \mathcal{H}^m(0,\tau;V) \mid u(\tau) = \cdots = u^{m-1}(\tau) = 0 \right\} .$$

We denote $X_m^d = \mathcal{D}((A^*)^m)$, with the graph norm.

Proposition 1. *Let $C \in \mathcal{L}(X_1, Y)$ be an admissible observation operator for $\mathbb{T}$. For $\tau > 0$ let Ψ_τ be the output map corresponding to the pair (A, C). Then, for each $\tau > 0$, Ψ_τ has a unique continuous extension $\Psi_\tau \in \mathcal{L}((Z_m^d)', [\mathcal{H}_R^m(0,\tau;Y)]')$, where*

$$Z_m^d = X_m^d + (\beta I - A^*)^{-1} C^* U, \tag{1}$$

for some $\beta \in \rho(A)$. Here and below, dualities are computed with respect to the Hilbert spaces X and $L^2([0,\tau];Y)$, respectively.

Moreover, assume that (A, C) is exactly observable in some time $\tau_0 > 0$. Then for each $\tau > \tau_0$, there exists a constant $m_\tau > 0$ such that, for every $f \in (Z_m^d)'$, we have

$$\|\Psi_\tau f\|_{[\mathcal{H}_R^m(0,\tau;Y)]'} \geqslant m_\tau \|f\|_{(Z_m^d)'} . \tag{2}$$

Theorem 2. *Let X, Y be Hilbert spaces and assume that the pair (A, C) is exactly observable in some time $\tau_0 > 0$ and that $\lambda \in \mathcal{H}^1(0,\tau)$ with $\lambda(0) \neq 0$. We define $\mathbb{F}_\tau f$ as the convolution (on $[0,\tau]$) of λ with $\Psi_\tau f$. Then for every $\tau > \tau_0$, $\mathbb{F}_\tau$ is one-to-one from $(Z_m^d)'$ to $[\mathcal{H}_R^{m-1}(0,\tau;Y)]'$ and there exists a positive constant $\tilde{\kappa}_\tau$ such that*

$$\|f\|_{(Z_m^d)'} \leqslant \tilde{\kappa}_\tau \|\mathbb{F}_\tau f\|_{(\mathcal{H}_R^{m-1}(0,\tau;Y))'} \qquad \forall\, f \in (Z_m^d)'. \tag{3}$$

The above theorem gives conditions to recover f from y, when the system is described by $\dot{z} = Az + \lambda f$, $z(0) = 0$, $y = Cz$ (for $t \in (0,\tau)$). An interesting particular case is when A corresponds to the wave equation and f is the control operator that corresponds to a Dirac mass (point source) acting on the velocity component of the state.