Sharp Convergence Rate of Glimm Scheme for General Nonlinear Hyperbolic Systems

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Joint work with **Andrea Marson**, University of Padova

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Outline

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General systems of conservation laws

$$\partial_t u + \partial_x f(u) = 0 \qquad x \in \mathbb{R}\,, t \geq 0$$
$$u(0, x) = u_0(x)$$

- $u = u(t, x) \in \mathbb{R}^n$ vector of the unknowns
- $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$, smooth flux
- The system is strictly hyperbolic

$$Df(u)r_k(u) = \lambda_k(u)r_k(u) \qquad k = 1, \dots, n$$

$$0 < \lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u) < 1$$

- No classical Lax assumptions on $\nabla\lambda_k \cdot r_k$
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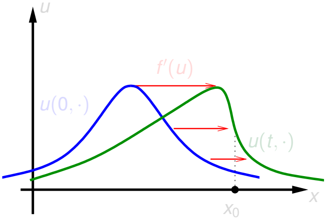
Loss of regularity

$$\partial_t u + \partial_x (f(u)) = 0, \quad u \in \mathbb{R},$$

smooth sol. $u = u(t,x)$ satisfy

$$\partial_t u + f'(u)\,\partial_x u = 0.$$

- Gradient Catastrophe ($f''(u) > 0$)



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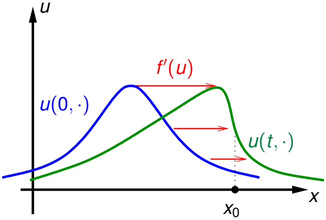

Existence theory

$\Rightarrow$ look for (discontinuous) weak sol'ns

$$\iint \{u\,\partial_t\phi + f(u)\,\partial_x\phi\}\; dxdt = 0$$

for all $\phi \in \mathcal{C}^1$ with compact support.

- Solutions $u(t,\cdot) \in BV$ are constructed by
  - Glimm scheme
  - Wave-front tracking algorithm
  - Vanishing viscosity approximations

$$\partial_t u^\varepsilon + \partial_x (f(u^\varepsilon)) = \varepsilon \partial^2_{xx} u^\varepsilon \qquad (\varepsilon \to 0)$$

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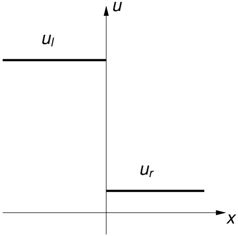
Riemann Problem (GNL systems)

$$\partial_t u + \partial_x f(u) = 0$$

$$u(0,x) = \begin{cases} u^L & \text{if} \quad x < 0 \\ u^R & \text{if} \quad x > 0 \end{cases}$$

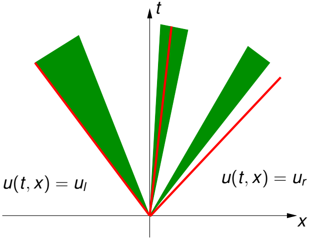


Riemann datum

Shocks

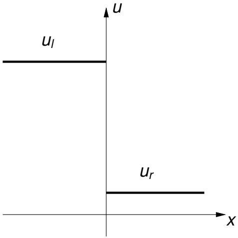
Rarefactions

Time goes on ....



Packs of *i*-th waves

Riemann problem (NGNL systems)



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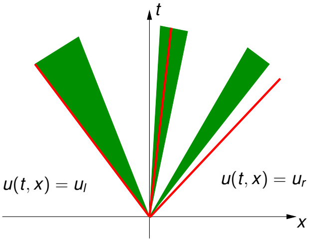


Packs of *i*-th waves

Outline

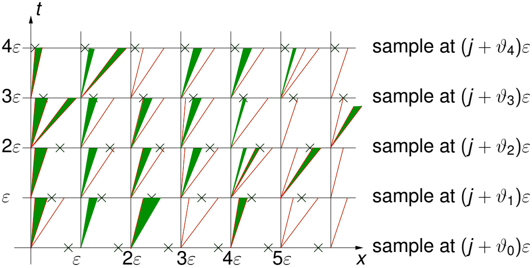
1. Introduction
   - General setting
   - Review of Glimm scheme and wave tracing
   - Convergence rate
2. Convergence rate for Glimm scheme
   - Error estimates
   - Differences with the Lax case
3. New interaction potentials
   - The quadratic part
   - The cubic part
   - Conclusion

The Glimm scheme

J. Glimm, CPAM 1965

- Mesh sizes $\Delta x = \Delta t = \varepsilon << 1 \quad (0 < \lambda_i(u) < 1)$
- Equidistributed sampling sequence $\{\vartheta_k\}_{k\in\mathbb{N}} \subset]0,1[$

$$\lim_{n\to\infty} \left| \lambda - \frac{\sharp\{k \le n : \vartheta_k \le \lambda\}}{n} \right| = 0 \qquad \forall \lambda \in]0,1[$$



$\Rightarrow$ solve the Riemann problems

Glimm Functional

A-priori bounds on total variation

$V(t)$ = Total strength of waves in $u^\varepsilon(t,\cdot) \approx$ Tot. Var.$\{u^\varepsilon(t,\cdot)\}$
$Q(t)$ = Wave interaction potential

- $t \mapsto V(t) + c\,Q(t)$ non increasing
- Uniform BV bounds $\implies$ compacteness
- $u^\varepsilon \to u$ in $\mathbb{L}^1_{loc}$ as $\varepsilon \to 0$
- $\{\vartheta_k\}_k$ equidistributed $\implies$ u weak solution

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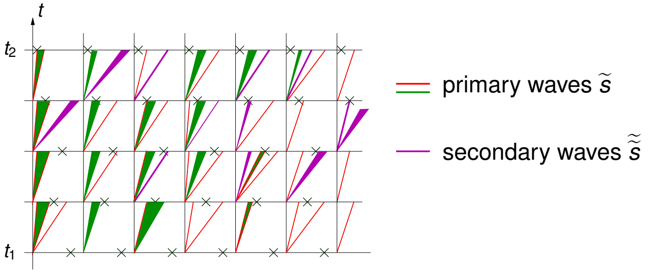
The wave tracing

T.P. Liu CMP 1977, Memoirs AMS 1981 - T.P. Liu, T. Yang, CMP 2002



- primary waves $\tilde{s}$ can be traced in a time interval $[t_1, t_2]$
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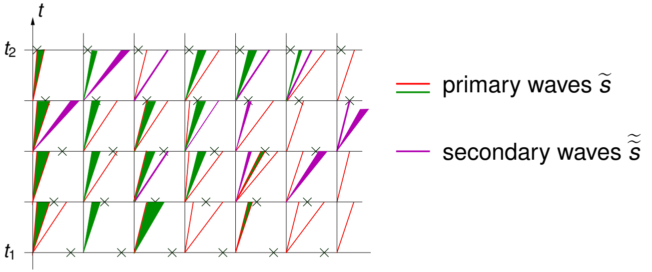

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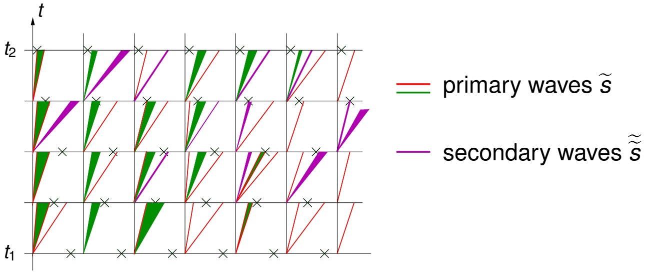

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Outline

Lax Systems

Choose a sampling sequence $\{\vartheta_k\}_{k\in\mathbb{N}} \subset]0,1[$ such that

$$\left|\lambda - \frac{\sharp\{k \le n : \vartheta_k \le \lambda\}}{n}\right| \le C\frac{1+\log n}{n} \qquad \forall \lambda \in]0,1[$$

J.G. Van der Corput, Compositio Math. 1935

u^ε Glimm approximate solution with mesh size $\varepsilon \doteq \Delta x$

$$\|u^\varepsilon(T,\cdot) - u(T,\cdot)\|_{\mathbb{L}^1} = o(1)\cdot\sqrt{\varepsilon}|\log\varepsilon|$$

GNL or LD systems: A. Bressan, A.M., ARMA 1998

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Goal

Analyze convergence rate of Glimm approximate solutions for general hyperbolic systems

$$\partial_t u + A(u)\partial_x u = 0$$

- no assumption on $A(u)$ besides strict hyperbolicity
- in non conservative case solutions are limits of vanishing viscosisty

$$\partial_t u^\varepsilon + A(u)\partial_x u^\varepsilon = \varepsilon\, \partial^2_{xx} u^\varepsilon \qquad (\varepsilon \to 0)$$

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Estimate of the error

A. Bressan, A.M., ARMA 1998

In a time interval $[t_1, t_2[$

$$\|u^\varepsilon(t_2,\cdot)-u(t_2,\cdot)\|_{\mathbb{L}^1} = \|u^\varepsilon(t_1,\cdot)-u(t_1,\cdot)\|_{\mathbb{L}^1} + \\ + \mathcal{O}(1)\cdot\Big[\sum_{\text{primary waves}} |\tilde{s}|\cdot|\text{change in speed of } \tilde{s}| + \\ + \sum_{\text{secondary waves}} |\tilde{\tilde{s}}| + (\text{error in speeds})\Big]\cdot(t_2-t_1)$$

$$\sum_{\text{secondary waves}} |\tilde{\tilde{s}}| = \mathcal{O}(1)\cdot\Big[\Delta_{[t_1,t_2[}V + C_0\Delta_{[t_1,t_2[}Q\Big]$$

$$\text{error in speeds} = \mathcal{O}(1)\cdot\frac{1+\log(t_2-t_1)/\varepsilon}{(t_2-t_1)/\varepsilon}$$

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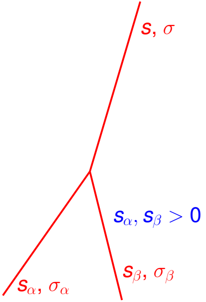
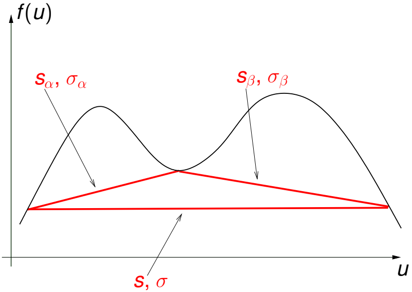
Estimate of the main term





$$s \approx s_\alpha + s_\beta \qquad \sigma \approx \frac{s_\alpha \sigma_\alpha + s_\beta \sigma_\beta}{s_\alpha + s_\beta}$$

(s_α, s_β size of waves, $\sigma_\alpha, \sigma_\beta$ speed of waves)

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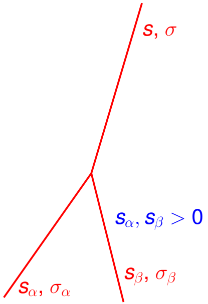

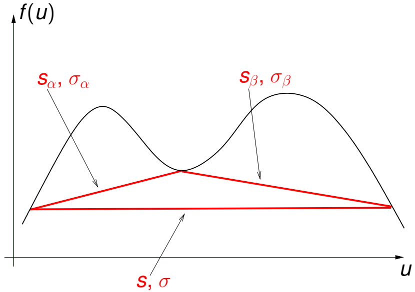

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$$|s_\alpha||\sigma_\alpha - \sigma| + |s_\beta||\sigma_\beta - \sigma| \approx \frac{|s_\alpha s_\beta||\sigma_\alpha - \sigma_\beta|}{|s_\alpha| + |s_\beta|} \ \Big(= \mathcal{O}(1) \cdot |s_\alpha s_\beta| \Big)$$

- For genuinely nonlinear systems

$$|\Delta Q| \geq \frac{|s_\alpha s_\beta|}{2}$$

- In a time interval $[t_1, t_2[$

$$\sum_{\text{primary waves}} |\tilde{s}| \cdot \big|\text{change in speed of } \tilde{s}\big| =$$

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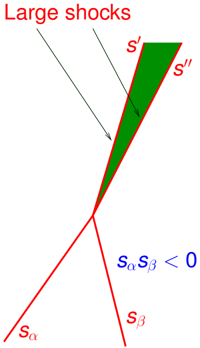
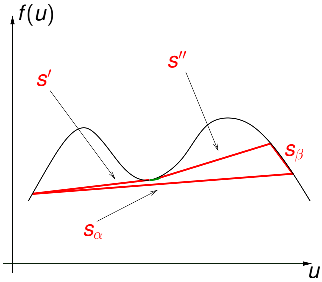
Outline

No global quadratic interaction potential

$$Q(u) = \sum_{\text{approaching waves}} |s_\alpha s_\beta|$$





$|s'|, |s''| >> |s_\beta|$ $\qquad \Delta Q \geq -|s_\alpha s_\beta| + |s' s''| > 0$

No global quadratic interaction potential

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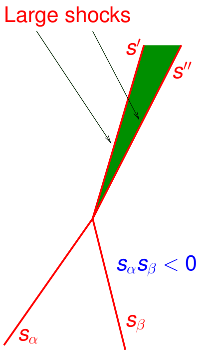

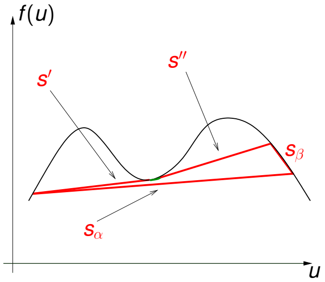

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New interaction potentials

AIM: define interaction potential Q for non Lax systems so that

- $t \mapsto V(t) + c\,Q(t)$ non increasing
- [wave size] $\times$ [change in speed] $= \mathcal{O}(1) \cdot |\Delta Q|$

Consider:

- Systems with finite number of connected linearly degenerate hypersurfaces $\{u \;:\; \nabla\lambda_k \cdot r_k(u) = 0\}$
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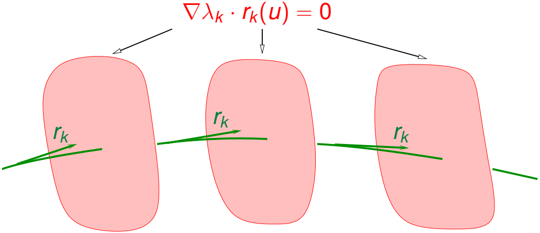
Finite number of linearly degenerate manifolds

We assume that

$\nabla\lambda_k \cdot r_k$ vanishes on finite number of hypersurfaces

transversal to r_k



Scalar case: f has a finite number of inflection points

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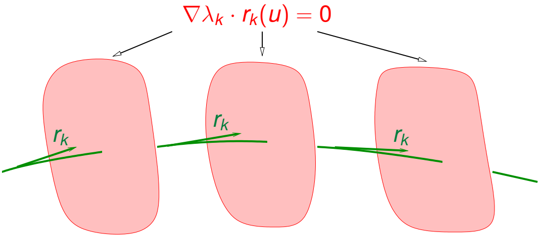
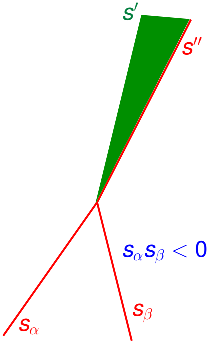


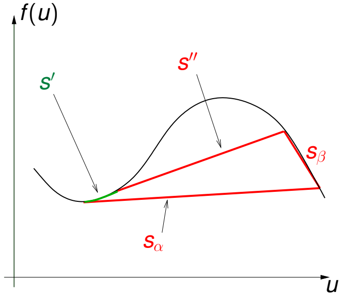
Scalar case: f has a finite number of inflection points

Outline

A locally quadratic interaction potential

For “one inflection point” systems





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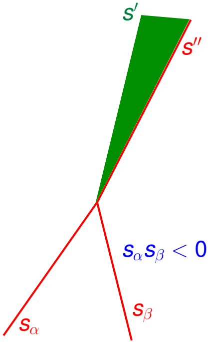

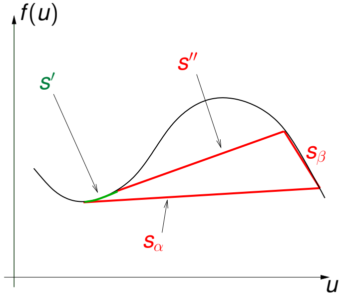

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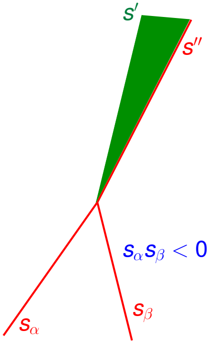

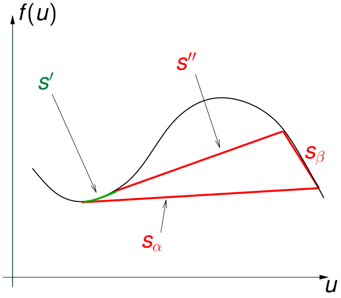

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A locally quadratic interaction potential

F.A, A.Marson, JDE 2001

$$Q(t) \doteq 2 \sum_{\substack{\text{same family} \\ s_\alpha s_\beta > 0}} |s_\alpha s_\beta| + \sum_{\text{rarefactions}} |s_\alpha|^2 +$$

$$+ k \Bigg[\sum_{\substack{\text{same family} \\ s_\alpha s_\beta < 0}} + \sum_{\substack{\text{approaching waves} \\ \text{different families}}} \Bigg] |s_\alpha s_\beta|$$

$$\Delta Q = 2|s' s''| + |s'|^2 - k|s_\alpha s_\beta| = \mathcal{O}(1) \cdot |s_\alpha s_\beta| - k|s_\alpha s_\beta|$$

$$\leq -\frac{1}{2}|s_\alpha s_\beta|, \qquad k >> 1$$

Construct an interaction potential quadratic in "small waves" crossing one inflection point

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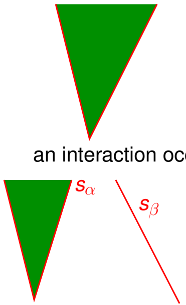
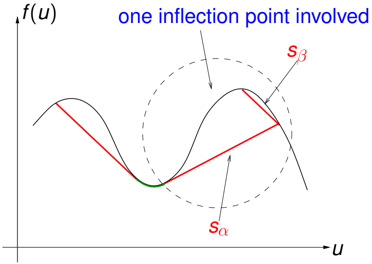
A deeper look at interactions





Choose δ_0 small so that if $|s_\alpha| < \delta_0$, then

$$\text{new rarefactions} = \mathcal{O}(1) \cdot |s_\beta|$$

$$u \in B_{\delta_0}(\text{inflection point}) \quad \Rightarrow \quad D^2\lambda_k \cdot r_k(u) \neq 0$$

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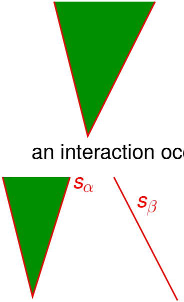

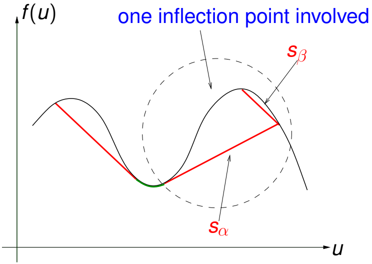

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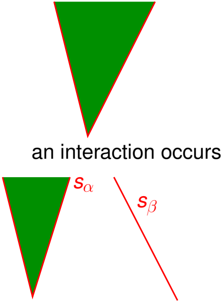

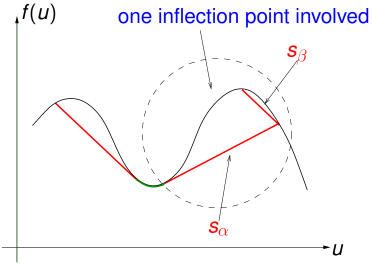

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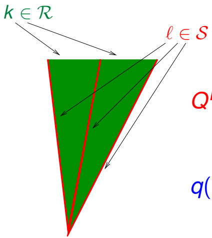

$$Q^I(s) = 2\sum_{\ell \neq k} |s_\ell s_k| + \sum_{\ell \in \mathcal{S}} q(s_\ell) + \sum_{k \in \mathcal{R}} |s_k|^2$$

$q(s_\ell) = 0$ if $|s_\ell| \leq \delta_0$, intrinsic potential

The quadratic part is

$$Q_q(t) = 2 \sum_{\substack{\text{same family} \\ s_\alpha s_\beta > 0}} |s_\alpha s_\beta| + \sum_\alpha Q^I(s_\alpha)$$

$$+ k\left[\sum_{\substack{\text{same family} \\ s_\alpha s_\beta < 0}} + \sum_{\substack{\text{approaching waves} \\ \text{different families}}} \right] |s_\alpha s_\beta|$$

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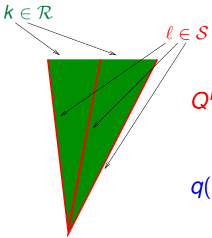

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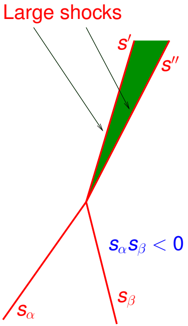
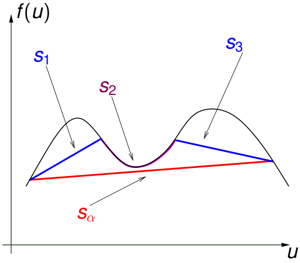
Intrinsic interaction potential

What's $q(\cdot)$?





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Takes into account of (possible) future splitting of s_α, $|s_\alpha| > \delta_0$

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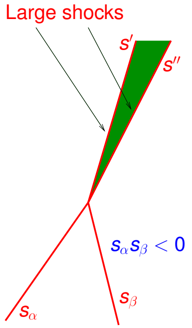

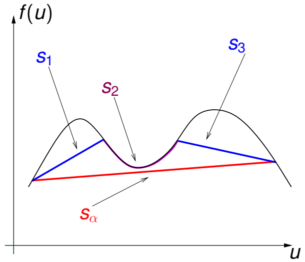

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Outline

Bianchini's interaction potential

S. Bianchini, DCDS 2003

$$\mathcal{Q}(t) = \sum_{\text{same family}} \left| \int_0^{s_\alpha} \int_0^{s_\beta} |\sigma_\alpha(\tau) - \sigma_\beta(\tau')| \, d\tau d\tau' \right|$$

- $\sigma_\alpha(\cdot)$ speed of the wave s_α associated to manifold of travelling profiles

$$-\sigma_i u_x + Df(u)u_x = u_{xx}, \qquad \sigma_i \approx \lambda_i$$

- If s_α, s_β are shocks

$$\left| \int_0^{s_\alpha} \int_0^{s_\beta} |\sigma_\alpha(\tau) - \sigma_\beta(\tau')| \, d\tau d\tau' \right| = |s_\alpha s_\beta| \cdot |\sigma_\alpha - \sigma_\beta|$$

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The new interaction potential

$$Q(t) = Q_q(t) + k\mathcal{Q}(t)$$

- Q_q decreases at interactions between "small" waves ($\leq \delta_0$) of the same family
- possible increase of Q_q due to interactions of "large" waves ($> \delta_0$) is controlled by the corresponding decrease of $\mathcal{Q}$

For suitable $c, k >> 1$ at each interaction

$$\Delta V + c\Delta Q \leq -\frac{1}{2}\Delta Q$$

$$\text{[wave size]} \times \text{[change in speed]} = \mathcal{O}(1) \cdot |\Delta Q|$$

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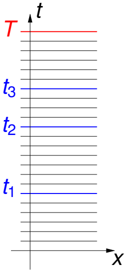
Outline

The final estimate



fix $\eta >> \varepsilon$

choose a partition of $[0, T]$ such that

$$\Delta_{[t_i,t_{i+1}[}V + C_0\Delta_{[t_i,t_{i+1}[}Q + (t_{i+1} - t_i) \approx \eta$$

number of subintervals $\approx 1/\eta$

$$\|u^\varepsilon(T,\cdot) - u(T,\cdot)\|_{\mathbb{L}^1} \approx \mathcal{O}(1) \cdot \frac{1}{\eta} \cdot \left[\eta^2 + \eta\frac{\log(\eta/\varepsilon)}{\eta/\varepsilon}\right]$$

If $\eta = \sqrt{\varepsilon}\log|\log\varepsilon|$, then

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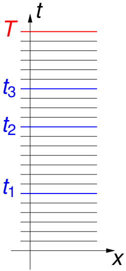
Same convergence rate, different potential:
J. Hua, Z. Jiang, T. Yang, preprint 2008

The final estimate



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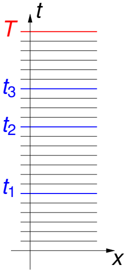

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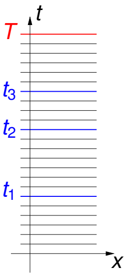
The final estimate



fix $\eta >> \varepsilon$

choose a partition of $[0, T]$ such that

$$\Delta_{[t_i, t_{i+1}[} V + C_0 \Delta_{[t_i, t_{i+1}[} Q + (t_{i+1} - t_i) \approx \eta$$

number of subintervals $\approx 1/\eta$

$$\|u^\varepsilon(T,\cdot) - u(T,\cdot)\|_{\mathbb{L}^1} \approx \mathcal{O}(1) \cdot \frac{1}{\eta} \cdot \left[\eta^2 + \eta \frac{\log(\eta/\varepsilon)}{\eta/\varepsilon}\right]$$

If $\eta = \sqrt{\varepsilon} \log|\log \varepsilon|$, then

$$\|u^\varepsilon(T,\cdot) - u(T,\cdot)\|_{\mathbb{L}^1} = o(1) \cdot \sqrt{\varepsilon}|\log \varepsilon|$$

Same convergence rate, different potential:

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Thank you for your attention!