Vibration testing for the detection of small perturbations of an interface.

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Overview of related results

- Detection of small conductivity inclusions from the asymptotic expansion of eigenvalue perturbations.
- Determination of small perturbations of an interface from the asymptotic expansion of the boundary voltage potential.

References

- H. Ammari and S. Moskow, Asymptotic expansions for eigenvalues in the presence of small inhomogeneities, *Math. Meth. Appl. Sci.*, **26** (2003) pp. 67–75.
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Outline

The direct problem

- Small perturbation of an interface
- Asymptotic expansion for eigenvalue perturbation

The inverse problem

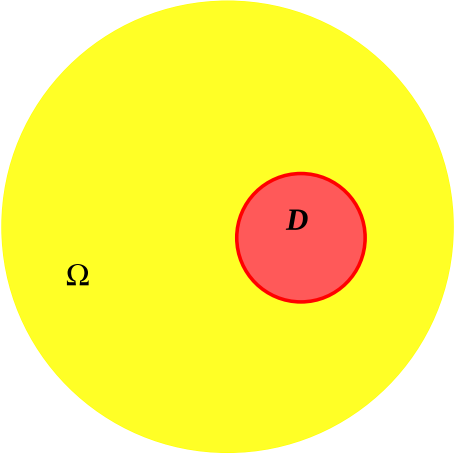
- Reconstruction formula
- Reconstruction algorithm
- Numerical experiments
- Concluding remarks

Part I

The direct problem

Small perturbations of the conducting interface

$\Omega \subset \mathbb{R}^2$ is a plane region occupied by a homogeneous isotropic conducting material containing an inclusion D strictly contained in Ω.
Let γ_e and γ_i be two positive constants representing the conductivities in $\Omega \setminus \overline{D}$ and D respectively.



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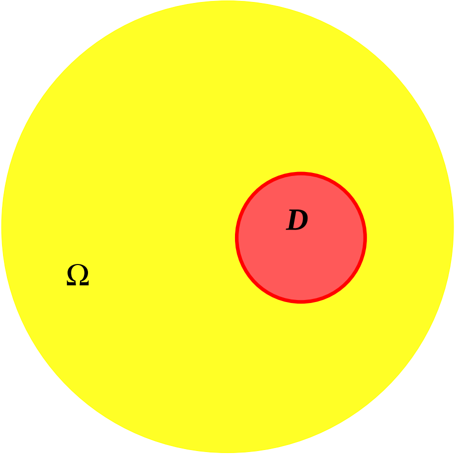

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The forward problem

Let u_0 be the solution of the following eigenvalue problem:

$$\begin{cases} \nabla \cdot (\gamma_D \nabla u_0) = -\omega_0^2 u_0 & \text{in} \quad \Omega, \\ \gamma_D \dfrac{\partial u_0}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \\ \displaystyle\int_\Omega u_0^2 = 1. \end{cases}$$

Then $u_0 \in H^1(\Omega)$ and it satisfies the transmission conditions

$$\begin{cases} u_0^i = u_0^e \\ \gamma_i \dfrac{\partial u_0^i}{\partial \nu} = \gamma_e \dfrac{\partial u_0^e}{\partial \nu} \end{cases} \quad \text{on } \partial D,$$

where

$$u_0^e = u_0|_{\Omega \setminus D} \quad \text{and} \quad u_0^i = u_0|_{\overline{D}}.$$

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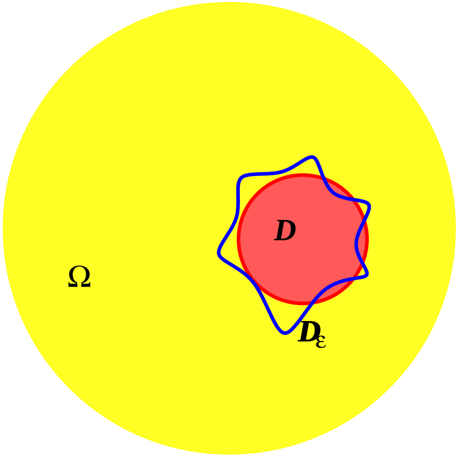
$$u_0^e = u_0|_{\Omega \setminus D} \quad \text{and} \quad u_0^i = u_0|_{\overline{D}}.$$

The perturbed problem

Let D_ϵ an ϵ-perturbation of the domain D with

$$\partial D_\epsilon = \left\{ x + \epsilon h(x) \nu(x), x \in \partial D \right\},$$

$\nu(x)$ unit outer normal vector to ∂D at x, h smooth function and ϵ positive small parameter.



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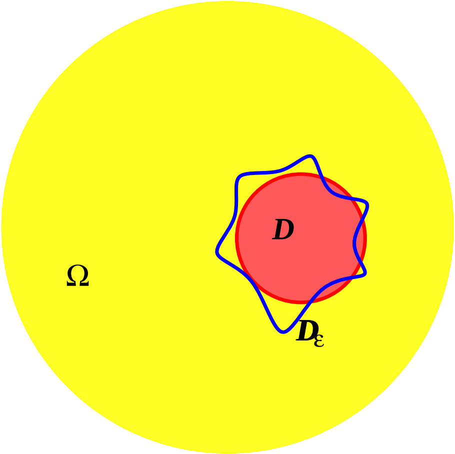

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Let $u_\epsilon \in H^1(\Omega)$ be the solution to

$$\begin{cases} \nabla \cdot (\gamma_{D_\epsilon} \nabla u_\epsilon) &= -\omega_\epsilon^2 u_\epsilon \quad \text{in} \quad \Omega, \\ \gamma_{D_\epsilon} \dfrac{\partial u_\epsilon}{\partial \nu} &= 0 \quad \text{on} \quad \partial\Omega, \\ \displaystyle\int_\Omega u_\epsilon^2 &= 1. \end{cases}$$

Then u_ϵ satisfies the transmission conditions across ∂D_ϵ.

Assumptions

- $d(D, \partial\Omega) \geq K$
- $\partial\Omega \in C^{1,\alpha}$ and $\partial D \in C^{2,\alpha}$
- $h \in C^{1,\alpha}$ and $\|h\|_{C^{1,\alpha}} \leq H$

Asymptotic expansion of eigenvalue perturbation

Theorem

Let ω_0^2 be a simple eigenvalue. Then, there exists ω_ϵ^2 such that $\omega_\epsilon^2 \longrightarrow \omega_0^2$ as $\epsilon \longrightarrow 0$ and

$$\omega_\epsilon^2 - \omega_0^2 = -\epsilon(\gamma_i - \gamma_e) \int_{\partial D} h(x) \left(\left(\frac{\partial u_0^e}{\partial \tau}(x) \right)^2 + \frac{\gamma_e}{\gamma_i} \left(\frac{\partial u_0^e}{\partial \nu}(x) \right)^2 \right) d\sigma_x + o(\epsilon)$$

where τ and ν are the tangential and outward normal vector to ∂D.

Proof

- Osborn result on convergence of eigenvalues of sequences of self-adjoint collectively compact operators.
- Gradient estimates for solutions of elliptic equations

Proof

Lemma [Osborn]

Let X be a real Hilbert space and let $T : X \to X$ and $T_\epsilon : X \to X$ be compact, self-adjoint linear operators such that $\{T_\epsilon\}_{\epsilon>0}$ are collectively compact and $T_\epsilon \to T$ pointwise as $\epsilon \to 0$. Let μ_0 be a nonzero simple eigenvalue of $T$and u_0 the corresponding eigenfunction. Then there is an eigenvalue μ_ϵ of T_ϵ such that $\mu_\epsilon \to \mu_0$ for $\epsilon \to 0$. Moreover there exists a constant C (independent of ϵ) such that

$$|\mu_0 - \mu_\epsilon - < (T - T_\epsilon)u_0, u_0 >| \leq C \, \|(T - T_\epsilon)u_0\|_X^2$$

and

$$\|u_\epsilon - u_0\|_X \leq C \, \|(T - T_\epsilon)u_0\|_X$$

where u_ϵ is an eigenfunction corresponding to μ_ϵ.

Proof

Let $X = \{f \in L^2(\Omega) : \int_\Omega f = 0\}$ and $T : X \to X$ the linear operator given by $Tf = v_0$ where v_0 is the solution to

$$\begin{cases} \nabla \cdot (\gamma_D \nabla v_0) = f & \text{in} \quad \Omega, \\ \gamma_D \dfrac{\partial v_0}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \\ \displaystyle\int_\Omega v_0 = 0. \end{cases}$$

Let $T_\epsilon : X \to X$ given by by $T_\epsilon f = v_\epsilon$, where v_ϵ is the solution to

$$\begin{cases} \nabla \cdot (\gamma_{D_\epsilon} \nabla v_\epsilon) = f & \text{in} \quad \Omega, \\ \gamma_{D_\epsilon} \dfrac{\partial v_\epsilon}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \\ \displaystyle\int_\Omega v_\epsilon = 0. \end{cases}$$

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Proof

- T_ϵ and T are compact and self-adjoint.
- $\{T_\epsilon\}_{\epsilon \geq 0}, (T_0 = T)$ are collectively compact *i.e.* $\{T_\epsilon f : \|f\|_X \leq 1, \epsilon \geq 0\}$ is sequentially compact. In fact , if $v \in \{T_\epsilon f : \|f\|_X \leq 1, \epsilon \geq 0\}$, then

$$\|v\|_{H^1(\Omega)} \leq C$$

where C is independent of ϵ.

- $T_\epsilon \rightarrow T$ pointwise in X as $\epsilon \rightarrow 0$. For $f \in X$, let $v_\epsilon = T_\epsilon(f)$ and $v_0 = Tf$. Then,

$$\|\nabla(v_\epsilon - v_0)\|_{L^2(\Omega)} \leq C\|\nabla v_0\|_{L^2(D_\epsilon \triangle D)}.$$

and by the Poincaré inequality

$$\|v_\epsilon - v_0\|_{H^1(\Omega)} \leq C\|\nabla v_0\|_{L^2(D_\epsilon \triangle D)}.$$

Finally, using the last inequality and the fact that $|D_\epsilon \triangle D| \rightarrow 0$ as $\epsilon \rightarrow 0$ and that $\nabla v_0 \in L^2(\Omega)$, $T_\epsilon \rightarrow T$ pointwise as $\epsilon \rightarrow 0$ in $L^2(\Omega)$.

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Proof

By Osborn's result we have that for small ϵ, there is an eigenvalue μ_ϵ of T_ϵ such that $\mu_\epsilon \to \mu_0$ and

$$\left|\mu_0 - \mu_\epsilon - < (T_\epsilon - T)u_0, u_0 > \right| \leq C\|(T_\epsilon - T)u_0\|^2_{L^2(\Omega)},$$

where u_0 is such that $Tu_0 = \mu_0 u_0$, $\int_\Omega u_0^2 = 1$ and $\mu_0 = -\frac{1}{\omega_0^2}$.
Moreover

$$\|u_\epsilon - u_0\|_{L^2(\Omega)} \leq C\|(T_\epsilon - T)u_0\|_{L^2(\Omega)},$$

where u_ϵ is the eigenfunction corresponding to $\mu_\epsilon = -\frac{1}{\omega_\epsilon^2}$ such that $\int_\Omega u_\epsilon^2 = 1$.

Proof

We are now left with estimating the terms in the formula

$$\left| \mu_0 - \mu_\epsilon - < (T_\epsilon - T)u_0, u_0 > \right| \leq C \| (T_\epsilon - T) u_0 \|^2_{L^2(\Omega)}.$$

We have

$$\begin{aligned} < (T - T_\epsilon)u_0, u_0 > &= < \mu_0 u_0 - \tilde{v}_\epsilon, u_0 > \\ &= -\mu_0 \int_\Omega (\gamma_{D_\epsilon} - \gamma_D) \nabla \tilde{v}_\epsilon \cdot \nabla u_0 \end{aligned}$$

where $\tilde{v}_\epsilon$ is the solution to

$$\begin{cases} \nabla \cdot (\gamma_{D_\epsilon} \nabla \tilde{v}_\epsilon) = u_0 & \text{in} \quad \Omega, \\ \gamma_{D_\epsilon} \dfrac{\partial \tilde{v}_\epsilon}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \\ \displaystyle\int_\Omega \tilde{v}_\epsilon = 0. \end{cases}$$

Proof

We now write

$$< (T - T_\epsilon)u_0, u_0 >= -\mu_0 \int_\Omega (\gamma_{D_\epsilon} - \gamma_D)\nabla \tilde{v}_\epsilon \cdot \nabla u_0$$
$$= -\mu_0 \int_{D_\epsilon \setminus D} (\gamma_i - \gamma_e)\nabla \tilde{v}_\epsilon^i \cdot \nabla u_0^e + \mu_0 \int_{D \setminus D_\epsilon} (\gamma_i - \gamma_e)\nabla \tilde{v}_\epsilon^e \cdot \nabla u_0^i.$$

Using gradient estimates established by Li and Vogelius we have that

$$\|u_0^i\|_{C^{1,\alpha'}(\bar{D})} \leq C$$

and

$$\|\tilde{v}_\epsilon^i\|_{C^{1,\alpha'}(\bar{D}_\epsilon)} \leq C$$

and analogous estimates hold in $\Omega \setminus \bar{D}$ and in $\Omega \setminus \bar{D}_\epsilon$ so that we can approximate ∇u_0 and $\nabla \tilde{v}_\epsilon$ at points inside $D_\epsilon \triangle D$ with ∇u_0 and $\nabla \tilde{v}_\epsilon$ at points of ∂D_ϵ.

Proof

Using the transmission conditions and the fact that $\|\tilde{v}_\epsilon - \mu_0 u_0\|_{H^1(\Omega)} \leq \epsilon^{1/2}$ we get that

$$< (T - T_\epsilon)u_0, u_0 >= -\mu_0^2 \epsilon(\gamma_i - \gamma_e) \int_{\partial D} h(x) \left(\left(\frac{\partial u_0^e}{\partial \tau}(x) \right)^2 + \frac{\gamma_e}{\gamma_i} \left(\frac{\partial u_0^e}{\partial \nu}(x) \right)^2 \right) d\sigma_x$$

Finally from a result by Capdeboscq and Vogelius we can estimate the remainder term using

$$\|(T_\epsilon - T)u_0\|^2_{L^2(\Omega)} = \|\tilde{v}_\epsilon - \mu_0 u_0\|^2_{L^2(\Omega)} \leq C\epsilon^{1+\eta}$$

for some $\eta > 0$.

Part II

The inverse problem

Problem:

From knowledge of eigenvalues and eigenfunctions we want to determine the perturbation ϵh

Dual asymptotic formula

Let u_0 be an eigenfunction of problem

$$\begin{cases} \nabla \cdot (\gamma_D \nabla u_0) = -\omega_0^2 u_0 & \text{in} \quad \Omega, \\ \gamma_D \dfrac{\partial u_0}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \\ \displaystyle\int_\Omega u_0^2 = 1. \end{cases}$$

For $g \in L^2(\partial\Omega)$ such that $\int_{\partial\Omega} g u_0 = 0$, let w_g be the solution to

$$\begin{cases} \nabla \cdot (\gamma_D \nabla w_g) = -\omega_0^2 w_g & \text{in} \quad \Omega, \\ \gamma_D \dfrac{\partial w_g}{\partial \nu} = g & \text{on} \quad \partial\Omega, \\ \displaystyle\int_\Omega w_g u_0 = 1. \end{cases}$$

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Dual asymptotic formula

Multiplying the first equation by u_ϵ and integrating over Ω, we get from the divergence theorem

$$\int_{\partial\Omega} g u_\epsilon + \omega_0^2 \int_\Omega w_g u_\epsilon = \int_\Omega \gamma_D \nabla u_\epsilon \cdot \nabla w_g .$$

Since $\int_{\partial\Omega} g u_0 = 0$ and

$$\omega_\epsilon^2 \int_\Omega w_g u_\epsilon = \int_\Omega \gamma_{D_\epsilon} \nabla u_\epsilon \cdot \nabla w_g ,$$

we obtain

Dual asymptotic formula

$$\int_{\partial\Omega} g(u_\epsilon - u_0) + (\omega_0^2 - \omega_\epsilon^2)\int_\Omega w_g u_\epsilon = -\int_\Omega (\gamma_{D_\epsilon} - \gamma_D)\nabla u_\epsilon \cdot \nabla w_g \, dx.$$

We now use

- Gradient estimates for u_ϵ ad w_g
- The asymptotic expansion of the eigenvalues

$$\omega_\epsilon^2 - \omega_0^2 = O(\epsilon)$$

- L^2 estimates for $u_\epsilon - u_0$

$$\|u_\epsilon - u_0\|_{L^2(\Omega)} \leq C\epsilon^{\frac{1}{2}+\eta}$$

  for some positive η

- Energy estimates and trasmission conditions

Dual asymptotic formula

We derive

Theorem

The following asymptotic formula holds as $\epsilon \to 0$:

$$\int_{\partial\Omega} g(u_\epsilon - u_0) + (\omega_0^2 - \omega_\epsilon^2)\int_\Omega w_g u_0$$
$$= \epsilon(\gamma_i - \gamma_e)\int_{\partial D} h(x)\left(\frac{\partial u_0^e}{\partial\tau}(x)\frac{\partial w_g^e}{\partial\tau}(x) + \frac{\gamma_e}{\gamma_i}\frac{\partial u_0^e}{\partial\nu}(x)\frac{\partial w_g^e}{\partial\nu}(x)\right)d\sigma_x$$
$$+O(\epsilon^{1+\beta})$$

(1)

for some $\beta > 0$.

Reconstruction algorithm

With the measurements $(\omega_\epsilon^2 - \omega_0^2, (u_\epsilon - u_0)|_{\partial\Omega})$ and a finite number of linearly independent functions $g_1, \ldots, g_L$ on $\partial\Omega$ satisfying $\int_{\partial\Omega} g_l u_0 d\sigma = 0$, define the functional $J(\epsilon h)$ by

$$J(\epsilon h) := \sum_{l=1}^{L} \Bigg| \int_{\partial\Omega} g_l (u_\epsilon - u_0) + (\omega_0^2 - \omega_\epsilon^2) \int_\Omega w_{g_l} u_0$$
$$- \epsilon \int_{\partial D} h(x)(\gamma_i - \gamma_e) \left(\frac{\partial u_0^e}{\partial \tau}(x) \frac{\partial w_{g_l}^e}{\partial \tau}(x) + \frac{\gamma_e}{\gamma_i} \frac{\partial u_0^e}{\partial \nu}(x) \frac{\partial w_{g_l}^e}{\partial \nu}(x) \right) d\sigma_x \Bigg|^2$$

Algorithm:

Find min $J(\epsilon h)$

Reconstruction algorithm

Advantage:

$$\epsilon \int_{\partial D} h(x)(\gamma_i - \gamma_e) \left(\frac{\partial u_0^e}{\partial \tau}(x) \frac{\partial w_{g_l}^e}{\partial \tau}(x) + \frac{\gamma_e}{\gamma_i} \frac{\partial u_0^e}{\partial \nu}(x) \frac{\partial w_{g_l}^e}{\partial \nu}(x) \right) d\sigma_x$$

filters less oscillations of h because of the flexibility we have in choosing w_g.

Multiple eigenvalues

The method for reconstructing the shape deformation in the case of a multiple eigenvalue ω_0^2 is to minimize the functional $J(\epsilon h)$ over ϵh where J is given by

$$J(\epsilon h) := \sum_{l=1}^{L} \left| \frac{1}{m} \sum_{j=1}^{m} \int_{\partial\Omega} g_l (u_\epsilon^j - u_{0,j}) - \frac{1}{m} \sum_{j=1}^{m} (\omega_\epsilon^j)^2 + \omega_0^2 \right.$$
$$\left. - \frac{\epsilon}{m} \sum_{j=1}^{m} \int_{\partial D} h(x)(\gamma_i - \gamma_e) \left(\frac{\partial u_{0,j}^e}{\partial \tau}(x) \frac{\partial w_{g_l}^e}{\partial \tau}(x) + \frac{\gamma_e}{\gamma_i} \frac{\partial u_{0,j}^e}{\partial \nu}(x) \frac{\partial w_{g_l}^e}{\partial \nu}(x) \right) \right|^2$$

where $\{u_{0,j}\}_{j=1,\dots,m}$ be L^2-orthonormal eigenfunctions corresponding to ω_0^2, $(\omega_\epsilon^j)^2$ be the associated eigenvalues and for u_ϵ^j be the associated eigenfunction (normalized with respect to L^2) such that $(\omega_\epsilon^j)^2 \to \omega_0^2$ and $u_\epsilon^j \to u_{0,j}$ as $\epsilon \to 0$ and finally $g_1, \dots, g_L$ are linearly independent functions.

Numerical experiments

- Ω is the unit disk at the origin and D is the disk centered at $(0, -0.2)$ with radius 0.4.
- We fix the conductivities:

$$\gamma_e = 1 \quad \text{and} \quad \gamma_i = k = 1.5.$$

- To acquire data we use boundary integral methods corresponding to the first and second eigenvalue.
- The function g is of the form

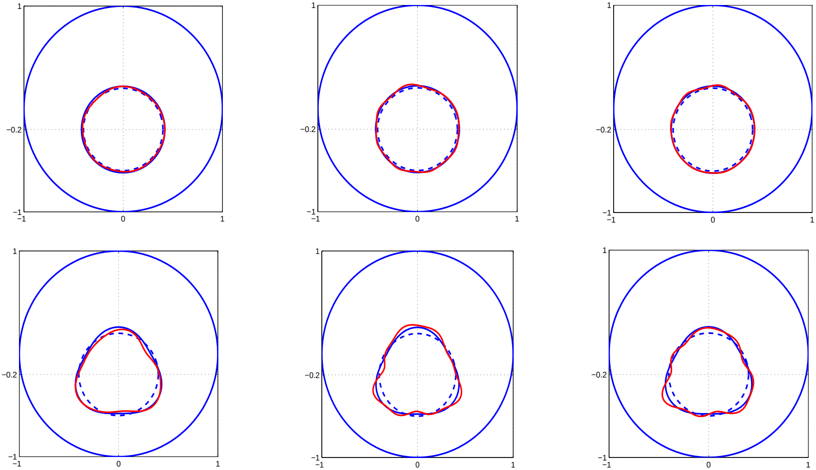
$$g_l = a_l + b_l \cos\theta + c_l \sin(l+1)\theta + d_l \cos(l+1)\theta, \ 1 \leq l \leq L(=8),$$

- We simulate the reconstruction method for the perturbation function h given by

$$h(\theta) = 1 - 2\sin(j\theta), \ j = 0, 3, 6, 9, \ \text{and } \epsilon = 0.02, \ 0.04.$$

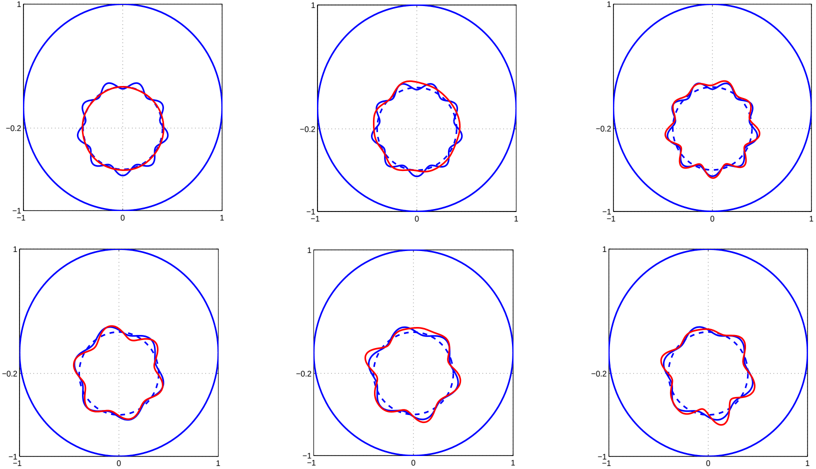
Example 1

$h(\theta) = 1 - 2\sin(j\theta),\ \ j = 0, 3,\ $ and $\epsilon = 0.02.$



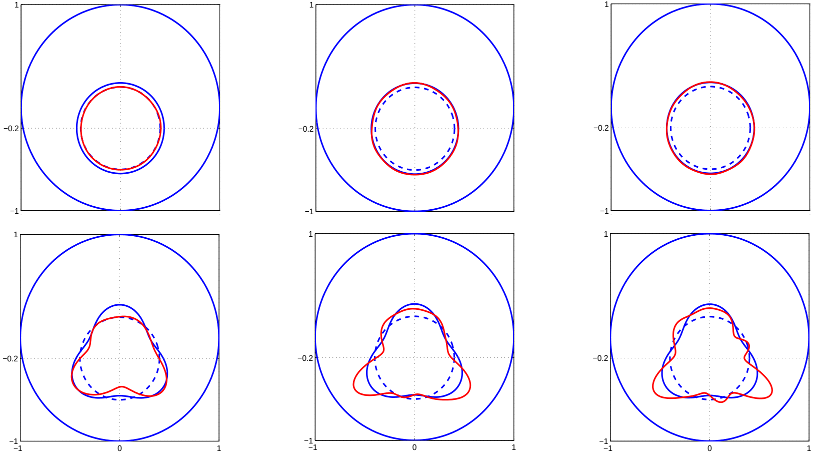
Example 1

$h(\theta) = 1 - 2\sin(j\theta),\ \ j = 6, 9,\ $ and $\epsilon = 0.02.$



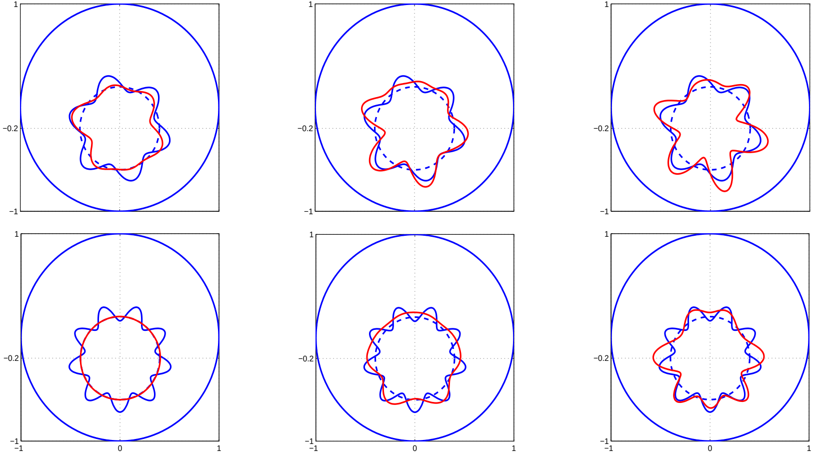
Example 2

$h(\theta) = 1 - 2\sin(j\theta),\ \ j = 0, 3$ and $\epsilon = 0.04$.



Example 2

$h(\theta) = 1 - 2\sin(j\theta)$, $j = 6, 9$, and $\epsilon = 0.04$.



Concluding remarks

- Resolution limit of our procedure.
- Resolution limit increases as the used eigenfrequency increases. Indeed, we have showed that multi-modal measurements yield better reconstruction than those obtained by only one pair of modal parameters.
- Extension to the elastic case.