Optimal control and regularity of boundary traces for some interactive PDE systems

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Introduction

Interactive PDE systems: composite systems of evolutionary PDEs (thermoelastic systems, PDE models for acoustic-structure or fluid-structure interactions, . . .)

Challenging features: they may comprise dynamics

- of different type (e.g., *hyperbolic/parabolic*),
- acting on manifolds of different dimensions,
- coupled by means of boundary traces.

A strong motivation for a PDE analysis: control problems

More recent contribution: analysis of some nonlinear coupled PDE systems, within the context of dynamical systems theory $\rightsquigarrow$ existence of global attractors, . . .

A linear model for fluid-solid interactions

Ω_f, $\Omega_s \subset \mathbb{R}^n$ (fluid and solid domains), Ω is the interior of $\overline{\Omega}_f \cup \overline{\Omega}_s$.

$\Gamma_s := \partial\Omega_s$ (interface), $\Gamma_f := \partial\Omega_f \setminus \partial\Omega_s$

$$\begin{cases} u_t - div\, \epsilon(u) + \nabla p = 0 & \text{in } \ \Omega_f \times (0,T) \\ div\, u = 0 & \text{in } \ \Omega_f \times (0,T) \\ w_{tt} - div\, \sigma(w) = 0 & \text{in } \ \Omega_s \times (0,T) \\ u = 0 & \text{on } \ \Gamma_f \times (0,T) \\ w_t = u & \text{on } \ \Gamma_s \times (0,T) \\ \sigma(w) \cdot \nu = \epsilon(u) \cdot \nu - p\nu - g & \text{on } \ \Gamma_s \times (0,T) \\ u(0,\cdot) = u_0 & \text{in } \ \Omega_f \\ w(0,\cdot) = w_0\,, w_t(0,\cdot) = w_1 & \text{in } \ \Omega_s\,. \end{cases} \tag{1}$$

u: velocity of the fluid, p: pressure; w: displacement of the solid.

ν: unit outward normal to Ω_s

σ: elastic stress tensor; ϵ: strain tensor

[Lions, 1969], [Du, Gunzburger, Hou & Lee, 2003]

The *uncontrolled* problem

Applications range from naval and aerospace engineering to cell biology and biomedical engineering.

Numerical studies: Shulkes, 1992; Errate, Dasser, 1995; Esteban & Maday, 1994; Farhat, Lesoinne & LeTallec, 1998, . . .

Existence of solutions has been explored in many papers:
San Martin, Starovoitov & Tucsnak, 2002;
Du *et al.*, 2003;
Da Veiga, 2004; Boulakia, 2004;
Feireisl, 2003; Coutand & Shkoller, 2005;
. . .

- Barbu, Grujić, Lasiecka & Tuffaha, 2007: existence of energy-level weak solutions (using a novel trace regularity result for the linear elastic equation);
- Avalos & Triggiani, 2007: they show *(i)* well-posedness, *(ii)* uniform stability properties, *(iii) backward uniqueness*.

A new trace regularity result

Lemma (Barbu *et al.*, 2007). *Let* (w, w_t) *be a solution to an elastic wave equation defined on* $\Omega \times (0, T)$*,*

$$w_{tt} - div\, \sigma(w) = 0\,,$$

driven by the following data:

$$w(0) \in H^1(\Omega_s)\,, \quad w_t(0) \in L_2(\Omega_s)\,, \quad w_t|_{\Gamma_s} \in L_2(0, T; H^{1/2}(\Gamma_s))\,.$$

Then

$$\sigma(w) \cdot \nu \in L_2((0, T) \times \Gamma_s) \oplus C([0, T), H^{-1/2}(\Gamma_s))\,.$$

Weak solutions

$H := \left\{u \in L_2(\Omega_f) : \; div\, u = 0\,,\; u \cdot \nu|_{\Gamma_f} = 0\right\}$, $Y = H \times H^1(\Omega_s) \times L_2(\Omega_s)$

$V := \left\{v \in H^1(\Omega_f) : \; div\, v = 0\,,\; u|_{\Gamma_f} = 0\right\}$ (Note: $(L_2)^n$, $(H^s)^n \rightsquigarrow L_2$, H^s)

Definition (Weak solution). *Let* $(u_0, w_0, w_1) \in H$ *and* $T > 0$. *We say that a triple* $(u, w, w_t) \in C([0,T], H \times H^1(\Omega_s) \times L_2(\Omega_s))$ *is a weak solution to the PDE system* (1) *if*

- $(u(\cdot,0), w(\cdot,0), w_t(\cdot,0)) = (u_0, w_0, w_1)$,
- $u \in L_2(0,T;V)$,
- $\sigma(w) \cdot \nu \in L_2(0,T;H^{-1/2}(\Gamma_s))$, $w_t|_{\Gamma_s} = u|_{\Gamma_s} \in L_2(0,T;H^{1/2}(\Gamma_s))$, *and*
- *the following variational system holds a.e. in* $t \in (0,T)$:

$$\begin{cases} \frac{d}{dt}(u,\phi)_f + (\epsilon(u), \epsilon(\phi))_f + \langle \sigma(w)\cdot\nu + g, \phi\rangle = 0 \\ \frac{d}{dt}(w_t,\psi)_s + (\sigma(w), \epsilon(\psi))_s - \langle \sigma(w)\cdot\nu, \psi\rangle = 0\,, \end{cases} \tag{2}$$

for all test functions $\phi \in V$ *and* $\psi \in H^1(\Omega_s)$.

Semigroup (abstract) formulation

state: $y(t) := (u(t), w(t), w_t(t)) \in Y \equiv H \times H^1(\Omega_s) \times L_2(\Omega_s)$

control: $g(t) \in U := L_2(\Gamma_s)$

The PDE problem (1) $\rightsquigarrow$ $\begin{cases} y'(t) = Ay(t) + Bg(t)\,, & 0 < t \le T \\ y(0) = y_0 \in Y \end{cases}$

where

- $A : D(A) \subset Y \to Y$ is the generator of a C_0-semigroup e^{At} on Y, $t \ge 0$ ([Barbu *et al.*, 2007]);
- $B \in \mathcal{L}(U, [\mathcal{D}(A^*)]')$; equivalently, $A^{-1}B \in \mathcal{L}(U, Y)$ ([Lasiecka & Tuffaha, 2008]).

Technical details

$$A = \begin{pmatrix} A_f & A_f N\sigma(\cdot)\cdot\nu & 0 \\ 0 & 0 & I \\ 0 & div\,\sigma & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} A_f N \\ 0 \\ 0 \end{pmatrix}$$

$$\mathcal{D}(A) = \Big\{ y = (u, w, z) \in H : \; u \in V,\; A_f(u + N\sigma(w)\,\nu) \in H,\; z \in H^1(\Omega_s)$$

$$div\,\sigma(w) \in L_2(\Omega_s),\; z|_{\Gamma_s} = u|_{\Gamma_s} \Big\}$$

where

- $A_f : V \to V'$ is defined by

$$(A_f u, \phi) = -(\epsilon(u), \epsilon(\phi)) \qquad \forall\, \phi \in V\,,$$

- while the (Neumann) map $N : L_2(\Gamma_s) \to H$ is defined as follows:

$$Ng = h \Leftrightarrow (\epsilon(h), \epsilon(\phi)) = \langle g, \phi \rangle \qquad \forall \phi \in V\,.$$

The linear-quadratic control problem

Cost functional:

$$J(g) = \int_0^T \left(|Ry(t)|_Z^2 + |g(t)|_U^2 \right) dt + |Gy(T)|_W^2 \,, \qquad (3)$$

$$R \in \mathcal{L}(Y, Z)\,,\; G \in \mathcal{L}(Y, W)$$

The problem:

$$\min_{g \in L_2(0,T;U)} J(g)\,, \quad \text{where } y \text{ solves} \quad \begin{cases} y' = Ay + Bg \\ y(0) = y_0\,. \end{cases}$$

One needs to explore the properties satisfied by *'the couple'* (A, B): the regularity of the operator $e^{At}B$ (equivalently, of $B^*e^{A^*t}$) plays a key role.

The application of abstract theories calls for a deep regularity analysis of a *dual* (uncontrolled) PDE problem.

Solvability of the optimal control problem

$Y_{-\alpha} := H \times H^{1-\alpha}(\Omega_s) \times H^{-\alpha}(\Omega_s)$, $0 < \alpha < \frac{1}{4}$

Theorem (Lasiecka & Tuffaha, *Preprint 2008*). *The semigroup e^{At} and the control operator B arising from the PDE problem* (1) *satisfy the following singular estimate: there exists a constant C such that*

$$|e^{At}Bg|_{Y_{-\alpha}} \le \frac{C}{t^{1/4+\epsilon}}|g|_{L_2(\Gamma_s)}\,, \quad 0 < t \le T\,. \tag{4}$$

Critical consequence: In view of (4), when $G \equiv 0$ the theory developed in [Avalos & Lasiecka, 1996], [Lasiecka & Triggiani, 2004] applies, provided that

$$R \in \mathcal{L}(Y_{-\alpha}, Z)\,; \tag{5}$$

for the Bolza problem, i.e. when $G \neq 0$, one needs both

$$R \in \mathcal{L}(Y_{-\alpha}, Z)\,, \qquad G \in \mathcal{L}(Y_{-\alpha}, W)$$

[Lasiecka & Tuffaha, 2008].

Then, one has, in particular ([Lasiecka & Triggiani, 2004], [Lasiecka & Tuffaha, 2008]),

(i) the *feedback synthesis* of the optimal control:

$$\hat{g}(t) = -B^* P(t)\, \hat{y}(t)\,, \qquad 0 \le t < T\,;$$

(ii) the operator $P(t)$ solves *the Differential Riccati Equation*

$$\frac{d}{dt}(P(t)x, z)_Y + (P(t)x, Az)_Y + (P(t)Ax, z)_Y + (Rx, Rz)_Y$$
$$-(B^* P(t)x, B^* P(t)z)_U = 0 \quad \forall x,\, z \in \mathcal{D}(A)\,,\ t \in [0, T)\,;$$

(iii) *the operator* $B^* P(t)$ is bounded: $Y \to U$, $0 \le t < T$.

Let us observe that the functional

$$J(g) = \int_0^T \left(|R_1 u(t)|^2_{0,\Omega_f} + |g(t)|^2_{0,\Gamma_s} \right) dt + |u(T)|^2_{0,\Omega_f} \,,$$

is allowed, with any $R_1 \in \mathcal{L}(L_2(\Omega_f))$, while

$$J(g) = \int_0^T \Big(\underbrace{|u(t)|^2_{0,\Omega_f} + (\sigma(w(t)), \epsilon(w(t)))_s + |w_t(t)|^2_{0,\Omega_s}}_{E(t)} + |g(t)|^2_{0,\Gamma_s} \Big) \, dt$$

is not allowed.

Our goal: ($G \equiv 0$) to remove the assumption (5) on R, yet showing well-posedness of Riccati equations.

Question: does the *couple* (A, B) satisfy the following conditions?

Assumptions (Acquistapace, B. & Lasiecka, 2005). *For each* $t \in [0, T]$, *the operator* $B^* e^{A^* t}$ *can be represented as*

$$B^* e^{A^* t} x = F(t)x + G(t)x, \qquad t \geq 0, \quad x \in \mathcal{D}(A^*), \tag{6}$$

where $F(t) : Y \to U$, $t > 0$, *and* $G(t) : \mathcal{D}(A^*) \to U$ *are bounded linear operators satisfying the following assumptions:*

(i) *there exists a constant* $\gamma \in (\frac{1}{2}, 1)$ *such that*

$$\|F(t)\|_{\mathcal{L}(Y,U)} \leq \frac{c_T}{t^\gamma} \qquad \forall t \in (0, T);$$

(ii) *the operator* $G(\cdot)$ *belongs to* $\mathcal{L}(Y, L^p(0, T; U))$ *for all* $p \in [1, \infty)$, *with*

$$\|G(\cdot)\|_{\mathcal{L}(Y,L^p(0,T;U))} \leq c_p < \infty \qquad \forall p \in [1, \infty);$$

(iii) *there is an* $\epsilon > 0$ *such that:*

(a) *the operator* $G(\cdot)A^{*-\epsilon}$ *belongs to* $\mathcal{L}(Y, C([0, T], U))$, *and in particular*

$$\|A^{-\epsilon} G(t)^*\|_{\mathcal{L}(U,Y)} \leq c < \infty \qquad \forall t \in [0, T]\,;$$

(b) *there exists* $q \in (1,2)$ *(which, in general, will depend on* ϵ*) such that the operator* $B^* e^{A^* \cdot} R^* R A^\epsilon$ *has an extension which belongs to* $\mathcal{L}(Y, L^q(0,T;U))$.

If so, we shall assume that the observation operator R is such that

(c) the operator $R^* R$ belongs to $\mathcal{L}(\mathcal{D}(A^\epsilon), \mathcal{D}(A^{*\epsilon}))$, i.e.

$$\|A^{*\epsilon} R^* R A^{-\epsilon}\|_{\mathcal{L}(Y)} \leq c < \infty\,.$$

Remarks:

- Assumption $(iii)(c)$ just requires that the observation operator *'maintains' regularity* (for instance, it allows $R = I$);
- Under assumption $(iii)(c)$, condition $(iii)(b)$ holds true if *there exists* $q \in (1,2)$ *such that the operator* $B^* e^{A^* \cdot} A^{*\epsilon}$ *has an extension which belongs to* $\mathcal{L}(Y, L^q(0,T;U))$.

The PDE counterpart

Regularity of the (unbounded) operator $B^* e^{A^* \cdot} \rightsquigarrow$ regularity of $u|_{\Gamma_s}$

Regularity of the operator $B^* e^{A^* \cdot} A^{*\epsilon} \rightsquigarrow$ regularity of $u_t|_{\Gamma_s}$

The corresponding Riccati theory (*in short*)

Under the listed assumptions, [Acquistapace, B. & Lasiecka, 2005] establishes the existence of a unique optimal pair $\{\hat{y}(\cdot), \hat{g}(\cdot)\}$, along with several properties. In particular,

(i) For each $y_0 \in Y$ the optimal pair $\{\hat{y}, \hat{g}\}$ satisfies

$$\hat{y}(\cdot) \in C([0,T], Y)\,, \quad \hat{g}(\cdot) \in \bigcap_{1\le p<\infty} L^p(0,T;U)\,.$$

(ii) The *gain operator* $B^*P(t)$ is bounded: $\mathcal{D}(A^\epsilon) \to C([0,T],U))$, and the *feedback synthesis* of the optimal control holds:

$$\hat{g}(t) = -B^*P(t)\,\hat{y}(t)\,, \qquad 0 \le t \le T\,.$$

(iii) The operator $P(t)$, which is selfadjoint and positive, solves the *Differential Riccati Equation*

$$\frac{d}{dt}(P(t)x,z)_Y + (P(t)x,Az)_Y + (P(t)Ax,z)_Y + (Rx,Rz)_Y$$
$$-(B^*P(t)x, B^*P(t)z)_U = 0 \quad \forall x,\, z \in \mathcal{D}(A)\,,\; t \in [0,T)\,.$$

The corresponding Riccati theory

Theorem (Acquistapace, B. & Lasiecka, 2005). *Under the listed assumptions, the following statements are valid.*

- *For each* $x \in Y$ *the optimal pair* $(\widehat{g}(\cdot, s; x), \widehat{y}(\cdot, s; x))$ *satisfies*

$$\widehat{y}(\cdot, s; x) \in C([s, T], Y), \quad \widehat{g}(\cdot, s; x) \in \bigcap_{1 \le p < \infty} L^p(s, T; U).$$

- *The operator* $\Phi(t, s)) \in \mathcal{L}(Y)$ *defined by*

$$\Phi(t, s)x = \widehat{y}(t, s; x) = e^{A(t-s)}x + [L_s \widehat{g}(\cdot, s; x)](t)\,, \ s \le t \le T\,, \ x \in Y\,, \tag{7}$$

is an evolution operator, i.e.

$$\Phi(t, t) = I_Y\,, \qquad \Phi(t, s) = \Phi(t, \tau)\Phi(\tau, s) \quad \text{for } s \le \tau \le t \le T\,.$$

- *For each* $t \in [0, T]$ *the operator* $P(t) \in \mathcal{L}(Y)$ *defined by*

$$P(t)x = \int_t^T e^{A^*(\tau - t)} R^* R \Phi(\tau, t)x \, d\tau\,, \qquad x \in Y\,,$$

is self-adjoint and positive; it belongs to $\mathcal{L}(Y, C([0, T], Y))$ *and is such that*

$$(P(s)x, x)_Y = J_s(\widehat{g}(\cdot, s; x), \widehat{y}(\cdot, s; x)) \qquad \forall s \in [0, T]\,.$$

- *The* *gain operator* $B^*P(t)$ *belongs to* $\mathcal{L}(\mathcal{D}(A^\epsilon), C([0,T],U))$ *and the optimal pair satisfies, for* $s \leq t \leq T$,

$$\hat{g}(t,s;x) = -B^*P(t)\hat{y}(t,s;x) \qquad \forall x \in Y\,.$$

- *The operator* $\Phi(t,s)$ *defined by* (7) *satisfies, for* $s < t \leq T$,

$$\frac{\partial \Phi}{\partial s}(t,s)x = -\Phi(t,s)(A - BB^*P(t))x \in L^{\frac{1}{\gamma}}(s,T;[\mathcal{D}(A^{*\epsilon})]')$$

for all $x \in \mathcal{D}(A)$, *and*

$$\frac{\partial \Phi}{\partial t}(t,s)x = (A - BB^*P(t))\Phi(t,s)x \in C([s,T],[\mathcal{D}(A^*)]')$$

for all $x \in \mathcal{D}(A^\epsilon)$.

- *The operator* $P(t)$ *satisfies the following* *Differential Riccati Equation* *on* $[0,T)$:

$$(\tfrac{d}{dt}P(t)x,y)_Y + (P(t)x,Ay)_Y + (P(t)Ax,y)_Y$$
$$+(R^*Rx,y)_Y - (B^*P(t)x,B^*P(t)y)_Y = 0 \quad \forall\, x,y \in \mathcal{D}(A).$$

The trace regularity results

Theorem (B. & Lasiecka)**.** *Consider the PDE system* (1) *with* $g \equiv 0$*. Let* $y(t) = (u(t), w(t), w_t(t))$ *be the solution corresponding to an initial datum* $y_0 = (u_0, w_0, w_1)$*. The fluid component* u *admits a splitting* $u(t) = u_1(t) + u_2(t)$*, and the following statements pertain to the regularity of the traces of* u_1*,* u_2 *and* u_t *on* Γ_s*, respectively.*

(i) *The component* u_1 *satisfies a pointwise (in time) "singular estimate", namely there exists a positive constant* C_T *such that*

$$|u_1(t)|_{\Gamma_s}|_{L_2(\Gamma_s)} \le \frac{C_T}{t^{1/4+\delta}} |y_0|_Y \qquad \forall y_0 \in Y\,, \quad \forall t \in (0, T]$$

(for arbitrarily small $\delta > 0$*).*

(ii) *The component* u_2 *satisfies the following regularity:*

(iia) *if* $y_0 \in Y$*, then* $u_2|_{\Gamma_s} \in L_p(0, T; L_2(\Gamma_s))$ *for all (finite)* $p \ge 1$*;*

(iib) *if* $y_0 \in \mathcal{D}(A^\epsilon)$*,* $\epsilon \in (0, \frac{1}{4})$*, then* $u_2|_{\Gamma_s} \in C([0, T], L_2(\Gamma_s))$*.*

(iii) *Let now* $y_0 \in \mathcal{D}(A^{1-\epsilon})$*, with* $\epsilon \in (0, \frac{1}{4})$*. Then, the fluid component* u *of the corresponding solution satisfies, with* $q \in (1, 2)$*,*

$$u_t|_{\Gamma_s} \in L_q(0, T; L_2(\Gamma_s))\,. \tag{8}$$

Remarks about the result

1. The trace regularity results **do not follow** from the interior regularity.

2. The proof exploits

   - the parabolic regularity of the fluid component,
   - sharp trace results pertaining to the 'solid' component obtained (by using *microlocal analysis* arguments) in [Barbu *et al.*, 2007] and [Lasiecka & Tuffaha, 2008],

   and utilizes interpolation techniques.

The ‘source of inspiration’

A *boundary control problem* for a system of thermoelastic plate equations:

$\Omega \subset \mathbb{R}^2$, $\Gamma = \partial\Omega$ smooth

$$
\begin{cases}
w_{tt} - \rho\Delta w_{tt} + \Delta^2 w + \Delta\theta = 0 & \text{in } (0,T] \times \Omega \\
\theta_t - \Delta\theta - \Delta w_t = 0 & \text{in } (0,T] \times \Omega \\
w = \frac{\partial w}{\partial \nu} = 0 \quad \text{(clamped B.C.)} & \text{on } (0,T] \times \Gamma \\
\theta = g \qquad \text{(Dirichlet boundary control)} & \text{on } (0,T] \times \Gamma \\
w(0,\cdot) = w^0,\ w_t(0,\cdot) = w^1; \quad \theta(0,\cdot) = \theta^0 & \text{in } \Omega\,.
\end{cases}
$$

[B. & Lasiecka, 2004], [Acquistapace, B. & Lasiecka, 2005]