Workshop

“Direct, Inverse And Control Problems For PDE’s”

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An ultraparabolic problem arising from age–dependent population diffusion

The problem.

Find $u : [0,T] \times \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}_+$, satisfying

- $u_t(t,a,x)+u_a(t,a,x)=-d(t,a,x)+\Delta u(t,a,x)$,
  $t \in (0,T),\ a > 0,\ x \in \Omega$,
- $u(t,0,x) = b(t,x), \qquad t \in [0,T],\ x \in \Omega$,
- $u(t,a,x) = 0, \qquad t \in (0,T),\ a > 0,\ x \in \partial\Omega$,
- $u(0,a,x) = u_0(a,x), \quad a > 0,\ x \in \Omega$.

$u(t,a,x) =$ density of members of age a and position x at time t. Hence

$$P(t,x) := \int_0^{+\infty} u(t,a,x)\,da, \qquad N(t) := \int_\Omega P(t,x)\,dx$$

are the spatial density of population and the number of individuals.

$\Rightarrow L^1(\mathbb{R}_+ \times \Omega) =$ state space for the density u.

The death term d and the birth process b are assumed to be of the form

- $d(t, a, x) = -\mu(t, a, x)u(t, a, x)$,

- $b(t, x) = \int_0^{+\infty} \beta(t, a, x)u(t, a, x)\, da$.

$\mu(t, a, x)$ and $\beta(t, a, x)$ represent the mortality rate and the fertility rate.

Hence we have a problem with a nonlocal boundary condition.

Method of characteristics : G. Di Blasio [1979], Kunisch, Schappacher and Webb [1985], J. Dyson, E. Sanchez, R. Villella–Bressan, G. F. Webb [2007], $\cdots$.

Sum of Operators : G. Di Blasio and L. Lamberti [1978], Iannelli and Busemberg [1983], Da Prato and Sinestrari [1987].

Evolution Operators : A. Rhandi and R. Schnaubelt, [1999], P. Magal and S. Ruan [2007].

Sum of Operators, Evolution Operators:

Rewrite the equation in the following form

$$u_t(t,a,x) = -u_a(t,a,x) - \mu(t,a,x)u(t,a,x) + \Delta u(t,a,x),$$

prove that the operator in the (a,x) variables

$$A := -\partial/\partial a - \mu I + \Delta$$

($+$ boundary and initial conditions) generates a strongly continuous semigroup. Finally solve the equation

$$u'(t) = Au(t)$$

The method of characteristics.

Find $u : [0,T] \to L^1(\mathbb{R}_+ \times \Omega)$, satisfying

- $Du(t,a,x) = -\mu(t,a,x)u(t,a,x) + \Delta u(t,a,x),$
  $t \in (0,T), a > 0, x \in \Omega$

- $u(t,0,x) = \int_0^{+\infty} \beta(t,a,x)u(t,a,x)\, da,$
  $t \in [0,T],\ x \in \Omega,$

- $u(t,a,x) = 0, \qquad t \in (0,T),\ a > 0,\ x \in \partial\Omega,$

- $u(0,a,x) = u_0(a,x), \quad a > 0,\ x \in \Omega,$

where

$$Du(t,a,\cdot) := \lim_{h \to 0} \frac{u(t+h,a+h,\cdot) - u(t,a,\cdot)}{h}.$$

u is called a strong solution of the original problem

Fixed point argument

Find a function u satisfying

(1) $Du(t,a,x) = -\mu(t,a,x)u(t,a,x) + \Delta u(t,a,x),$
$t \in (0,T), a > 0, x \in \Omega$

(2) $u(t,0,x) = b(t,x), \quad t \in [0,T],\ x \in \Omega,$

(3) $u(t,a,x) = 0, \quad t \in (0,T),\ a > 0,\ x \in \partial\Omega,$

(4) $u(0,a,x) = u_0(a,x), \quad a > 0,\ x \in \Omega,$

supplemented by the condition

(5) $b(t,x) = \int_0^{+\infty} \beta(t,a,x)u_b(t,a,x)\, da$

Denoting by u_b the solution of (1)–(4), for a given b, and by $b \to S(b)$ the operator

$$S(b) := \int_0^{+\infty} \beta(t,a,x)u_b(t,a,x)\, da$$

find a fixed point of S.

Solve (1)–(4)

Set, for fixed $t, a > 0$,

- $m_1(s,x) := \mu(t-a+s, s, x); u_{0,1}(x) := b(t-a, x)$, if $a \le t$,

- $m_2(s,x) := \mu(s, a-t+s, x); u_{0,2}(x) := u_0(a-t, x)$, if $a > t$

if u_i are solutions of the following problems, $i = 1, 2$,

$$\begin{cases} u_s(s,x) = -m_i(s,x)u(s,x) + \Delta u(s,x), & s > 0, x \in \Omega, \\ u(s,x) \;\; = 0, & s > 0, \; x \in \partial\Omega, \\ u(0,x) \;\; = u_{0,i}(x), & x \in \Omega. \end{cases}$$

then the solution of (1)–(4) is given by

$$u(t,a,x) := \begin{cases} u_1(t,x), & \text{if } a \le t \\ u_2(a,x), & \text{if } a > t. \end{cases}$$

Nonautonomous Parabolic equations in $L^1(\Omega)$

$$\begin{cases} v_s(s,x) = -m(s,x)v(s,x) + \Delta v(s,x) + f(s,x), \\ \qquad\qquad s \in (0,T),\ x \in \Omega, \\ v(s,x) = 0, \qquad\qquad s > 0,\ x \in \partial\Omega, \\ v(0,x) = v_0(x), \qquad x \in \Omega. \end{cases}$$

Assumptions: $m : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ is measurable and there exists $M \in L^1_{loc}(\mathbb{R}_+)$ verifying the properties

- $m(s,x) \leq M(s) \quad s \in \mathbb{R}_+,\ x \in \Omega$;
- $|m(s,x) - m(s,y)| \leq M(s)d(x,y),\ s \in \mathbb{R}_+,\ x,y \in \Omega$.

In addition, for some $\theta \in (0,1/2)$,

- $f \in L^1_{loc}(\mathbb{R}_+; W^{2\theta,1}(\Omega))$, $v_0 \in W^{2\theta,1}(\Omega)$.

Then there exists a "regular" solutiont v, for each $T > 0$.

more precisely: v satisfies the properties

- $v \in C(0,T;W^{2\theta,1}(\Omega))$;
- $v_s, \Delta v \in L^1(0,T;W^{2\theta,1}(\Omega))$

In addition there exists $c = c(\theta)$, satisfying

- $\|v(s)\|_{W^{2\theta,1}(\Omega)} \le e^{\mathcal{M}(s)}[\,\|v_0\|_{W^{2\theta,1}(\Omega)} + \int_0^s \|f(\sigma)\|_{W^{2\theta,1}(\Omega)}\,d\sigma\,]$,
- $\|v_s\|_{L^1(0,T;W^{2\theta,1}(\Omega))}, \le ce^{\mathcal{M}(T)}[\,\|v_0\|_{W^{2\theta,1}(\Omega)} + \|f\|_{L^1(0,T;W^{2\theta,1}(\Omega))}\,]$,
- $\|\Delta v\|_{L^1(0,T;W^{2\theta,1}(\Omega))} \le ce^{\mathcal{M}(T)}[\,\|v_0\|_{W^{2\theta,1}(\Omega)} + \|f\|_{L^1(0,T;W^{2\theta,1}(\Omega))}\,]$

where

$$\mathcal{M}(s) := \int_0^s M(\sigma)\,d\sigma$$

Assumptions

$\mu \geq 0$ and $\beta \geq 0$ are measurable, continuous with respect to t and there exist $\mu_0 \in L^1_{loc}(\mathbb{R}_+)$, $\beta_0 \in L^\infty(\mathbb{R}_+)$ and $b_0 > 0$ satisfying, for $t > 0$, $a > 0$ and $x, y \in \Omega$

- $\mu(t,a,x) \leq \mu_0(a)$; $|\mu(t,a,x) - \mu(t,a,y)| \leq \mu_0(a) d(x,y)$;
- $\beta(t,a,x) \leq \beta_0(a)$; $|\beta(t,a,x) - \beta(t,a,y)| \leq \beta_0(a) d(x,y)$,
- $\beta_0(a) e^{\int_0^a \mu_0(\sigma)\, d\sigma} \leq b_0$.

In addition, for some $\theta \in (0, 1/2)$

- $u_0 \in L^1(\mathbb{R}_+; W^{2\theta,1}(\Omega))$,
- $\int_0^{+\infty} e^{\int_a^{a+T} \mu_0(\sigma)\, d\sigma} \|u_0(a,\cdot)\|_{W^{2\theta,1}}\, da < +\infty$

Existence 1

Theorem 1. Given $b \in C(0,T;L^1(\Omega)) \cap B(0,T;W^{2\theta,1}(\Omega))$ there exists $u = u_b \in C(0,T;L^1(\mathbb{R}_+ \times \Omega))$ satisfying (1)–(4). Furthermore

- $\int_0^{+\infty} \|u(t,a,\cdot)\|\,da \leq \int_0^t \|b(s,\cdot)\|\,ds + \int_0^{+\infty} \|u_0(a,\cdot)\|\,da.$

- $\int_0^{+\infty} \|u(t,a,\cdot)\|_{W^{2\theta,1}}\,da \leq \int_0^t e^{\int_0^{t-s} \mu_0(\sigma)\,d\sigma} \|b(s,\cdot)\|_{W^{2\theta,1}}\,ds$
  $+ \int_0^{+\infty} e^{\int_a^{a+t} \mu_0(\sigma)\,d\sigma} \|u_0(a,\cdot)\|_{W^{2\theta,1}}\,da$

and

- $\int_0^T \int_0^{+\infty} \|\Delta u(t,a,\cdot)\|_{W^{2\theta,1}}\,da\,dt \leq \cdots$

- $\int_0^T \int_0^{+\infty} \|Du(t,a,\cdot)\|_{W^{2\theta,1}}\,da\,dt \leq \cdots$

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Proof. The assertions follow from results on parabolic equations $\cdots$.

Existence 2

Theorem 2 Let the assumptions on μ, β and u_0 hold. Then there exists a unique $u \in C(0, T; L^1(\mathbb{R}_+ \times \Omega))$, verifying

- $u \in B(0, T; L^1(\mathbb{R}_+; W^{2\theta,1}(\Omega))$
- $\Delta u \in L^1(0, T; L^1(\mathbb{R}_+; W^{2\theta,1}(\Omega))$
- $Du \in L^1(0, T; L^1(\mathbb{R}_+; W^{2\theta,1}(\Omega))$

solution of (1)–(4) satisfying

$$u(t, 0, x) = \int_0^{+\infty} \beta(t, a, x) u(t, a, x)\, da.$$

Morover u satisfies estimates similar to those of u_b, given by Theorem 1.

Proof. We use a fixed point argument and the results of Theorem 1.