Equivalent states in viscoelasticity and applications to semi-group theory

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The classical initial boundary value problem related with the differential system of viscoelasticity is given in the domain $Q = \Omega \times (0, \infty)$ by

$$\begin{aligned} \frac{\partial^2}{\partial t^2}\mathbf{u}(x,t) &= \nabla \cdot \mathbf{G}_0 \nabla \mathbf{u}(x,t) \\ &\quad + \nabla \cdot \int_0^\infty \dot{\mathbf{G}}(x,s) \nabla \mathbf{u}^t(x,s) ds + \mathbf{f}(x,t), \end{aligned} \qquad (1)$$

where $\mathbf{u}$ is the displacement, $\nabla \mathbf{u}^t(x,s) = \nabla \mathbf{u}(x, t-s)$ is the history of the strain $\nabla \mathbf{u}$ and $\dot{\mathbf{G}}(x,\cdot) \in H^1(0,\infty)$.

To this equation we must associate the initial conditions

$$\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \dot{\mathbf{u}}(x,0) = \dot{\mathbf{u}}_0(x), \quad \forall x \in \Omega, \qquad (2)$$

$$\mathbf{u}^{t=0}(x,s) = \mathbf{u}^0(x,s) \quad \forall\, (x,s) \in \Omega \times (0,\infty) \qquad (3)$$

together with the boundary conditions, which, for example, can be expressed by

$$\mathbf{u}(x,t)|_{\partial\Omega} = \mathbf{0} \tag{4}$$

It is easy to show that the equation (1) can be written in the form

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}\mathbf{u}(x,t) &= \nabla\cdot\left[\mathbf{G}_0(x)\nabla\mathbf{u}(x,t) + \int_0^t \dot{\mathbf{G}}(x,s)\nabla\mathbf{u}^t(x,s)ds\right] \\
&\quad +\nabla\cdot\int_t^\infty \dot{\mathbf{G}}(x,s)\nabla\mathbf{u}^t(x,s)ds + \mathbf{f}(x,t) \\
&= \nabla\cdot\left[\mathbf{G}_0(x)\nabla\mathbf{u}(x,t) + \int_0^t \dot{\mathbf{G}}(x,s)\nabla\mathbf{u}^t(x,s)ds\right] \\
&\quad +\nabla\cdot\mathbf{F}(x,t) + \mathbf{f}(x,t) \qquad (5)
\end{aligned}$$

where we have put

$$\begin{aligned}\mathbf{F}(x,t) &= \int_t^\infty \dot{\mathbf{G}}(x,s)\nabla\mathbf{u}^t(x,s)ds \\ &= \int_0^\infty \dot{\mathbf{G}}(x,t+s)\nabla\mathbf{u}^0(x,s)ds. \qquad (6)\end{aligned}$$

For the problem (5), the initial condition is

$$\mathbf{u}(x,0)=\mathbf{u}_0(x), \quad \dot{\mathbf{u}}(x,0)=\dot{\mathbf{u}}_0(x), \quad \forall x\in\Omega, \qquad (7)$$

It is evident, by virtue of (6), that two different initial histories $\mathbf{u}_1^0(x,s)$ and $\mathbf{u}_2^0(x,s)$ yield the same solution if $\forall\tau\in[0,\infty)$

$$\int_0^\infty \dot{\mathbf{G}}(\tau+s)\nabla\mathbf{u}_1^0(s)ds=\int_0^\infty \dot{\mathbf{G}}(\tau+s)\nabla\mathbf{u}_2^0(s)ds, \qquad (8)$$

because they provide the same quantity $\mathbf{F}(x,t)$.

Therefore, we arrive at the notion of equivalence between initial histories, which can be characterized by the condition (8).

Consequently, it clearly appears that these different histories must be considered as a unique state for the viscoelastic material.

The concept of equivalence among states was introduced by Noll in his axiomatic formulation of continuum mechanics W.Noll, *Arch. Rational Mech. Anal.* (1972), on the ground of considerations related to the definition of material, considered as a dynamic system.

In Noll's theory, in particular, denoting by $P : [0, d_P) \to Sym$ a process of duration d_P, defined by $P(\tau) = \nabla \dot{\mathbf{u}}_P(\tau)$

$\forall \tau \in [0, d_P)$, two states σ_1 and σ_2 are equivalent if for any process P the response of the material, represented by the stress tensor $\mathbf{T}$, is such that

$$\mathbf{T}(\sigma_1 * P) = \mathbf{T}(\sigma_2 * P), \tag{9}$$

where $(\sigma * P)$ denotes the continuation of the state σ with the process P.

This requires, in the linear case, that two histories are equivalent if the conditions (8) hold, i.e.

$$\int_0^\infty \dot{\mathbf{G}}(\tau + s)\nabla \mathbf{u}_1^0(s)ds = \int_0^\infty \dot{\mathbf{G}}(\tau + s)\nabla \mathbf{u}_2^0(s)ds,$$

for every $\tau \in [0, \infty)$.

Therefore, following Noll's view point, one attains the same conclusions obtained by studying the differential problem directly.

From these observations it appears natural to introduce a new notion of state, called **minimal state**, defined by $\sigma(t) = \mathbf{I}_r^t = (\nabla \mathbf{u}(t), \mathbf{I}^t)$, where $\mathbf{I}^t$ denotes the equivalence class of the strain histories $\nabla \mathbf{u}^t(s)$ defined by

$$\mathbf{I}^t(\tau) = \int_0^\infty \dot{\mathbf{G}}(\tau + s) \nabla \mathbf{u}^t(s) ds, \ \ \tau \in [0, \infty) \tag{10}$$

while

$$\begin{aligned} \mathbf{I}_r^t(\tau) &= \int_0^\infty \dot{\mathbf{G}}(\tau + s) \nabla \mathbf{u}_r^t(s) ds \qquad (11) \\ &= \mathbf{G}(\tau) \nabla \mathbf{u}(t) + \int_0^\infty \dot{\mathbf{G}}(\tau + s) \nabla \mathbf{u}^t(s) ds. \end{aligned}$$

where $\nabla \mathbf{u}_r^t(s) = \nabla \mathbf{u}(t) - \nabla \mathbf{u}^t(s)$ is the relative strain history. Of course

$$\mathbf{T}(t) = \mathbf{I}_r^t(0)$$

Then, $\mathbf{I}_r^t(\tau)$ is the stress response to a constant process

The reason why we use the name **minimal** state is connected with the minimal information required to identify the state.

Consider the kernel $\dot{\mathbf{G}}(s) = \mathbf{A}\mathrm{e}^{-\alpha s}$, a history $\nabla \mathbf{u}^t(s)$ and the function

$$\begin{aligned}
\mathbf{I}^t(\tau) &= \int_0^\infty \mathbf{A}\mathrm{e}^{-\alpha(\tau+s)} \nabla \mathbf{u}^t(s) ds \\
&= \mathbf{A}\mathrm{e}^{-\alpha\tau} \int_0^\infty \mathrm{e}^{-\alpha s} \nabla \mathbf{u}^t(s) ds \\
&= \mathrm{e}^{-\alpha\tau} (\mathbf{T}(\nabla \mathbf{u}^t) - \mathbf{G}_0 \nabla \mathbf{u}(t))
\end{aligned}$$

Therefore, the state $\sigma(t)$ is given by

$$\sigma(t) = (\nabla \mathbf{u}(t), \mathbf{T}(t))$$

For a kernel $\dot{\mathbf{G}}(s) = \sum_{i=1}^{n} \mathbf{A}_i \mathrm{e}^{-\alpha_i s}$ the state is

$$\sigma(t) = (\nabla \mathbf{u}(t), \mathbf{T}(t), \mathbf{T}^{(1)}(t), ..., \mathbf{T}^{(n-1)}(t))$$

The introduction of this notion of equivalence can have meaningful improvements in the study of stability problems. In fact, the new notion of state influences the norm of the state space and then the conditions for the stability are remarkably affected by the the topology chosen in that space.

Therefore, we shall refer to this formulation and we shall try to express both the normed spaces and the free energies as functions of the minimal state.

Semi-group theory

The differential system (1) can be rewritten in the form

$$\begin{aligned}
\frac{d}{dt}\mathbf{u}(\cdot,t) &= \mathbf{v}(\cdot,t) \\
\frac{d}{dt}\mathbf{v}(\cdot,t) &= \nabla\cdot\left[\mathbf{G}_\infty\nabla\mathbf{u}(\cdot,t)+\int_0^\infty\dot{\mathbf{G}}(\cdot,s)\nabla\mathbf{u}_r^t(\cdot,s)ds\right] \\
\frac{d}{dt}\nabla\mathbf{u}_r^t(\cdot,s) &= -\frac{d}{ds}\nabla\mathbf{u}_r^t(\cdot,s)-\nabla\mathbf{v}(\cdot,t)
\end{aligned} \tag{12}$$

This system is supplemented by Dirichlet boundary conditions

$$\mathbf{u}|_{\partial\Omega}=0 \tag{13}$$

The problem (12-13) can be studied using semi-group theory, where the state is given by the triple

$$\chi=(\mathbf{u},\mathbf{v},\nabla\mathbf{u}_r^t)\in\mathcal{G}=H_0^1(\Omega)\times L^2(\Omega)\times\mathcal{D}_G.$$

The space $\mathcal{D}_G$ is the domain of definition of Graffi-Volterra free energy ψ_G, D.Graffi, *Rend. Sem. Mat. Univ. Padova* (1968)

$$\psi_G(\nabla \mathbf{u}^t) = \frac{1}{2}\mathbf{G}_\infty \nabla \mathbf{u}(t) \cdot \nabla \mathbf{u}(t) + \frac{1}{2}\int_0^\infty \dot{\mathbf{G}}(s)\nabla \mathbf{u}_r^t(s) \cdot \nabla \mathbf{u}_r^t(s) ds. \tag{14}$$

given by

$$\mathcal{D}_G = \Big\{\mathbf{u}^t(s) \in H_0^1(\Omega),\ s \in \mathbb{R}^+ \int_0^\infty \dot{\mathbf{G}}(s)\nabla \mathbf{u}_r^t(s) \cdot \nabla \mathbf{u}_r^t(s) ds < \infty\Big\}$$

The functional (14) is a free energy under the following restrictions on the kernel $\dot{\mathbf{G}}$

- for all $(x,s) \in \Omega \times \mathbb{R}^+$

$$\dot{\mathbf{G}}(x,s) < 0\ , \quad \ddot{\mathbf{G}}(x,s) \geq 0\,, \tag{15}$$

- there exists $\alpha \in \mathbb{R}^{++}$ such that

$$\ddot{\mathbf{G}}(x,s) + \alpha\dot{\mathbf{G}}(x,s) \geq 0, \quad \text{for all } (x,s) \in \Omega \times \mathbb{R}^+ \tag{16}$$

Theorem. *(M.F. & B. Lazzari, Arch. Rational Mech. Anal. (1992). Under the hypotheses (15), (16), for any initial condition $\chi_0 \in \mathcal{G}$, there exists a solution $\chi = (\mathbf{u}, \mathbf{v}, \mathbf{I}_r^t)$ such that*

$$\| \mathbf{v}(t) \|_{L^2} + \psi_G(\nabla \mathbf{u}^t) \leq M e^{-\mu t}[\| \mathbf{v}(0) \|_{L^2} + \psi_G(\nabla \mathbf{u}^0)] \tag{17}$$

where M and μ are suitable constants.

It is easy to prove that the norm related to the Graffi-Volterra free energy (14) introduces a non-natural separation between histories. In fact, two equivalent histories have a not zero distance, in contrast with the impossibility of distinguishing the future effects of the two initial histories.

When we use the function $\mathbf{I}_r^t(\cdot,\tau)$, the system (12) can be reduced to the following problem

$$\begin{aligned}
\frac{d}{dt}\mathbf{u}(\cdot,t) &= \mathbf{v}(\cdot,t) \\
\frac{d}{dt}\mathbf{v}(\cdot,t) &= \nabla\cdot[\mathbf{G}_\infty(\cdot)\nabla\mathbf{u}(\cdot,t)] - \nabla\cdot\mathbf{I}_r^t(\cdot,0) \qquad (18) \\
\frac{d}{dt}\mathbf{I}_r^t(\cdot,\tau) &= \frac{\partial}{\partial\tau}\mathbf{I}_r^t(\cdot,\tau) - \tilde{\mathbf{G}}(\tau)\nabla\mathbf{v}(\cdot,t)
\end{aligned}$$

with the boundary condition (13), while the initial conditions are given by

$$\mathbf{u}(x,0) = \mathbf{u}_0(x) \ , \ \mathbf{v}(x,0) = \mathbf{v}_0(x) \ , \qquad (19)$$

$$\mathbf{I}_r^{t=0}(x,\tau) = \mathbf{I}_r^0(x,\tau) \, , \, \tau \in \mathbb{R}^+. \qquad (20)$$

For this problem, the state is given by the triple $\chi = (\mathbf{u}, \mathbf{v}, \mathbf{I}_r^t)$, which is an element of the Hilbert space $\mathcal{F} := H_0^1(\Omega) \times L^2(\Omega) \times \mathcal{D}_I$.

The problem now is to find the domain $\mathcal{D}_I$ on which is defined the function $\mathbf{I}_r^t$. Let me remember that in the previous system the domain of definition of the relative history is given by the domain of the Graffi free energy.

Also in this problem the domain $\mathcal{D}_I$ will be related with the free energy connected with the function $\mathbf{I}_r^t$. Therefore, it is crucial to find a new free energy.

In order to obtain such a free energy, let me remember its definition. The free energy is a function that must satisfy the inequality

$$\dot{\psi}(\sigma) \leq \mathbf{T}(\sigma)\cdot\dot{\mathbf{E}} \tag{21}$$

In the papers M.F.,*Wave and Stability* (2004), L.Deseri, M.F., M.Golden, *Arch. Rational Mech. Anal.* (2006) we have proved that

$$\begin{aligned}\psi_I(\sigma) = &\frac{1}{2}\int_\Omega \mathbf{G}_\infty(x)\nabla\mathbf{u}(x,t)\cdot\nabla\mathbf{u}(x,t)dx \\ &- \frac{1}{2}\int_0^\infty\int_\Omega \dot{\mathbf{G}}^{-1}(x,\tau)\mathbf{I}_{r\tau}^t(x,\tau)\cdot\mathbf{I}_{r\tau}^t(x,\tau)dxd\tau\end{aligned} \tag{22}$$

is a free energy. In the following the domain of the free energy $\psi_I(\mathbf{I}_r^t)$ will be denoted by $\mathcal{D}_I$. Then, the Hilbert space $\mathcal{F} := H_0^1(\Omega) \times L^2(\ \Omega)\times\mathcal{D}_I$ is well defined.

Theorem. *Under the hypotheses (15), (16), for any initial condition* $\chi_0 \in \mathcal{F}$ *, there exists a solution* $\chi = (\mathbf{u}, \mathbf{v}, \mathbf{I}_r^t)$ *such that*

$$\|\mathbf{v}(t)\|_{L^2} + \Psi_I(\mathbf{I}^t) \leq Me^{-\mu t}(\|\mathbf{v}(0)\|_{L^2} + \Psi_I(\mathbf{I}^0)) \qquad (23)$$

where M *and* μ *are suitable constants.*

Proof. Consider the functional total energy ζ defined by

$$\zeta(\mathbf{v}(t), \mathbf{I}^t) = \frac{1}{2}\mathbf{v}^2(x,t) + \Psi_I(\mathbf{I}^t) \qquad (24)$$

which satisfies the equality

$$
\begin{aligned}
& \dot{\zeta}(x,t) - \mathbf{v}(x,t)\cdot\dot{\mathbf{v}}(x,t) - \mathbf{T}(x,t)\cdot\nabla\mathbf{v}(x,t) \\
= \;& \frac{1}{2}\dot{\mathbf{G}}^{-1}(x,0)\mathbf{I}^{t}_{r\tau}(x,0)\cdot\mathbf{I}^{t}_{r\tau}(x,0) \\
- \;& \frac{1}{2}\int_0^\infty \dot{\mathbf{G}}^{-1}(x,\tau)\ddot{\mathbf{G}}(x,\tau)\dot{\mathbf{G}}^{-1}(x,\tau)\mathbf{I}^{t}_{r\tau}(x,\tau)\cdot\mathbf{I}^{t}_{r\tau}(x,\tau)d\tau.
\end{aligned}
$$

By means of hypotheses (15), (16) and relation (19) we obtain

$$
\begin{aligned}
\int_\Omega \dot{\zeta}(x,t)dx &\leq \frac{\alpha_1}{2}\int_\Omega\int_0^\infty \dot{\mathbf{G}}^{-1}(x,\tau)\mathbf{I}^{t}_{r\tau}(x,\tau)\cdot\mathbf{I}^{t}_{r\tau}(x,\tau)d\tau dx \\
&\leq 0. \qquad (25)
\end{aligned}
$$

Thus, if we introduce the total energy

$$\mathcal{E}(t) = \int_\Omega \zeta(x,t)dx \qquad (26)$$

then

$$0 \leq \mathcal{E}(t) \leq \mathcal{E}(0). \tag{27}$$

Moreover integrating (25) on $(0,\infty)$, we have

$$-\frac{\alpha_1}{2}\int_0^\infty \int_0^\infty \int_\Omega \dot{\mathbf{G}}^{-1}(x,\tau)\mathbf{I}^t_{r\tau}(x,\tau)\cdot \mathbf{I}^t_{r\tau}(x,\tau)dxd\tau dt \leq \mathcal{E}(0). \tag{28}$$

By Theorem 9.1 of L.Deseri, M.F., M.Golden, *Arch. Rational Mech. Anal.* (2006), we have that the solution $\mathbf{u}$ is an element of $H^{\frac{3}{2}}(\mathbb{R}^+; L^2(\Omega)) \cap L^2(\mathbb{R}^+; H^1_0(\Omega))$ and

$$\begin{aligned}
\int_0^\infty \mathcal{E}(t)dt &= \int_0^\infty \int_\Omega \left\{\frac{1}{2}\mathbf{v}^2(x,t) + \Psi_I(x,\mathbf{I}^t)\right\} dxdt \\
&= \frac{1}{2}\int_0^\infty \int_\Omega \Big\{(\mathbf{v}^2(x,t) + \mathbf{G}_\infty(x)\nabla\mathbf{u}(x,t)\cdot\nabla\mathbf{u}(x,t)) \\
&\quad - \int_0^\infty \dot{\mathbf{G}}^{-1}(x,\tau)\mathbf{I}^t_{r\tau}(x,\tau)\cdot\mathbf{I}^t_{r\tau}(x,\tau)d\tau\Big\} dxdt < \infty.
\end{aligned} \tag{29}$$

Now, we write the system (18) in the form

$$\dot{\chi}(t) = A\chi(t) \tag{30}$$

where A denotes the operator represented by the right-hand side of (18), which is defined on the domain

$$\mathcal{F}(A) = \Big\{\chi = (\mathbf{u}, \mathbf{v}, \mathbf{I}_r^t) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \\ -\int_0^\infty \int_\Omega \dot{\mathbf{G}}^{-1}(x,\tau)\mathbf{I}_r^t(x,\tau) \cdot \mathbf{I}_r^t(x,\tau) dx d\tau < \infty\Big\}.$$

Lemma. *Under the hypotheses* (15) *and* (16), *the operator* $A : \mathcal{D}_I(A) \to \mathcal{F}$ *is a maximal dissipative operator on* $\mathcal{F}$ *, i.e.*

a. $\langle A\chi, \chi\rangle \leq 0$ *, for any* $\chi \in \mathcal{D}_I(A)$;

b. the range of $A - I$ *is* $\mathcal{F}$*, where* I *is the identity operator.*

Proof. We have, from (22), (24), (26) and (30),

$$\mathcal{E}(t) = \frac{1}{2}\langle \chi(t), \chi(t)\rangle\,, \qquad \frac{d}{dt}\mathcal{E}(t) = \langle A\chi(t), \chi(t)\rangle\,. \quad (31)$$

Integrating (31) over Ω, we obtain

$$\begin{aligned}\langle A\chi(t), \chi(t)\rangle \; &= \frac{1}{2}\int_\Omega \dot{\mathbf{G}}^{-1}(x,0)\mathbf{I}^t_{r\tau}(x,0)\cdot\mathbf{I}^t_{r\tau}(x,0)dx \\ &-\frac{1}{2}\int_0^\infty\int_\Omega \dot{\mathbf{G}}^{-1}(x,\tau)\ddot{\mathbf{G}}(x,\tau)\dot{\mathbf{G}}^{-1}(x,\tau)\mathbf{I}^t_{r\tau}(x,\tau)\cdot\mathbf{I}^t_{r\tau}(x,\tau)d\tau \\ &\leq 0.\end{aligned}$$

Moreover, under the hypotheses (16), we have from (25) that

$$\begin{aligned} \langle A\chi(t), \chi(t)\rangle &\leq \frac{\alpha_1}{2}\int_0^\infty \int_\Omega \dot{\mathbf{G}}^{-1}(x,\tau)\mathbf{I}_\tau^t(x,\tau)\cdot \mathbf{I}_\tau^t(x,\tau)dxd\tau \\ &\leq 0 \end{aligned} \tag{32}$$

for any solution χ.

The proof of point b. is analogous to the case considered by C.M. Dafermos, *Arch. Rational Mech. Anal.* (1970).

Hence, by means of the Lumer-Phillips Theorem (see: A. Pazy, Lect. Notes in Math., 10. Univ. Maryland. 1974.) the operator A generates a strongly continuous semigroup of linear contraction operators $S(t)$ on $\mathcal{F}$, so that the solutions of the system (18), (19) have the form

$$\chi(t) = S(t)\chi_0.$$

Moreover, from (29) we obtain that the total energy

$$\mathcal{E}(t) = \frac{1}{2} \langle S(t)\chi_0, S(t)\chi_0 \rangle$$

satisfies the restriction

$$\int_0^\infty \langle S(t)\chi_0, S(t)\chi_0 \rangle \, dt < \infty \tag{33}$$

for any $\chi_0 \in \mathcal{F}$. Then $\mathcal{D}(A)$ is dense in $\mathcal{F}$.

Now we recall the following Lemma proved by R. Datko, *J. Math. Anal. Appl* (1970).

Lemma. *Given a strongly continuous semigroup of linear operators $S(t)$ on a Hilbert space $\mathcal{F}$, then for any $\chi_0 \in \mathcal{F}$, there exist two constants C, γ such that*

$$\langle S(t)\chi_0, S(t)\chi_0 \rangle \leq C e^{-\gamma t} \langle \chi_0, \chi_0 \rangle \tag{34}$$

if and only if the integral

$$\int_0^\infty \langle S(t)\chi_0, S(t)\chi_0\rangle \, dt$$

is convergent for any $\chi_0 \in \mathcal{F}$*.*

Indeed, for any initial condition χ_0 such that

$$\frac{1}{2}\langle \chi_0, \chi_0\rangle = \mathcal{E}(0) < \infty$$

we have from (33) that

$$\int_0^\infty \langle S(t)\chi_0, S(t)\chi_0\rangle \, dt < \infty,$$

and the inequality (34) follows.

What of two theorems is more convenient or suitable?

REMARK. The domain of definition of the initial states $\mathcal{D}_I$ is larger compared to the domain $\mathcal{D}_G$.

In order to obtain such a proposition, it is crucial to prove that

$$\Psi_I(\sigma) \leq \Psi_G(\sigma), \quad \sigma \in \mathcal{D}_G \tag{35}$$

Proof.

$$2\Psi_I(\mathbf{I}^t) = \mathbf{G}_\infty \nabla \mathbf{u}(t) \cdot \nabla \mathbf{u}(t) - \int_0^\infty \Big[\dot{\mathbf{G}}^{-1}(s)$$

$$\cdot \int_0^\infty \ddot{\mathbf{G}}(s+\tau_1) \nabla \mathbf{u}_r^t(\tau_1)\, d\tau_1 \cdot \int_0^\infty \ddot{\mathbf{G}}(s+\tau_2) \nabla \mathbf{u}_r^t(\tau_2)\, d\tau_2 \Big]\, ds$$

$$\leq \mathbf{G}_\infty \nabla \mathbf{u}(t) \cdot \nabla \mathbf{u}(t) + \int_0^\infty \left| \dot{\mathbf{G}}^{-1}(s) \right| \left| \int_0^\infty \ddot{\mathbf{G}}(s+\tau) \mathbf{E}_r^t(\tau)\, d\tau \right|^2 ds$$

$$\leq \mathbf{G}_\infty \nabla \mathbf{u}(t) \cdot \nabla \mathbf{u}(t)$$

$$+ \int_0^\infty \left| \dot{\mathbf{G}}^{-1}(s) \right| \left| \int_0^\infty \ddot{\mathbf{G}}(s+\tau)\, d\tau \right| \int_0^\infty \ddot{\mathbf{G}}(s+\tau) \nabla \mathbf{u}_r^t(\tau) \cdot \nabla \mathbf{u}_r^t(\tau)\, d\tau\, ds$$

$$= \mathbf{G}_\infty \nabla \mathbf{u}(t) \cdot \nabla \mathbf{u}(t) - \int_0^\infty \dot{\mathbf{G}}(\tau) \nabla \mathbf{u}_r^t(\tau) \cdot \nabla \mathbf{u}_r^t(\tau)\, d\tau = 2\Psi_G(\nabla \mathbf{u}^t).$$

Of course from this inequality it follows

$$\mathcal{D}_G \subset \mathcal{D}_I$$

From this proposition, you can see as the Theorem 2 may be applied to a larger set of the initial states.

In order to understand that $\mathcal{D}_I$ is bigger than $\mathcal{D}_G$ let me consider as a kernel $\dot{\mathbf{G}}(s) = \tilde{\mathbf{G}}_0 \mathrm{e}^{-\alpha s}$.

The stress $\mathbf{T}$ at the fixed time t_0, is given

$$\begin{aligned}\mathbf{T}(\nabla\mathbf{u}^{t_0}) &= \mathbf{G}_0\nabla\mathbf{u}(t_0)+\int_0^\infty \dot{\mathbf{G}}(s)\nabla\mathbf{u}(t_0-s)\,ds \\ &= \mathbf{G}_0\nabla\mathbf{u}(t_0)-\alpha\tilde{\mathbf{G}}_0\int_0^\infty \mathrm{e}^{-\alpha s}\nabla\mathbf{u}(t_0-s)\,ds\end{aligned}$$

This integral converges if

$$\begin{aligned}\nabla\mathbf{u}^{t_0}(s) &= \nabla\mathbf{u}(t_0-s)=\nabla\mathbf{u}_0(t_0-s)\mathrm{e}^{-\alpha(t_0-s)} \\ &= \nabla\mathbf{u}_0(t_0-s)\mathrm{e}^{-\alpha t_0}\mathrm{e}^{\alpha s}; \qquad \nabla\mathbf{u}_0(\tau)\in L^1(t_0,\infty)\end{aligned}$$

So that

$$\mathbf{T}(\nabla\mathbf{u}^{t_0})=\mathbf{G}_0\nabla\mathbf{u}(t_0)-\alpha\tilde{\mathbf{G}}_0\mathrm{e}^{-\alpha t_0}\int_0^\infty \nabla\mathbf{u}_0(t_0-s)\,ds$$

Moreover, if we consider the Graffi free energy, we have

$$\begin{aligned}\Psi_G &= \mathbf{G}_0 \nabla\mathbf{u}(t_0)\cdot\nabla\mathbf{u}(t_0) \\ &\quad + \int_0^\infty \alpha\tilde{\mathbf{G}}_0 \mathrm{e}^{-\alpha s}\nabla\mathbf{u}(t_0-s)\cdot\nabla\mathbf{u}(t_0-s)\,d\tau\end{aligned}$$

This integral converges if

$$\begin{aligned}\nabla\mathbf{u}(t_0-s) &= \nabla\mathbf{u}_0(t_0-s)\mathrm{e}^{-\frac{\alpha}{2}(t_0-s)} \\ &= \nabla\mathbf{u}_0(t_0-s)\mathrm{e}^{-\frac{\alpha}{2}t_0}\mathrm{e}^{\frac{\alpha}{2}s}; \qquad \nabla\mathbf{u}_0(\tau)\in L^2(t_0,\infty)\end{aligned}$$

So that

$$\begin{aligned}\Psi_G &= \mathbf{G}_0 \nabla\mathbf{u}(t_0)\cdot\nabla\mathbf{u}(t_0) \\ &\quad + \alpha\tilde{\mathbf{G}}_0 \mathrm{e}^{-\alpha t_0}\int_0^\infty \nabla\mathbf{u}_0(t_0-s)\cdot\nabla\mathbf{u}_0(t_0-s)\,ds\end{aligned}$$

then $\mathcal{D}_G \subset \mathcal{D}_T$.

Instead, the Ψ_I free energy

$$\Psi_I(\mathbf{I}^{t_0}) = \frac{1}{2}\mathbf{G}_0\nabla\mathbf{u}(t_0)\cdot\nabla\mathbf{u}(t_0)+\mathbf{T}(\mathbf{I}^{t_0})\cdot\nabla\mathbf{u}(t_0) - \int_0^\infty \dot{\mathbf{G}}^{-1}(s)$$
$$\cdot\int_0^\infty \ddot{\mathbf{G}}(s+\tau_1)\nabla\mathbf{u}(t_0-\tau_1)\,d\tau_1\cdot\int_0^\infty \ddot{\mathbf{G}}(s+\tau_2)\nabla\mathbf{u}(t_0-\tau_2)\,d\tau_2\,ds$$
$$= \frac{1}{2}\mathbf{G}_0\nabla\mathbf{u}(t_0)\cdot\nabla\mathbf{u}(t_0)+\mathbf{T}(\mathbf{I}^{t_0})\cdot\nabla\mathbf{u}(t_0)$$
$$+\alpha^2\mathbf{G}_0\mathrm{e}^{-\alpha t_0}\left(\int_0^\infty \nabla\mathbf{u}_0(t_0-s)\,ds\right)^2$$

converges if

$$\begin{aligned}\nabla\mathbf{u}(t_0-s) &= \nabla\mathbf{u}_0(t_0-s)\mathrm{e}^{-\alpha(t_0-s)}\\ &= \nabla\mathbf{u}_0(t_0-s)\mathrm{e}^{-\alpha t_0}\mathrm{e}^{\alpha s}; \qquad \nabla\mathbf{u}_0(\tau)\in L^1(t_0,\infty)\end{aligned}$$

Then in such a case we have

$$\mathcal{D}_G \subset \mathcal{D}_I = \mathcal{D}_T.$$

Conclusions

There are three reasons for which it is more convenient to use the Ψ_I free energy.

The former is that the state $\mathbf{I}_r^t = (\nabla \mathbf{u}(t), \mathbf{I}^t)$ is the minimal state.

The second reason is that the topology connected with Ψ_G considers two equivalent histories as separate states.

The last reason is that the domain $\mathcal{D}_I$ of the Ψ_I free energy is larger than the domain of Ψ_G. This result provides a stability theorem in a wider domain of perturbations.