Incompressible limits of the full Navier-Stokes-Fourier system on large domains

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Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p = \mathrm{div}_x \mathbb{S}$$

Entropy production

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma$$

Gibbs' equation

$$\vartheta D s(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right)$$

Physical space

$$\Omega_\varepsilon \subset R^3$$

$$\varepsilon \mathrm{dist}[x, \partial\Omega_\varepsilon] \to \infty \text{ for } \varepsilon \to 0$$

Total mass conservation

• The total mass of the fluid contained in Ω_ε is a constant of motion. Moreover,

$$\int_{\Omega_\varepsilon} \Big(\varrho_\varepsilon(t,\cdot) - \overline{\varrho} \Big) \ \mathrm{d}x = 0$$

Energy dissipation

• The total energy of the system is non-increasing in time, specifically,

$$E(t) = \int_{\Omega_\varepsilon} \left(\frac{\varepsilon^2}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) \right)(t) \mathrm{d}x$$

$$\leq E_0 = \int_{\Omega_\varepsilon} \left(\frac{\varepsilon^2}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + \varrho_{0,\varepsilon} e(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) \right) \mathrm{d}x$$

Entropy production

• The total entropy of the system is non-decreasing in time, specifically,

$$\int_{\Omega_\varepsilon} \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)(t,\cdot)\ \mathrm{d}x = \int_{\Omega_\varepsilon} \varrho_{0,\varepsilon} s(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon})\ \mathrm{d}x + \sigma_\varepsilon[[0,t]\times\overline{\Omega}_\varepsilon]$$

$$\sigma_\varepsilon \geq \frac{1}{\vartheta}\left(\varepsilon^2 \mathbb{S}_\varepsilon : \nabla_x \mathbf{u}_\varepsilon - \frac{\mathbf{q}_\varepsilon \cdot \nabla_x \vartheta}{\vartheta}\right)$$

Ill-prepared initial data

$$\varrho_\varepsilon(0,\cdot) = \varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon}, \ \overline{\varrho} > 0$$

$$\vartheta_\varepsilon(0,\cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon}, \ \overline{\varrho} > 0$$

$$\mathbf{u}_\varepsilon(0,\cdot) = \mathbf{u}_{0,\varepsilon}$$

$$\{\varrho^{(1)}_{0,\varepsilon}\}_{\varepsilon>0}, \ \{\vartheta^{(1)}_{0,\varepsilon}\}_{\varepsilon>0}, \ \{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^1 \cap L^\infty$$

$$\int_{\Omega_\varepsilon} \varrho^{(1)}_{0,\varepsilon} \, \mathrm{d}x = \int_{\Omega_\varepsilon} \vartheta^{(1)}_{0,\varepsilon} \, \mathrm{d}x = 0$$

Total dissipation balance

$$\int_{\Omega_\varepsilon} \frac{1}{2}\left[\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2+\frac{1}{\varepsilon^2}\Big(H(\varrho_\varepsilon,\vartheta_\varepsilon)-\partial_\varrho H(\overline{\varrho},\overline{\vartheta})(\varrho_\varepsilon-\overline{\varrho})-H(\overline{\varrho},\overline{\vartheta})\Big)\right](t)\,\mathrm{d}x$$

$$+\frac{\overline{\vartheta}}{\varepsilon^2}\sigma_\varepsilon[[0,t]\times\overline{\Omega}_\varepsilon]$$

$$\leq \int_{\Omega_\varepsilon} \frac{1}{2}\left[\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2+\frac{1}{\varepsilon^2}\Big(H(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon})-\partial_\varrho H(\overline{\varrho},\overline{\vartheta})(\varrho_{0,\varepsilon}-\overline{\varrho})-H(\overline{\varrho},\overline{\vartheta})\Big)\right]\mathrm{d}x$$

Free energy functional

$$H(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \overline{\vartheta} \varrho s(\varrho, \vartheta)$$

$$\frac{\partial^2 H(\varrho, \overline{\vartheta})}{\partial \varrho^2} = \frac{1}{\varrho} \frac{\partial p(\varrho, \overline{\vartheta})}{\partial \varrho}, \quad \frac{\partial H(\varrho, \vartheta)}{\partial \vartheta} = \frac{\varrho}{\vartheta} (\vartheta - \overline{\vartheta}) \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta}$$

Uniform estimates for linearly viscous fluids

$$\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \approx \text{ bounded in } L^\infty(0,T;L^2(\Omega_\varepsilon))$$

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \approx \text{ bounded in } L^\infty(0,T;L^2(\Omega_\varepsilon)) \cap L^2(0,T;W^{1,2}(\Omega_\varepsilon))$$

$$\mathbf{u}_\varepsilon \approx \text{ bounded in } L^\infty(0,T;L^2(\Omega;R^3)) \cap L^2(0,T;W^{1,2}(\Omega;R^3))$$

Acoustic equation, I

$$\varepsilon\partial_t[Z_\varepsilon]_\delta + A\mathrm{div}_x[\mathbf{V}_\varepsilon]_\delta = \varepsilon\mathrm{div}_x\left(\mathbf{G}^1_{\varepsilon,\delta} + \mathbf{G}^2_{\varepsilon,\delta}\right)$$

$$\varepsilon\partial_t[\mathbf{V}_\varepsilon]_\delta + \nabla_x[Z_\varepsilon]_\delta = \varepsilon\mathrm{div}_x\left(\mathbb{H}^1_{\varepsilon,\delta} + \mathbb{H}^2_{\varepsilon,\delta}\right),$$

$$\mathbf{V}_\varepsilon = \varrho_\varepsilon\mathbf{u}_\varepsilon,$$

$$[\mathbf{v}]_\delta(t,x) = \int_{R^3}\eta_\delta(x-y)\mathbf{v}(t,y)\ \mathrm{d}y$$

$\mathbf{G}^1, \mathbb{H}^1$ bounded in $L^2(0,T;W^{k,1})$, $\mathbf{G}^2, \mathbb{H}^2$ bounded in $L^2(0,T;W^{k,2})$

Acoustic equation, II

$$\Psi_\varepsilon = \Delta^{-1}\mathrm{div}_x[\mathbf{V}_\varepsilon]_\delta,\ z_\varepsilon = -[Z_\varepsilon]_\delta$$

$$\varepsilon\partial_t z_\varepsilon - A\Delta\Psi_\varepsilon = \varepsilon(g_\varepsilon^1 + g_\varepsilon^2)$$

$$\varepsilon\partial_t\Psi_\varepsilon - z_\varepsilon = \varepsilon(h_\varepsilon^1 + h_\varepsilon^2),$$

Duhamel's formula

$$\left[\begin{array}{c} z_\varepsilon \\ \Psi_\varepsilon \end{array} \right](t) = S\Big(\frac{t}{\varepsilon}\Big) \left[\begin{array}{c} z_{0,\varepsilon} \\ \Psi_{0,\varepsilon} \end{array} \right] + \int_0^t S\Big(\frac{t-s}{\varepsilon}\Big) \left[\begin{array}{c} (g^1_\varepsilon + g^2_\varepsilon)(s) \\ (h^1_\varepsilon + h^2_\varepsilon)(s) \end{array} \right] \mathrm{d}s$$

Local energy decay

For any function $\chi \in \mathcal{D}(R^3)$, there is a constant $c = c(\chi)$ such that

$$\int_{-\infty}^{\infty} \left\| \chi \, S(t) \left[\begin{array}{c} z_0 \\ \Psi_0 \end{array} \right] \right\|^2_{L^2(R^3) \times D^{1,2}(R^3)} \mathrm{d}t \leq c \left\| \left[\begin{array}{c} z_0 \\ \Psi_0 \end{array} \right] \right\|^2_{L^2(R^3) \times D^{1,2}(R^3)}$$

J.L. Metcalfe: Global Strichartz estimates for solutions to the wave equation exterior to a convex obstacle. *Trans. Amer. Math. Soc.*, **356**:4839–4855, 2004

N. Burq: Global Strichartz estimates for nontrapping geometries: About and article by H.Smith and C.Sogge. *Commun. Partial Differential Equations*, **28**:1675–1683, 2003

Dispersive estimates, I

$$\int_{-\infty}^{\infty} \left\| S\left(\frac{t}{\varepsilon}\right) \left[\begin{array}{c} z_{0,\varepsilon} \\ \Psi_{0,\varepsilon} \end{array} \right] \right\|^2_{L^2(B)\times D^{1,2}(B)} \mathrm{d}t \leq \varepsilon c \left\| \left[\begin{array}{c} z_{0,\varepsilon} \\ \Psi_{0,\varepsilon} \end{array} \right] \right\|^2_{L^2(R^3)\times D^{1,2}(R^3)}$$

Dispersive estimates, II

$$\int_0^T \left\| \int_0^t S\left(\frac{t-s}{\varepsilon}\right) \left[\begin{array}{c} g_\varepsilon^2(s) \\ h_\varepsilon^2(s) \end{array} \right] \mathrm{d}s \right\|^2_{L^2(B)\times D^{1,2}(B)} \mathrm{d}t$$

$$\leq c(T) \int_0^T \int_{-\infty}^{\infty} \left\| S\left(\frac{t}{\varepsilon}\right) S\left(\frac{-s}{\varepsilon}\right) \left[\begin{array}{c} g_\varepsilon^2(s) \\ h_\varepsilon^2(s) \end{array} \right] \right\|^2_{L^2(B)\times D^{1,2}(B)} \mathrm{d}t \ \mathrm{d}s$$

$$\leq \varepsilon c(T) \int_0^T \left\| S\left(\frac{-s}{\varepsilon}\right) \left[\begin{array}{c} g_\varepsilon^2(s) \\ h_\varepsilon^2(s) \end{array} \right] \right\|^2_{L^2(R^3)\times D^{1,2}(R^3)} \mathrm{d}s$$

$$= \varepsilon c(T) \int_0^T \left\| \left[\begin{array}{c} g_\varepsilon^2(s) \\ h_\varepsilon^2(s) \end{array} \right] \right\|^2_{L^2(R^3)\times D^{1,2}(R^3)} \mathrm{d}s,$$

Strong convergence of the velocity

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } L^2((0,T) \times B; R^3)$$

for any bounded ball $B \subset R^3$

For results for the isentropic Navier-Stokes system based on classical Strichartz estimates see
B. Desjardins and E. Grenier: Low Mach number limit of viscous compressible flows in the whole space. *Proc. R. Soc. London A*, **455**:2271–2279, 1999.