Long term behavior of binary fluid mixture flows in 2D

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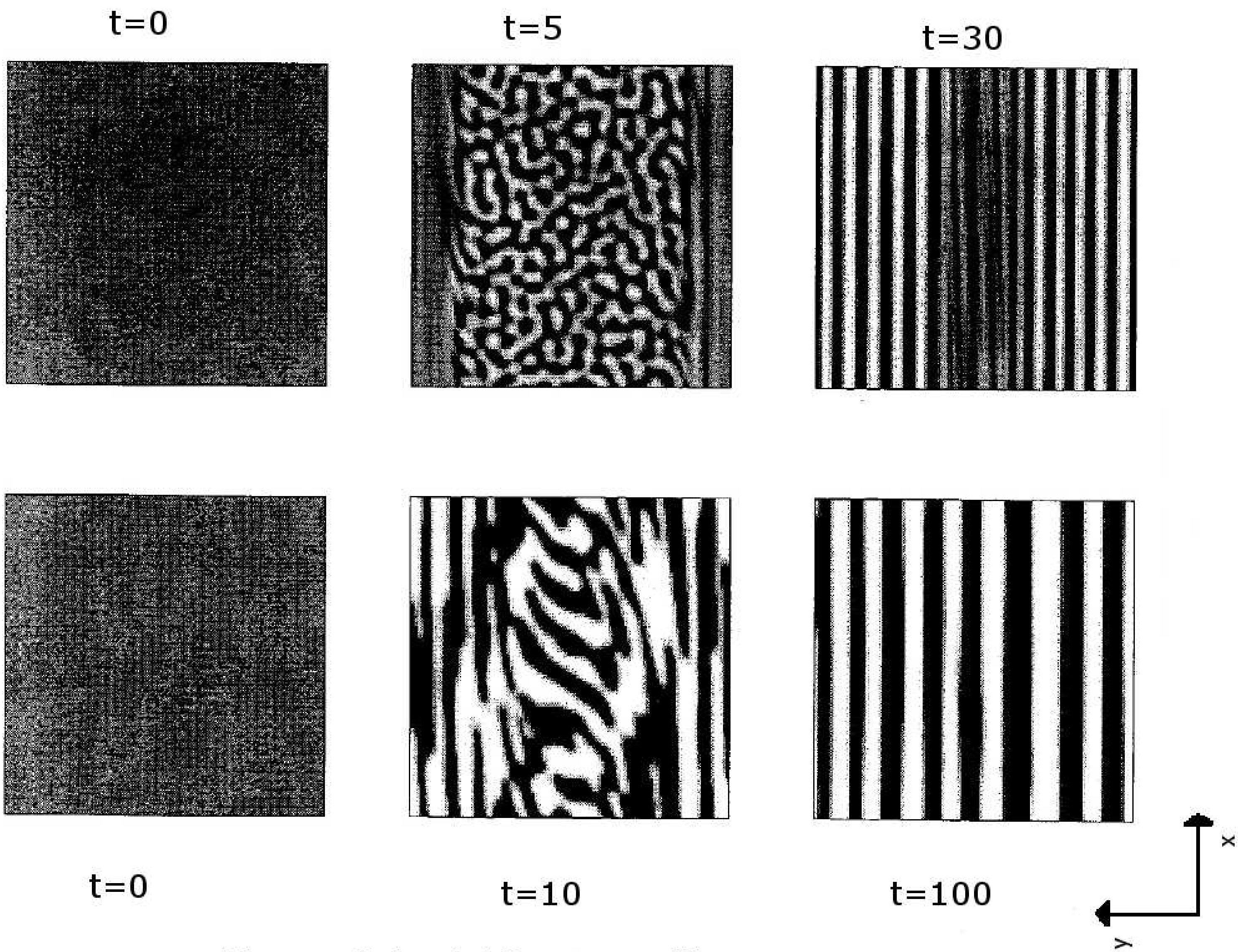

Figure: Spinodal Decomposition

- A model for the study of incompressible multi-phase flows: consider a mixture composed of two incompressible fluids (or phases) of mass densities $\rho_A(x)$ and $\rho_B(x)$.
- Define an order parameter function

$$\phi(x) = \frac{\rho_A(x) - \rho_B(x)}{\rho_A(x) + \rho_B(x)} \in [-1, +1],$$

such that

$$\begin{aligned} \phi(x) &= -1 \text{ if and only if the fluid } A \text{ is present at point } x, \\ \phi(x) &= +1 \text{ if and only if the fluid } B \text{ is present at point } x. \end{aligned}$$

- Suppose that each fluid possesses its own velocity field $\boldsymbol{v}_j$ $(j = A, B)$. Define the mean velocity field by $\boldsymbol{u} = (\boldsymbol{u}_A + \boldsymbol{u}_B)/2$.

- The pair $(\boldsymbol{u}, \phi)$ satisfies the set of equations:

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = \mathcal{K} \mu \nabla \phi + \boldsymbol{g}, \text{ in } \Omega \times (0, +\infty), \tag{1.1}$$

$$\nabla \cdot \boldsymbol{u} = 0, \text{ in } \Omega \times (0, +\infty), \tag{1.2}$$

and

$$\partial_t \phi + \boldsymbol{u} \cdot \nabla \phi + A_K \mu = 0, \text{ in } \Omega \times (0, +\infty), \tag{1.3}$$

$$\mu = -\varepsilon \Delta \phi + \alpha f(\phi), \text{ in } \Omega \times (0, +\infty), \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N = 2, 3$, with smooth boundary $\Gamma = \partial \Omega$ and $\boldsymbol{g}$ is an external volumic force (gravity force, for example).

- **Two cases**:

$$\begin{aligned} \text{If } K &= AC, \text{ then let } A_{AC} = I \\ &\Rightarrow (1.3) \text{ is a convective Allen-Cahn equation.} \\ \text{If } K &= CH, \text{ then let } A_{AC} = -\Delta \\ &\Rightarrow (1.3) \text{ is a convective Cahn-Hilliard equation.} \end{aligned}$$

- μ is the chemical potential of the theoretical uniform mixture of composition ϕ and is obtained as a variational derivative of the following free energy functional

$$\mathcal{F}(\phi) = \int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla \phi|^2 + \alpha F(\phi) \right) dx, \tag{1.5}$$

  where $F^{'}(s) = f(s)$ and ε, $\alpha > 0$ are two physical parameters describing the interaction between the two phases.

- Two typical examples of a potential function F: either

$$F(s) = c_1 s^4 - c_2 s^2,$$

  or

$$F(s) = c_1 \left((1+s) \log(1+s) + (1-s) \log(1-s) \right) + c_2 \left(1 - s^2\right), \ \ c_1, c_2 > 0.$$

- **Focus** on the case $K = AC$. Take Dirichlet boundary conditions

$$\boldsymbol{u} = \mathbf{0}, \;\; \phi = 0, \text{ on } \Gamma \times (0, +\infty),$$

  and initial conditions

$$\boldsymbol{u}_{|t=0} = \boldsymbol{u}_0, \; \phi_{|t=0} = \phi_0 \text{ in } \Omega.$$

- Set

$$\mathbb{H} = \left\{ \boldsymbol{u} \in \mathbb{L}^2 \left(\Omega, dx/\mathcal{K} \right) : \nabla \cdot \boldsymbol{u} = 0 \text{ in } \Omega, \;\; \boldsymbol{u} = 0 \text{ on } \Gamma \right\},$$

$$\mathbb{V} = \left\{ \boldsymbol{u} \in \mathbb{H}^1_0 \left(\Omega \right) : \nabla \cdot \boldsymbol{u} = 0 \text{ in } \Omega \right\}.$$

- We let

$$A_0 \boldsymbol{u} = -\boldsymbol{\Delta} \boldsymbol{u}, \;\; \forall \boldsymbol{u} \in D\left(A_0\right) = \mathbb{H}^2\left(\Omega\right) \cap \mathbb{V},$$

$$A_1 \phi = -\Delta \phi, \quad \forall \phi \in D\left(A_1\right) = H^2\left(\Omega\right) \cap H^1_0\left(\Omega\right).$$

- We formulate the class of problems we want to solve.

Problem *P*. Let $\dim \Omega = 2$. *For* $\boldsymbol{g} \in \mathbb{V}^*$ and *any given pair of initial data*

$$(\boldsymbol{u}_0, \phi_0) \in \mathbb{Y} := \mathbb{H} \times H_0^1(\Omega), \tag{1.7}$$

find $(\boldsymbol{u}(t), \phi(t)) \in C([0, +\infty); \mathbb{Y})$ *with*

$$\partial_t \boldsymbol{u}(t) \in L^{4/3}([0, +\infty); \mathbb{V}^*), \; \partial_t \phi(t) \in L^2\left([0, +\infty); L^2(\Omega)\right)$$

such that

$$\begin{cases} \frac{d\boldsymbol{u}}{dt} + \nu A_0 \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \mathcal{K} \varepsilon A_1 \phi \nabla \phi = \boldsymbol{g} \text{ in } \mathbb{V}^*, \; a.e. \text{ in } (0, +\infty), \\ \mu = \varepsilon A_1 \phi + \alpha f(\phi), \; a.e. \text{ in } \Omega \times (0, +\infty) \\ \frac{d\phi}{dt} + \mu + \boldsymbol{u} \cdot \nabla \phi = 0, \; a.e. \text{ in } \Omega \times (0, +\infty), \end{cases} \tag{1.8}$$

and fulfills the initial conditions. Here and below $\mathbb{Y}$ is a Hilbert space endowed with the obvious norm given by

$$\|(\boldsymbol{u}, \phi)\|_{\mathbb{Y}} := \left(|\boldsymbol{u}|^2 + \varepsilon |\nabla \phi|_{L^2}^2\right)^{1/2}.$$

- Concerning the nonlinearity of our system, we suppose that $f \in C^1(\mathbb{R})$ satisfies

$$\begin{cases} \lim_{|s|\to\infty} \inf f'(s) > 0, \\ |f'(s)| \leq c_f \left(1 + |s|^m\right), \ \forall s \in \mathbb{R},\, m \geq 1. \end{cases} \tag{1.9}$$

Proposition 1. *Let $f \in C^1(\mathbb{R})$ satisfy (1.9). If $(\boldsymbol{u}(t), \phi(t))$ is a smooth solution of problem $\mathbf{P}$, then the following estimate holds:*

$$\begin{aligned} &\|(\boldsymbol{u}(t), \phi(t))\|_{\mathbb{Y}}^2 + \int_t^{t+1} \left(\nu \|\boldsymbol{u}(s)\|^2 + |\mu(s)|_{L^2}^2 + |F(\phi(s))|_{L^1}\right) ds \\ &+ \int_t^{t+1} \left(\|\partial_t \boldsymbol{u}(s)\|_{\mathbb{V}^*}^{4/3} + |A_1\phi(s)|_{L^2}^2 + |\partial_t\phi(s)|_{L^2}^2\right) ds \\ \leq \ & Q\left(\|(\boldsymbol{u}(0), \phi(0))\|_{\mathbb{Y}}^2\right) e^{-\rho t} + C\left(\nu, \varepsilon, \alpha, \mathcal{K}, \|\boldsymbol{g}\|_{\mathbb{V}^*}\right), \ \forall t \geq 0, \end{aligned}$$

where the monotone non-decreasing function Q and the positive constants ρ and C are independent of t and the initial conditions.

Uniqueness of weak solutions to **P** follows from the following continuous dependence result.

Lemma 2. *Let* $(\boldsymbol{u}_i(t), \phi_i(t))$ *be the solution corresponding to the initial data* $(\boldsymbol{u}_i(0), \phi_i(0)) \in \mathbb{Y}$, $i = 1, 2$. *Then, for any* $t \geq 0$, *the following estimate holds*

$$
\begin{aligned}
&|\boldsymbol{u}_1(t) - \boldsymbol{u}_1(t)|^2 + |\nabla(\phi_1(t) - \phi_2(t))|_{L^2}^2 \\
&+ \int_0^t \left[\nu \|\boldsymbol{u}_1(s) - \boldsymbol{u}_1(s)\|^2 + \varepsilon |A_1(\phi_1(s) - \phi_2(s))|_{L^2}^2\right] ds \\
&\leq Ce^{Lt}\left(|\boldsymbol{u}_1(0) - \boldsymbol{u}_1(0)|^2 + |\nabla(\phi_1(0) - \phi_2(0))|_{L^2}^2\right),
\end{aligned}
\tag{1.10}
$$

where C *and* L *are positive constants depending only on the norms of the initial data in* $\mathbb{Y}$, *on* Ω *and on the parameters of the problem, but are both independent of time.*

- the existence of a compact absorbing set.

Lemma 3. *Let the assumptions of Proposition 1 be satisfied. Then, there is a positive constant* $\mathcal{C}$ *(independent of time and initial data, but which depends only on the physical parameters of the problem) such that, for any* $R > 0$*, there exists* $t_1 = t_1(R) > 0$ *such that*

$$\|(\boldsymbol{u}(t), \phi(t))\|_{\mathbb{V}\times D(A_1)} \leq \mathcal{C}, \qquad \forall\, t \geq t_1, \tag{1.11}$$

for any $(\boldsymbol{u}_0, \phi_0) \in \mathcal{B} \subset \mathbb{Y}$.

- We can also prove the following.

Proposition 4. *If* $\boldsymbol{g} \in \mathbb{H}$ *and* $f \in C^2(\mathbb{R})$ *satisfies (1.9), then there exists a* $D(A_0) \times D(A_1^{3/2})$*- bounded absorbing set, for the semigroup* $\mathcal{S}(t)$*. More precisely, there exist a time* $t_2 \geq t_1$ *and a positive constant* $\mathcal{C}'$ *such that*

$$|A_0 \boldsymbol{u}(t)|^2 + \left|A_1^{3/2}\phi(t)\right|_{L^2}^2 \leq \mathcal{C}', \quad \forall t \geq t_2. \tag{1.12}$$

1 The attractors

- The global attractor.

Theorem 5. *Let the assumptions of Proposition 1 be satisfied. Then there exists a connected compact global attractor* $\mathcal{A} \subset \mathbb{Y}$ *for the semigroup* $\mathcal{S}(t)$. *Moreover, if* $\boldsymbol{g} \in \mathbb{V}^*$ *and* $f \in C^1(\mathbb{R})$, *the global attractor* $\mathcal{A}$ *is bounded in* $\mathbb{V} \times D(A_1)$, *whereas* $\mathcal{A}$ *is bounded in* $D(A_0) \times D(A_1^{3/2})$ *when* $\boldsymbol{g} \in \mathbb{H}$ *and* $f \in C^2(\mathbb{R})$.

Remark 1. *Suppose that* $\boldsymbol{g} \in \mathbb{H}_d^{s-1}(\Omega)$, $\mathbb{H}_d^0(\Omega) := \mathbb{H}$ *and* $f \in C^{s+1}(\mathbb{R})$, $s \geq 0$, *satisfies (1.9). Then, we can prove as in Proposition 4 that any functional invariant set for the semigroup* $\mathcal{S}(t)$ *is in fact bounded in* $D(A_0^{(s+1)/2}) \times D(A_1^{(s+2)/2})$, *provided that the boundary* Γ *of* Ω *is smooth enough.*

- The existence of finite dimensional exponential attractors.

Theorem 6. *Let* $\boldsymbol{g} \in \mathbb{H}$ *and* $f \in C^2(\mathbb{R})$ *satisfy the assumptions of Theorem 5. Then,* $\mathcal{S}(t)$ *possesses an exponential attractor* $\mathcal{M} \subset \mathbb{Y}$*, namely,*

(i) $\mathcal{M}$ *is compact and semi-invariant with respect* $\mathcal{S}(t)$ *, i.e.,*

$$\mathcal{S}(t)(\mathcal{M}) \subset \mathcal{M}, \qquad \forall\, t \geq 0. \tag{1.13}$$

(ii) The fractal dimension $\dim_F(\mathcal{M}, \mathbb{Y})$ *of* $\mathcal{M}$ *is finite.*

(iii) $\mathcal{M}$ *attracts exponentially fast any bounded subset* B *of* $\mathbb{Y}$*, that is, there exist a positive nondecreasing function* Q *and a constant* $\rho > 0$ *such that*

$$dist_{\mathbb{Y}}(\mathcal{S}(t)B, \mathcal{M}) \leq Q(\|B\|_{\mathbb{Y}})e^{-\rho t}, \qquad \forall\, t \geq 0. \tag{1.14}$$

Here $dist_{\mathbb{Y}}$ *denotes the non-symmetric Hausdorff distance between sets in* $\mathbb{Y}$ *and* $\|B\|_{\mathbb{Y}}$ *stands for the size of* B *in* $\mathbb{Y}$*.*

Remark 2. *Theorem 6 entails that* $\mathcal{A}$ *has finite fractal dimension.*

- Physical bounds on the fractal dimension of the global attractor. We have the following.

Theorem 7. *We consider the dynamical system* $(\mathcal{S}(t), \mathbb{Y})$ *associated with problem (1.8), when* $A_{AC} = I$. *Let* $\overline{n}$ *be the first integer such that*

$$\overline{n} - 1 < c\sqrt{\frac{a_1 + (a_2)^2 (1 + \log a_2)}{a_3}} \leq \overline{n},$$

where a_i, $i = 1, 2, 3$, *are computed explicitly in terms of the physical parameters* ε, ν, $\mathcal{K}$, $|\Omega|$ *and* c *is a non-dimensional positive constant that depends only on* Ω *and* m. *Then, the corresponding global attractor* $\mathcal{A}$ *defined by Theorem 5 has a Hausdorff dimension less than or equal to* $\overline{n}$ *and a fractal dimension less than or equal to* $2\overline{n}$.

2 Convergence to equilibria

Proposition 8. *Let the hypotheses of Theorem 5 hold. Assume that* $\mathbf{g} = \mathbf{0}$*. The semigroup* $\mathcal{S}(t)$ *has a (strict) Lyapunov functional defined by the free energy, namely,*

$$\mathcal{L}(\mathbf{u}_0, \phi_0) = \frac{1}{2}\left[\varepsilon \left|\nabla \phi_0\right|_{L^2}^2 + \left|\boldsymbol{u}_0\right|^2\right] + \alpha \int\limits_{\Omega} F\left(\phi_0\right) dx,$$

where F *is the primitive of* f*. In particular, we have, for all* $t > 0$,

$$\frac{d}{dt}\mathcal{L}(\boldsymbol{u}\left(t\right), \phi\left(t\right)) = -\nu \left\|\boldsymbol{u}\left(t\right)\right\|^2 - \left|\mu\left(t\right)\right|_{L^2}^2 . \tag{1.15}$$

- The set of equilibria: the stationary problem corresponding to problem **P** is

$$\begin{cases} \boldsymbol{v} = 0 \text{ in } \Omega, \\ -\varepsilon \Delta \psi + \alpha f(\psi) = 0 \text{ in } \Omega, \\ \psi = 0 \text{ on } \Gamma. \end{cases}$$

- The asymptotic behavior of single trajectories.

Theorem 9. *Let the assumptions of Proposition 1 hold. Suppose, in addition, that the nonlinearity F is real analytic. For any given initial datum $(\boldsymbol{u}_0, \phi_0) \in \mathbb{Y}$, the solution $(\boldsymbol{u}(t), \phi(t))$ to* $\mathbf{P}$ *converges to a single equilibrium $(\mathbf{0}, \psi)$ in the topology of $\mathbb{Y}$, that is,*

$$\lim_{t \to +\infty} (|\boldsymbol{u}(t)| + |\phi(t) - \psi|_{H^1}) = 0. \tag{1.16}$$

Moreover, there exist $C \geq 0$ and $\xi \in (0, 1/2)$ depending on $(\mathbf{0}, \psi)$ such that

$$|\boldsymbol{u}(t)| + |\phi(t) - \psi|_{H^1} \leq C(1+t)^{-\xi/(1-2\xi)}, \tag{1.17}$$

for all $t \geq 0$.

3 Singular potentials

- Physicists have often proposed to consider functions like

$$f(s) = c_1 \log \frac{1+s}{1-s} - c_2 s, \;\; c_1, c_2 > 0.$$

  where we suppose that the order parameter ϕ is normalized in such a way that the two pure phases of the fluid are -1 and $+1$, respectively.

- General assumptions on the singular potential $f \in C^1(-1,1)$:

$$\lim_{s \to \pm 1} f(s) = \pm\infty \text{ and } \lim_{s \to \pm 1} f'(s) = +\infty. \tag{1.18}$$

- Define the quantity

$$D[\phi] = \left[1 - |\phi|_{L^\infty(\overline{\Omega})}\right]^{-1}$$

  and the following Banach space

$$\mathbb{X} := \left\{ (\boldsymbol{u}_0, \phi_0) \in \mathbb{H} \times \left(H_0^1(\Omega) \cap L^\infty(\Omega) \right) : 0 < D[\phi_0] < +\infty \right\},$$

  endowed with the metric topology of $\mathbb{Y} = \mathbb{H} \times H_0^1(\Omega)$.

We have

Theorem 10. *Suppose that f satisfies (1.18). Let $(\boldsymbol{u}_0, \phi_0) \in \mathbb{X}$ and $\boldsymbol{g} \in \mathbb{V}^*$. If $(\boldsymbol{u}(t), \phi(t))$ is a regular solution of problem $\mathbf{P}$, then the dissipative estimate holds. In addition, the order parameter $\phi(t)$ is strictly separated from the singular values ± 1 of the function f, that is, there exists a positive constant $\delta \in]0,1[$ which depends only on $D[\phi_0]$ such that*

$$|\phi(t)|_{L^\infty(\overline{\Omega})} \leq \delta < 1, \text{ for all } t \geq 0. \tag{1.19}$$

Finally, associated with problem $\mathbf{P}$, we can define a semigroup $\overline{\mathcal{S}}(t)$, by the standard expression

$$\overline{\mathcal{S}}(t)(\boldsymbol{u}_0, \phi_0) = (\boldsymbol{u}(t), \phi(t)), \overline{\mathcal{S}}(t) : \mathbb{X} \to \mathbb{X},$$

where $(\boldsymbol{u}(t), \phi(t))$ is the unique solution to $\mathbf{P}$.

Theorem 11. *Let the assumptions of Theorem 10 be satisfied. Then there exists a connected compact global attractor $\overline{\mathcal{A}} \subset \mathbb{X}$ for the semigroup $\overline{\mathcal{S}}(t)$. Moreover, the global attractor $\overline{\mathcal{A}}$ is bounded in $\mathbb{V} \times D(A_1)$.*

- Convergence to single equilibria.

Theorem 12. *Let* $\boldsymbol{g} = \mathbf{0}$ *and let* f *satisfy the assumptions of (1.18). Suppose, in addition, that the nonlinearity* f *is real analytic in* $(-1, 1)$*. For any given initial datum* $(\boldsymbol{u}_0, \phi_0) \in \mathbb{X}$*, the solution* $(\boldsymbol{u}(t), \phi(t))$ *to* $\mathbf{P}$ *converges to a single equilibrium* $(\mathbf{0}, \psi)$ *in the topology of* $\mathbb{Y}$*, that is,*

$$\lim_{t \to +\infty} \left(|\boldsymbol{u}(t)| + |\phi(t) - \psi|_{H^1} \right) = 0. \tag{1.20}$$

4 Open questions

- The $3D$ treatment of our models.
- **Conjecture:** For every

$$(\boldsymbol{u}_0, \phi_0) \in \mathbb{Y} = \mathbb{H} \times H_0^1(\Omega), \ \boldsymbol{g} = \boldsymbol{g}(x) \in \mathbb{V}^*,$$

the problem $\mathbf{P}$ possesses at least one global weak energy solution

$$(\boldsymbol{u}(t), \phi(t)) \in L^\infty(\mathbb{R}_+; \mathbb{Y}) \cap L_b^2\left(\mathbb{R}_+; \mathbb{V} \times \left(H^2(\Omega) \cap H_0^1(\Omega)\right)\right) =: \mathcal{Z}_b,$$

which satisfies an energy inequality of the form:

$$\partial_t E(t) + \kappa E(t) + c\left(\nu \|\boldsymbol{u}(t)\|_{\mathbb{V}}^2 + |\phi(t)|_{H^2}^2 + |\mu(t)|_{L^2}^2\right)$$

$$+c\left(|f(\phi(t))|, 1 + |\phi(t)|\right)_{L^2} \leq c\left(1 + \|\boldsymbol{g}\|_{\mathbb{V}^*}^2\right),$$

for some positive constant c which depends only on the physical parameters of the problem, but is independent of initial data and time.

- Here, $0 < \kappa < \xi$ are two small constants and

$$E(t) := \|(\boldsymbol{u}(t), \phi(t))\|_{\mathbb{Y}}^2 + 2\alpha (F(\phi(t)), 1)_{L^2} + \xi |\phi(t)|_{L^2}^2 + c.$$

- This energy inequality implies that the following estimate holds:

$$\|(\boldsymbol{u}(t), \phi(t))\|_{\mathbb{Y}}^2 + \int_{\tau}^{t} e^{-\kappa(t-s)} \left(\nu \|\boldsymbol{u}(s)\|_{\mathbb{V}}^2 + |\phi(s)|_{H^2}^2\right) ds \tag{1.21}$$

$$\leq \|(\boldsymbol{u}(\tau), \phi(\tau))\|_{\mathbb{Y}}^2 e^{-\kappa(t-\tau)} + c_* \left(1 + \|\boldsymbol{g}\|_{\mathbb{V}^*}^2\right),$$

for almost every $t \geq \tau \in \mathbb{R}_+$, where $c_* > 0$ depends only on Ω, ε, α, ν and $\mathcal{K}$.

- Define the trajectory phase-space, as

$$Tr(\mathcal{Z}_b) := \{(\boldsymbol{u}(t), \phi(t)) \in \mathcal{Z}_b : (\boldsymbol{u}(t), \phi(t)) \text{ solves } \mathbf{P} \text{ and satisfies } (1.21)\}.$$

- Define a shift-semigroup $S_t : Tr(\mathcal{Z}_b) \to Tr(\mathcal{Z}_b)$, which acts continuously on $Tr(\mathcal{Z}_b)$, by

$$S_t(\boldsymbol{u}(s), \phi(s)) = (\boldsymbol{u}(t+s), \phi(t+s)), \ t \geq 0, \ s \in \mathbb{R}.$$

- The trajectory dynamical system $(S_t, Tr(\mathcal{Z}_b))$ is well-defined. The dissipative estimate (1.21) implies

$$\|S_t(\boldsymbol{u}, \phi)\|^2_{\mathcal{Z}_b} \leq c \|(\boldsymbol{u}, \phi)\|^2_{L^\infty(\mathbb{R}_+;\mathbb{Y})} e^{-kt} + c_* \left(1 + \|\boldsymbol{g}\|^2_{\mathbb{V}^*}\right), \tag{1.22}$$

for every $t \geq 0$ and $(\boldsymbol{u}(t), \phi(t)) \in \mathcal{Z}_b$.

- Defining the ball $B_R(\mathcal{Z}_b)$ as

$$B_R(\mathcal{Z}_b) := \left\{(\boldsymbol{u}(t), \phi(t)) \in \ \mathcal{Z}_b : \|(\boldsymbol{u}, \phi)\|_{\mathcal{Z}_b} \leq R\right\}. \tag{1.23}$$

- We have that $B_R(\mathcal{Z}_b)$ is a $\mathcal{Z}_b$- absorbing ball for the trajectory semigroup S_t on $\mathcal{Z}_b$.
- $B_R(\mathcal{Z}_b)$ is compact in

$$\mathcal{Z}_{loc} := L^\infty_{w^*,loc}(\mathbb{R}_+; \mathbb{Y}) \cap L^2_{w,loc}\left(\mathbb{R}_+; \mathbb{V} \cap \left(H^2(\Omega) \cap H^1_0(\Omega)\right)\right),$$

  where w and w^* denote the weak and weak* topologies, respectively.
- The trajectory dynamical system $(S_t, Tr(\mathcal{Z}_b))$ possesses a trajectory attractor $\mathcal{A}_{tr}$ and

$$\mathcal{A}_{tr|t=0} = \mathcal{A}^{m-v}.$$