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Direct, Inverse and Control Problems for PDE's

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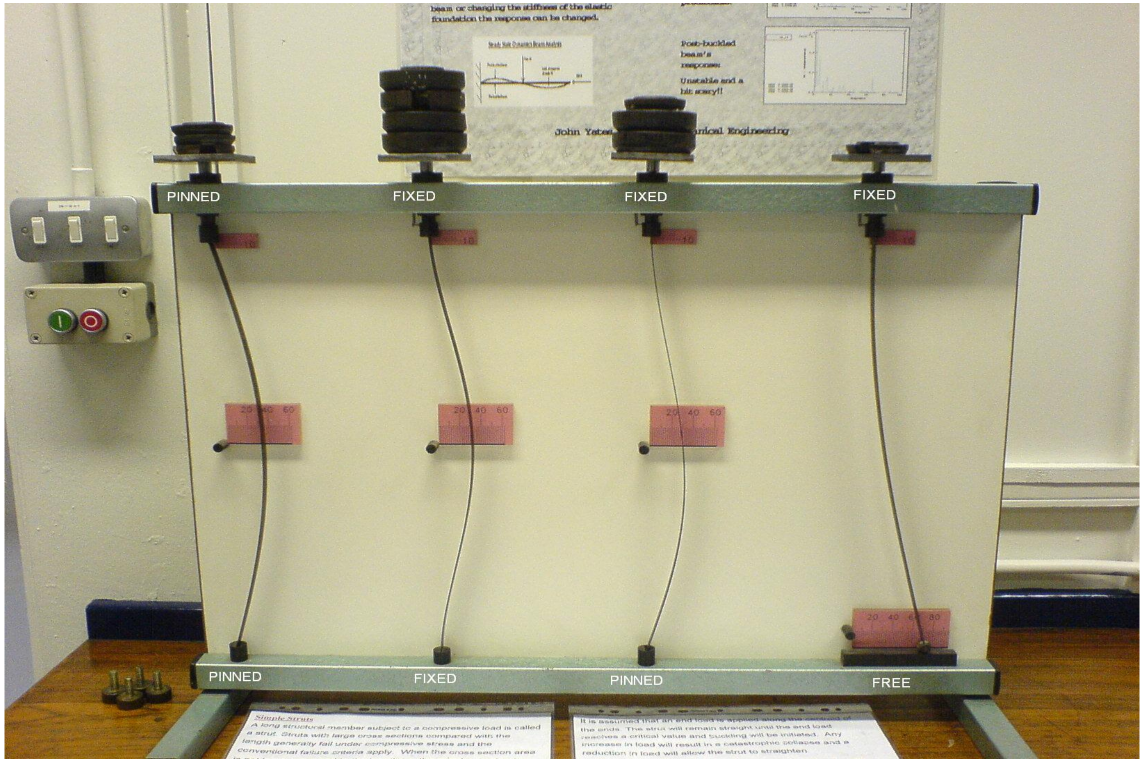
On the extensible thermoelastic beam

C. Giorgi[1] – M.G. Naso[1] – V. Pata[2]

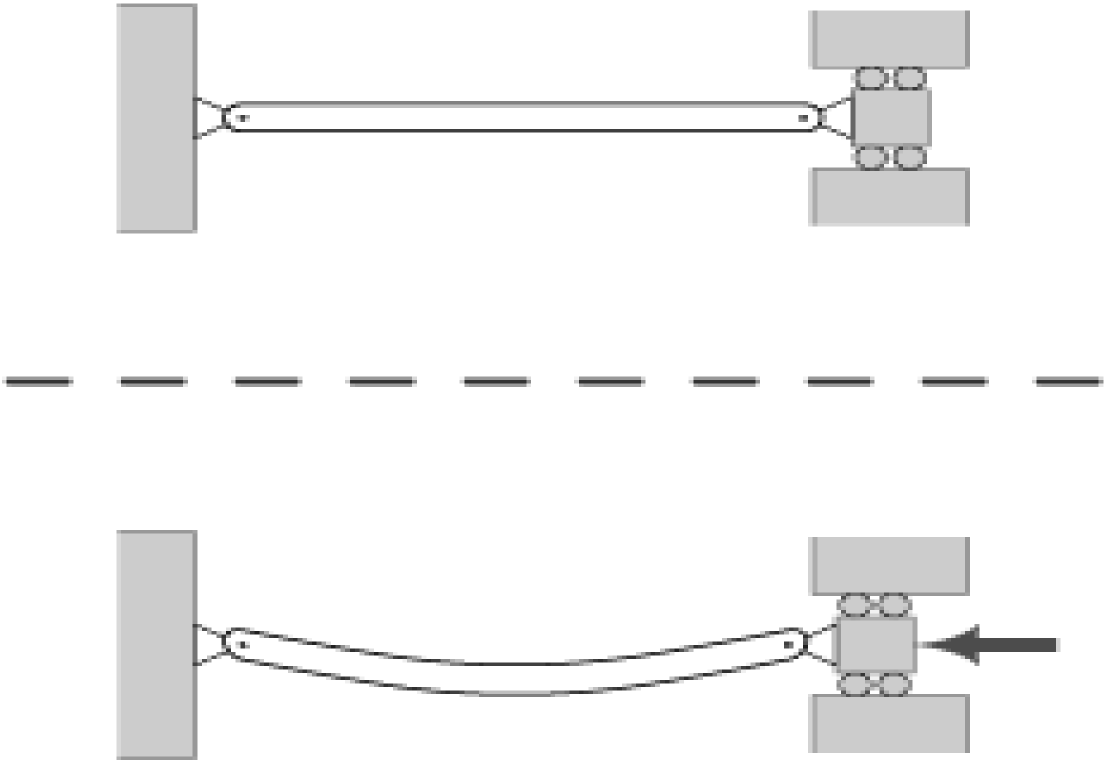
[1] Dipartimento di Matematica, Università di Brescia

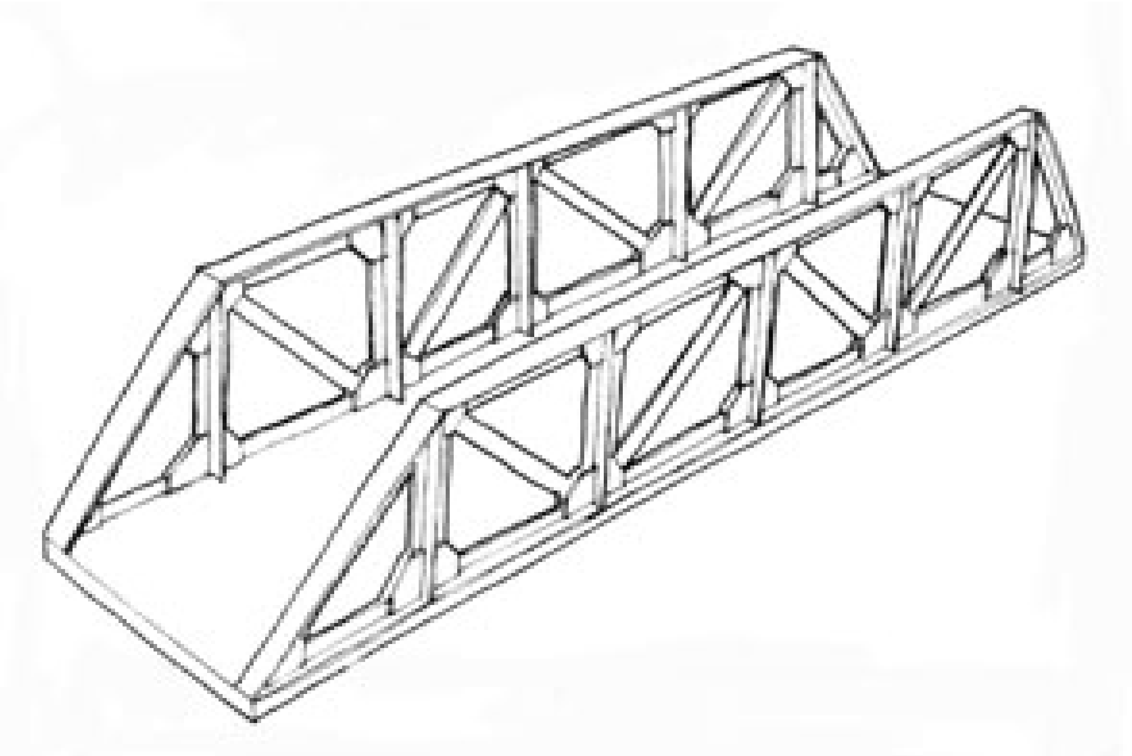
[2] Dipartimento di Matematica, Politecnico di Milano



Fixed and/or hinged ends

Buckling





A beam or “girder” bridge

Ancient "edge rails"

Modern "edge rails"

Contents:

- Derivation of the **full-model** equations of the thermoelastic beam. (the isothermal motion reduces to the von Kármán model)

- Derivation of a **reduced model** (accounting for elongation) concerning deflection and transversal thermal diffusion, only. (the isothermal motion reduces to the Woinowsky-Krieger model)

  [GNP.1] C.G. - M.G.Naso - V.Pata, *A mathematical model of the extensible thermoelastic beam*, work in progress.

- Global longtime dynamics of the reduced model with hinged (pinned) ends: post-buckling dynamics under (Fourier) thermal dissipation;

  [GNP.2] C.G. - M.G.Naso - V.Pata. *Global attractors for the extensible thermoelastic beam system*, submitted.

Previous results:

- Analysis of the steady states (Euler and thermal buckling);

  [CZGP] M.Coti Zelati - C.G. - V.Pata, *Steady states of the hinged extensible beam with external load*, submitted.

- Global longtime dynamics of the viscoelastic isothermal case;

  [GNV] C.G. - V.Pata - E.Vuk. *On the extensible viscoelastic beam*, Nonlinearity, 21 (2008) 713–733.

We present the derivation and the analysis of the longtime behavior of the following nonlinear system

$$\begin{cases} \partial_{tt}u + \partial_{xxxx}u + \partial_{xx}\theta - \boxed{\left(\beta + \int_0^1 |\partial_\xi u(\xi,\cdot)|^2 \mathrm{d}\xi\right)\partial_{xx}u} = f, \\ \partial_t\theta - \partial_{xx}\theta - \partial_{xxt}u = g, \end{cases} \tag{1}$$

where

$u = u(x,t) : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$: vertical deflection of the beam;

$\theta = \theta(x,t) : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$: vertical temperature gradient.

B.C. $\quad u(0,t) = u(1,t) = \boxed{u_{xx}(0,t) = u_{xx}(1,t) = 0}$, $\theta(0,t) = \theta(1,t) = 0,$

I.C. $\quad u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x), \quad \theta(x,0) = \theta_0(x),$

The solutions to problem (1) describes the mechanical and thermal evolution (in the transversal direction) of an extensible thermoelastic beam of unitary natural length with hinged ends.

- The **static counterpart** of (1) reduces to the uncoupled BV problem

$$\begin{cases} \partial_{xxxx}u - \left(\beta + \int_0^1 |\partial_\xi u(\xi,\cdot)|^2 \mathrm{d}\xi\right) \partial_{xx}u = f + g, \\ \partial_{xx}\theta = -g, \\ u(0) = u(1) = \partial_{xx}u(0) = \partial_{xx}u(1) = 0 \\ \theta(0) = \theta(1) = 0 \end{cases} \tag{2}$$

  The buckled stationary states for u are scrutinized in [CZGP] for a general value of $\beta \in \mathbb{R}$ and for a source $f + g$ with a general shape.

- **Boundary conditions.** Different boundary conditions for u are physically significant, such as

  - both ends of the beam are hinged (pinned),

  - both ends are clamped,

  - one end is hinged and the other one clamped.

  All these boundary conditions are allowed as well, without substantial changes in the model.

  On the contrary, the so-called cantilever boundary condition (one end clamped and the other one free) does not involve the extensibility of the beam.

- **Open question.** If and how this model could be extended to account for shear deformations and thermo-mechanical coupling in plates.

At a generic point $(x, y) \in [0, \ell] \times \left[-\frac{h}{2}, \frac{h}{2}\right]$ of the vertical section of the beam

$$\mathfrak{U}(x, y, t) = (W(x, y, t), U(x, y, t))\,, \quad \text{displacement vector field}$$

$$\Theta(x, y, t), \quad \text{absolute temperature field}$$

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix} = \frac{1}{2}\left[\nabla\mathfrak{U} + (\nabla\mathfrak{U})^\top\right] + \boxed{\frac{1}{2}(\nabla\mathfrak{U})^\top \nabla\mathfrak{U}} \quad \text{strain tensor.}$$

Let

$\Theta_0 > 0$ the reference-temperature value,

$\rho > 0$ the reference mass density.

- **The stress-strain relation** (see Carlson)

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \frac{E}{1+\nu}\left[\boldsymbol{\varepsilon} + \frac{\nu}{1-2\nu}\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{1}\right] - \frac{E}{1-2\nu}\boxed{\boldsymbol{\varepsilon}^{\Theta}}, \qquad \text{stress tensor}$$

where $\boxed{\boldsymbol{\varepsilon}^{\Theta} = \alpha\,(\Theta - \Theta_0)\mathbf{1}}$ is the thermal strain tensor,

$E > 0$ is the Young's modulus

$\nu \in (0, \frac{1}{2})$ is the Poisson ratio

$\alpha > 0$, is the coefficient of thermal expansion

- **The entropy density (per unit mass)** (see Chadwick)

$$S = \frac{E\alpha}{\rho\,(1-2\nu)}\,\mathrm{tr}(\boldsymbol{\varepsilon}) + \frac{c_v}{\Theta_0}(\Theta - \Theta_0),$$

where $c_v > 0$ is the beam heat capacity at constant strain.

- **The entropy balance equation** (see Lagnese-Lions)

$$\boxed{\rho\,\Theta\,\partial_t S = -\nabla\cdot\boldsymbol{q} + \rho\,r}$$

where $r(x, y, t)$ is the heat supplied (per unit mass) and

$$\boldsymbol{q} = -k_0\nabla\Theta, \qquad k_0 > 0 \quad \text{(Fourier law)}.$$

It follows from the Gibbs relation and the internal energy balance (no approximation!)

- **The approximation scheme** (consistent with large deformations)

  Geometrical nonlinearities, due to kinematics, are taken into account.

  Kinematic assumptions

  - the thinness of the beam: $\boxed{h \ll \ell}$,
  - the Kirchhoff hypothesis: any cross section remains perpendicular to the deformed longitudinal axis of the beam,
  - $\boxed{W(x,y,t) = w(x,t) - y\,\partial_x u(x,t)}$, $\boxed{U(x,y,t) = u(x,t)}$, where $w(x,t) = W(x,0,t)$ and $u(x,t) = U(x,0,t)$.
    (rigorously justified in large deflection theory by Ciarlet)

- **The approximation scheme**

  Linearization of the temperature field and source with respect to the transversal direction ($|y| < h \ll \ell$).

  Thermal assumptions

  - $\boxed{\Theta(x, y, t) - \Theta_0 = \vartheta(x, t) + y\,\theta(x, t)}$, where
    $\vartheta(x, t) = \Theta(x, 0, t) - \Theta_0$, and $\theta(x, t) = \partial_y \Theta(x, 0, t)$.
  - $\boxed{r(x, y, t) = g_0(x, t) + y\, g(x, t)}$, where
    $g_0(x, t) = r(x, 0, t)$, and $g(x, t) = \partial_y r(x, 0, t)$.

- **The approximation scheme** (consequences)

$$\sigma_{11} = \frac{E}{1-\nu^2}\,\varepsilon_{11} - \alpha\,\frac{E}{1-\nu}[\vartheta(x,t) + y\,\theta(x,t)],$$

$$\sigma_{22} = \sigma_{12} = \sigma_{21} = 0,$$

$$S = \frac{E\alpha}{\rho\,(1-\nu)}\,\varepsilon_{11} + \varpi[\vartheta(x,t) + y\,\theta(x,t)]$$

where

$$\varepsilon_{11}(x,y,t) = \partial_x w(x,t) - y\,\partial_{xx} u(x,t) + \boxed{\frac{1}{2}\,|\partial_x u(x,t)|^2},$$

$$\varpi = \frac{E\alpha^2(1+\nu)}{\rho\,(1-2\nu)(1-\nu)} + \frac{c_v}{\Theta_0} > 0.$$

From the entropy balance equation we obtain

- **The heat equations**

$$\begin{cases} \rho\,\partial_t\vartheta - \dfrac{k_0}{\Theta_0\varpi}\,\partial_{xx}\vartheta + \dfrac{E\alpha}{(1-\nu)\varpi}\partial_t\left[\partial_x w + \dfrac{1}{2}\left|\partial_x u\right|^2\right] = \dfrac{\rho}{\Theta_0\varpi}g_0, \\ \rho\,\partial_t\theta - \dfrac{k_0}{\Theta_0\varpi}\,\partial_{xx}\theta - \dfrac{E\alpha}{(1-\nu)\varpi}\partial_{xxt}u = \dfrac{\rho}{\Theta_0\varpi}g. \end{cases}$$

B.C. $\vartheta(0,t) = \vartheta(\ell,t) = 0$, $\theta(0,t) = \theta(\ell,t) = 0$,

I.C. $\vartheta(x,0) = \vartheta_0(x)$, $\theta(x,0) = \theta_0(x)$.

- **The motion equations** (variational derivation)

$$\begin{cases} \rho\,\partial_{tt}w - \dfrac{E}{1-\nu^2}\partial_x\left\{\partial_x w + \dfrac{1}{2}|\partial_x u|^2 - \alpha(1+\nu)\vartheta\right\} = 0, \\ \rho\,\partial_{tt}u - \dfrac{\rho\,h^2}{12}\partial_{xxtt}u + \dfrac{Eh^2}{12(1-\nu^2)}\partial_{xxxx}u + \dfrac{E\alpha h^2}{12(1-\nu)}\partial_{xx}\theta \\ \quad - \dfrac{E}{1-\nu^2}\partial_x\left\{\left[\partial_x w + \dfrac{1}{2}|\partial_x u|^2 - \alpha(1+\nu)\vartheta\right]\partial_x u\right\} = \dfrac{\rho\,f}{h}. \end{cases}$$

B.C. $\quad u(0,t) = u(\ell,t) = \partial_{xx}u(0,t) = \partial_{xx}u(\ell,t) = 0$, and

$\quad w(0,t) = 0,\ w(\ell,t) = \boxed{C \geq 0}$,

I.C. $\quad u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x)$,

$\quad w(x,0) = w_0(x),\ \partial_t w(x,0) = w_1(x)$.

Isothermal case $\boxed{\theta = \vartheta = 0}$: it reduces to the von Kármán system.

$w(\ell, t) = 0$

$w(\ell, t) = C < 0$

Stationary solutions

$$\begin{cases} \partial_{xx}\vartheta = -\dfrac{\rho}{k_0} g_0 & \Longleftrightarrow \qquad \vartheta(x) = \widehat{\vartheta}(x) \\ \partial_{xx}\theta = -\dfrac{\rho}{k_0} g & \Longleftrightarrow \qquad \theta(x) = \widehat{\theta}(x) \\ \partial_{xxxx}u - \dfrac{12}{h^2}\left[\beta + \dfrac{1}{2\ell}\displaystyle\int_0^\ell |\partial_x u(x)|^2 \,\mathrm{d}x\right] \partial_{xx}u = \dfrac{12(1-\nu^2)\rho}{h^3 E} f + \dfrac{\alpha(1+\nu)\rho}{k_0} g \\ \partial_x w + \dfrac{1}{2}|\partial_x u|^2 - \alpha(1+\nu)\widehat{\vartheta} = \beta + \dfrac{1}{2\ell}\displaystyle\int_0^\ell |\partial_x u(x)|^2 \,\mathrm{d}x \end{cases}$$

where

$$\begin{aligned} \beta &= \frac{C}{\ell} - \frac{\alpha(1+\nu)}{\ell}\int_0^\ell \widehat{\vartheta}(x)\,\mathrm{d}x \\ &= \frac{C}{\ell} + \frac{\alpha(1+\nu)\rho}{k_0\ell}\int_0^\ell \int_0^x \left[\int_0^\xi g_0(\eta)\,\mathrm{d}\eta - \frac{\ell}{2} g_0(\xi)\right] \mathrm{d}\xi\,\mathrm{d}x. \end{aligned}$$

As established in [CZGP], no buckling occurs when

$$\beta \geq -\pi^2 h^2/12\,\ell^2$$

When $\boxed{C = 0}$. No buckling occurs when the mean value of $\tilde{\vartheta}$ is "small"

$$\frac{1}{\ell}\int_0^\ell \tilde{\vartheta}(x)\,\mathrm{d}x \leq \frac{\pi^2 h^2}{12\alpha(1+\nu)\ell^2}.$$

When $\boxed{C \neq 0}$. Unlike the purely mechanical case, buckling can occur even under axial tension ($C > 0$) because of the thermal axial expansion produced by the external heating. Indeed, the no-buckling condition reads

$$C \geq \alpha(1+\nu)\int_0^\ell \widehat{\vartheta}(x)\,\mathrm{d}x - h^2\pi^2/12\ell.$$

- **Further approximations**

  We remove the dependence on ϑ and w.

  Kinematic and thermal assumptions

  – the axial velocity component is negligible: $\boxed{\partial_t w \equiv 0}$
  (physically justified by the hinged ends)

  – the temperature diffusion in the axial direction is negligible:
  $\boxed{\partial_{xx}\vartheta(x,t) \equiv 0}$
  (physically justified by Zener in 1938)

  – the external heat supply vanishes on the x-axis: $\boxed{g_0 \equiv 0}$.

The reduced system

$$\begin{cases} \boxed{\partial_t \left\{ \vartheta + \frac{E\alpha}{(1-\nu)\varpi\rho} \left[\partial_x w + \frac{1}{2} |\partial_x u|^2 \right] \right\} = 0}, \\ \rho\,\partial_t\theta - \frac{k_0}{\Theta_0\varpi}\,\partial_{xx}\theta - \frac{E\alpha}{(1-\nu)\varpi}\partial_{xxt}u = \frac{\rho}{\Theta_0\varpi}g, \\ \boxed{\partial_x \left\{ \partial_x w + \frac{1}{2} |\partial_x u|^2 - \alpha(1+\nu)\vartheta \right\} = 0}, \\ \rho\,\partial_{tt}u - \frac{\rho\,h^2}{12}\partial_{xxtt}u + \frac{Eh^2}{12(1-\nu^2)}\partial_{xxxx}u + \frac{E\alpha h^2}{12(1-\nu)}\partial_{xx}\theta \\ \quad - \frac{E}{1-\nu^2}\partial_x \left\{ \left[\partial_x w + \frac{1}{2} |\partial_x u|^2 - \alpha(1+\nu)\vartheta \right] \partial_x u \right\} = \frac{\rho\,f}{h}. \end{cases}$$

$$\begin{cases} \vartheta + \dfrac{E\alpha}{(1-\nu)\varpi\rho}\left[\partial_x w + \dfrac{1}{2}|\partial_x u|^2\right] = \boxed{\phi(x)} \\ \rho\,\partial_t\theta - \dfrac{k_0}{\Theta_0\varpi}\,\partial_{xx}\theta - \dfrac{E\alpha}{(1-\nu)\varpi}\partial_{xxt}u = \dfrac{\rho}{\Theta_0\varpi}g \\ \partial_x w + \dfrac{1}{2}|\partial_x u|^2 - \alpha(1+\nu)\vartheta = \boxed{\psi(t)} \\ \rho\,\partial_{tt}u - \dfrac{\rho\,h^2}{12}\partial_{xxtt}u + \dfrac{Eh^2}{12(1-\nu^2)}\partial_{xxxx}u + \dfrac{E\alpha h^2}{12(1-\nu)}\partial_{xx}\theta - \dfrac{E}{1-\nu^2}\boxed{\psi(t)}\,\partial_{xx}u = \dfrac{\rho\,f}{h} \end{cases}$$

$$\boxed{\psi(t)} = \frac{C}{\ell} + \frac{1}{2\ell}\int_0^\ell |\partial_x u(x,t)|^2\,\mathrm{d}x - \frac{\alpha(1+\nu)}{\ell}\int_0^\ell \vartheta(x,t)\mathrm{d}x$$

$$\boxed{\phi(x)} = \vartheta_0 + \frac{E\alpha}{(1-\nu)\varpi\rho}\left[\partial_x w_0 + \frac{1}{2}|\partial_x u_0|^2\right]$$

The reduced model [GNP.1].

$$\begin{cases} \rho\,\partial_t\theta - \dfrac{k_0}{\Theta_0\varpi}\,\partial_{xx}\theta - \dfrac{E\alpha}{(1-\nu)\varpi}\partial_{xxt}u = \dfrac{\rho}{\Theta_0\varpi}g \\ \rho\,\partial_{tt}u - \dfrac{\rho\,h^2}{12}\partial_{xxtt}u + \dfrac{Eh^2}{12(1-\nu^2)}\partial_{xxxx}u + \dfrac{E\alpha h^2}{12(1-\nu)}\partial_{xx}\theta \\ \qquad - \dfrac{E}{\ell\,(1-\nu^2)}\left[\lambda_0 + \lambda_1 \displaystyle\int_0^\ell \left|\partial_\xi u(\xi,\cdot)\right|^2 \mathrm{d}\xi\right]\partial_{xx}u = \dfrac{\rho\,f}{h} \end{cases}$$

$$\begin{aligned} \lambda_0 &= C - \alpha(1+\nu)\left[\int_0^\ell \vartheta_0(x)\mathrm{d}x + \frac{E\alpha}{2\rho\varpi(1-\nu)}\int_0^\ell \left|\partial_x u_0(x)\right|^2 \mathrm{d}x\right], \\ \lambda_1 &= \frac{1}{2} + \frac{\alpha^2(1+\nu)E}{2\rho\,\varpi(1-\nu)} \quad \boxed{>0} \end{aligned}$$

We scrutinize the global longtime behavior of the IBVP (reduced model)

$$\begin{cases} \partial_{tt}u - \boxed{\partial_{xxtt}u} + \partial_{xxxx}u + \partial_{xx}\theta - \left(\beta + \int_0^1 |\partial_\xi u(\xi,\cdot)|^2 \mathrm{d}\xi\right)\partial_{xx}u = f, \\ \partial_t\theta - \partial_{xx}\theta - \partial_{xxt}u = g, \\ \theta(0,t) = \theta(1,t) = 0, \\ u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \\ \theta(x,0) = \theta_0(x), \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x). \end{cases} \tag{3}$$

for $\beta \in \mathbb{R}$ and neglecting $\boxed{\partial_{xxtt}u}$.

We consider the abstract Cauchy problem

$$\begin{cases} \partial_{tt}u + Au - A^{1/2}\theta + \big(\beta + \|u\|_1^2\big)A^{1/2}u = f(t), \quad t > 0, \\ \partial_t\theta + A^{1/2}\theta + A^{1/2}\partial_t u = g(t), \quad t > 0, \\ u(0) = u_0, \quad \partial_t u(0) = u_1, \quad \theta(0) = \theta_0, \end{cases} \tag{4}$$

on the product Hilbert space

$$\mathcal{H} = H^2 \times H \times H$$

where $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ is a real Hilbert space and $A : \mathcal{D}(A) \Subset H \to H$ a strictly positive selfadjoint operator: $H^r = \mathcal{D}(A^{r/4})$, $\|u\|_r = \|A^{r/4}u\|$.

Remark. Problem (3) is just a particular case of (4): $H = L^2(0,1)$ and

$$A = \partial_{xxxx}, \quad \mathcal{D}(\partial_{xxxx}) = \big\{w \in H^4(0,1) : w(0) = w(1) = w''(0) = w''(1) = 0\big\}.$$

Proposition 1. Assume that

$$f \in L^1_{\text{loc}}(\mathbb{R}^+, H), \qquad g \in L^1_{\text{loc}}(\mathbb{R}^+, H) + L^2_{\text{loc}}(\mathbb{R}^+, H^{-1}).$$

Then, for all initial data $z = (u_0, u_1, \theta_0) \in \mathcal{H}$, problem (4) admits a unique solution

$$(u(t), \partial_t u(t), \theta(t)) \in \mathcal{C}(\mathbb{R}^+, \mathcal{H})$$

which continuously depends on the initial data.

- We define the *solution operator* $S(t) \in \mathcal{C}(\mathcal{H}, \mathcal{H})$, $\forall t \geq 0$, as

$$z = (u_0, u_1, \theta_0) \mapsto S(t)z = (u(t), \partial_t u(t), \theta(t)).$$

  In the autonomous case, when both f and g are time-independent, S is a *strongly continuous semigroup*.

- For any given $z = (u_0, u_1, \theta_0) \in \mathcal{H}$, we define the *energy* by

$$\mathcal{E}(t) = \frac{1}{2}\|S(t)z\|_{\mathcal{H}}^2 + \frac{1}{4}\big(\beta + \|u(t)\|_1^2\big)^2.$$

- Multiplying the first equation of (4) by $\partial_t u$ and the second one by θ, we obtain the *energy identity*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} + \|\theta\|_1^2 = \langle \partial_t u, f\rangle + \langle \theta, g\rangle.$$

- The energy is bounded. For every $T > 0$, there exist a positive increasing function $\mathcal{Q}_T$ such that

$$\boxed{\mathcal{E}(t) \leq \mathcal{Q}_T(\mathcal{E}(0))} \qquad \forall t \in [0, T].$$

Existence of an absorbing set $\mathfrak{B}$ in $\mathcal{H}$.

Theorem 2. Let $f \in L^\infty(\mathbb{R}^+, H)$, and let $\partial_t f$ and g be translation bounded functions in $L^2_{\mathrm{loc}}(\mathbb{R}^+, H^{-1})$, that is,

$$\sup_{t\geq 0} \int_t^{t+1} \{\|\partial_t f(\tau)\|^2_{-1} + \|g(\tau)\|^2_{-1}\}\mathrm{d}\tau < \infty. \tag{5}$$

Then, for every $R \geq 0$, there exist $R_0 > 0$ and $t_0 = t_0(R) \geq 0$ such that

$$\mathcal{E}(t) \leq R_0, \qquad \forall t \geq t_0,$$

whenever $\mathcal{E}(0) \leq R$. Both R_0 and t_0 can be explicitly computed.

$\mathfrak{B}$ can be chosen to be the ball of $\mathcal{H}$ centered at zero of radius $1 + R_0$.

The proof makes use of the functional

$$\Lambda(t) = \mathcal{E}(t) + 2\varepsilon\Big\{\langle \partial_t u(t), u(t)\rangle + 2\langle \partial_t u(t), \theta(t)\rangle_{-1}\Big\} - \langle u(t), f(t)\rangle + C$$

and heavily relies on the following

Lemma. (Gatti - Pata - Zelik) Let $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy, for some $K \geq 0$, $Q \geq 0$, $\varepsilon_0 > 0$ and every $\varepsilon \in (0, \varepsilon_0]$, the differential inequality

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\Lambda(t) + \varepsilon\Lambda(t) \leq K\varepsilon^2[\Lambda(t)]^{3/2} + \varepsilon^{-2/3}\varphi(t)}$$

where $\varphi \in L^1_{\mathrm{loc}}(\mathbb{R}^+, \mathbb{R}^+)$ is such that $\sup_{t\geq 0} \int_t^{t+1} \varphi(\tau)\mathrm{d}\tau \leq Q$.

Then, there exist $R_1 > 0$ and $\kappa > 0$ such that, for every $R \geq 0$, it follows that

$$\Lambda(t) \leq R_1, \qquad \forall t \geq R^{1/\kappa}(1 + \kappa Q)^{-1},$$

whenever $\Lambda(0) \leq R$. Both R_1 and κ can be explicitly computed in terms of K, Q and ε_0.

The non-autonomous case The decomposition from [GPV] fails to work. Other techniques (such as the α-contraction method) should be employed to establish asymptotic compactness.

The autonomous case An equivalent problem. Denoting

$$\theta_g = A^{-1/2}g, \qquad \omega(t) = \theta(t) - \theta_g,$$

it is apparent that $(u(t), \partial_t u(t), \omega(t))$ solves the abstract IVP

$$\begin{cases} \partial_{tt}u + Au - A^{1/2}\omega + \left(\beta + \|u\|_1^2\right)A^{1/2}u = h, \\ \partial_t\omega + A^{1/2}\omega + A^{1/2}\partial_t u = 0, \end{cases}$$

where $h = f + g \in H$, with the initial conditions

$$\zeta = (u(0), \partial_t u(0), \omega(0)) = z - z_g, \qquad z_g = (0, 0, \theta_g).$$

It generates a strongly continuous semigroup $S_0(t)$ on $\mathcal{H}$, such that

$$S(t)(\zeta + z_g) = z_g + S_0(t)\zeta, \qquad \forall \zeta \in \mathcal{H}.$$

Thus, if $\mathfrak{B}$ is the absorbing set of S, $S_0(t)$ possesses the absorbing set

$$\mathfrak{B}_0 = -z_g + \mathfrak{B} \stackrel{\text{def}}{=} \{\zeta \in \mathcal{H} : \zeta = z - z_g\,,\ \ z \in \mathfrak{B}\}$$

The functional

$$\boxed{\mathcal{L}_0(t) = \mathcal{E}_0(t) - \langle h, u(t)\rangle}$$

is a Lyapunov functional for $S_0(t)$: it satisfies the differential equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_0 + \|\omega\|_1^2 = 0.$$

The existence of the global attractor, jointly with its optimal regularity, have been addressed in [GNP.2]

Theorem 3. Let $f, g \in H$ and $\beta \in \mathbb{R}$. Then, the semigroup $S_0(t)$ acting on $\mathcal{H}$ possesses the (connected) global attractor $\mathfrak{A}_0$ bounded in

$$\mathcal{V} = H^4 \times H^2 \times H^2 \Subset \mathcal{H}.$$

Accodingly, the semigroup $S(t)$ acting on $\mathcal{H}$ possesses the (connected) global attractor $\mathfrak{A}$, where

$$\mathfrak{A} = z_g + \mathfrak{A}_0.$$

The regularity of $\mathfrak{A}_0$ and $\mathfrak{A}$ is optimal.

Remark. $\mathfrak{A}$ is as regular as f and g permit. For instance, if $f, g \in H^n$, then each component of $\mathfrak{A}$ belongs to H^n for every $n \in \mathbb{N}$.

Steps of the proof of Theorem 3.

– A suitable (exponential) asymptotic compactness property of the semigroup is obtained exploiting a particular decomposition of $S_0(t)$ (see [GPV]).

– Due to such a decomposition, we can prove the existence of regular exponential attractors for $S_0(t)$ having finite fractal dimension in $\mathcal{H}$ (e.g., see Efendiev, Miranville, Zelik, *Exponential attractors for a nonlinear reaction-diffusion system in* $\mathbb{R}^3$, C.R. Acad. Sci. Paris, 2000).

– Since the global attractor is the *minimal* closed attracting set, we conclude that the fractal dimension of $\mathfrak{A}_0$ in $\mathcal{H}$ is finite as well.

– Since $\mathfrak{A}_0$ is bounded in $\mathcal{V} = H^4 \times H^2 \times H^2$, its regularity is optimal.

- **The structure of the global attractor**. Let

$$\mathcal{S} = \left\{ \hat{z} \in \mathcal{H} : S(t)\hat{z} = \hat{z},\ \forall t \geq 0 \right\}$$

be the (nonempty) set of stationary points of $S(t)$: $\hat{z} = (\hat{u}, 0, \theta_g)$, where $\hat{u} \in H^4$ is a solution to the elliptic problem

$$\boxed{A\hat{u} + \left(\beta + \|\hat{u}\|_1^2\right) A^{1/2}\hat{u} = f + g.}$$

Let $\mathcal{S}_0 = -z_g + \mathcal{S}$ be the set of stationary points of $S_0(t)$, namely

$$\hat{\zeta} = \hat{z} - z_g = (\hat{u}, 0, 0).$$

Theorem 4. Characterization of $\mathfrak{A}_0$ ($\mathfrak{A}$).

The global attractor $\mathfrak{A}_0$ ($\mathfrak{A}$) coincides with the unstable set of $\mathcal{S}_0$ ($\mathcal{S}$).

- **Exponential stability**.

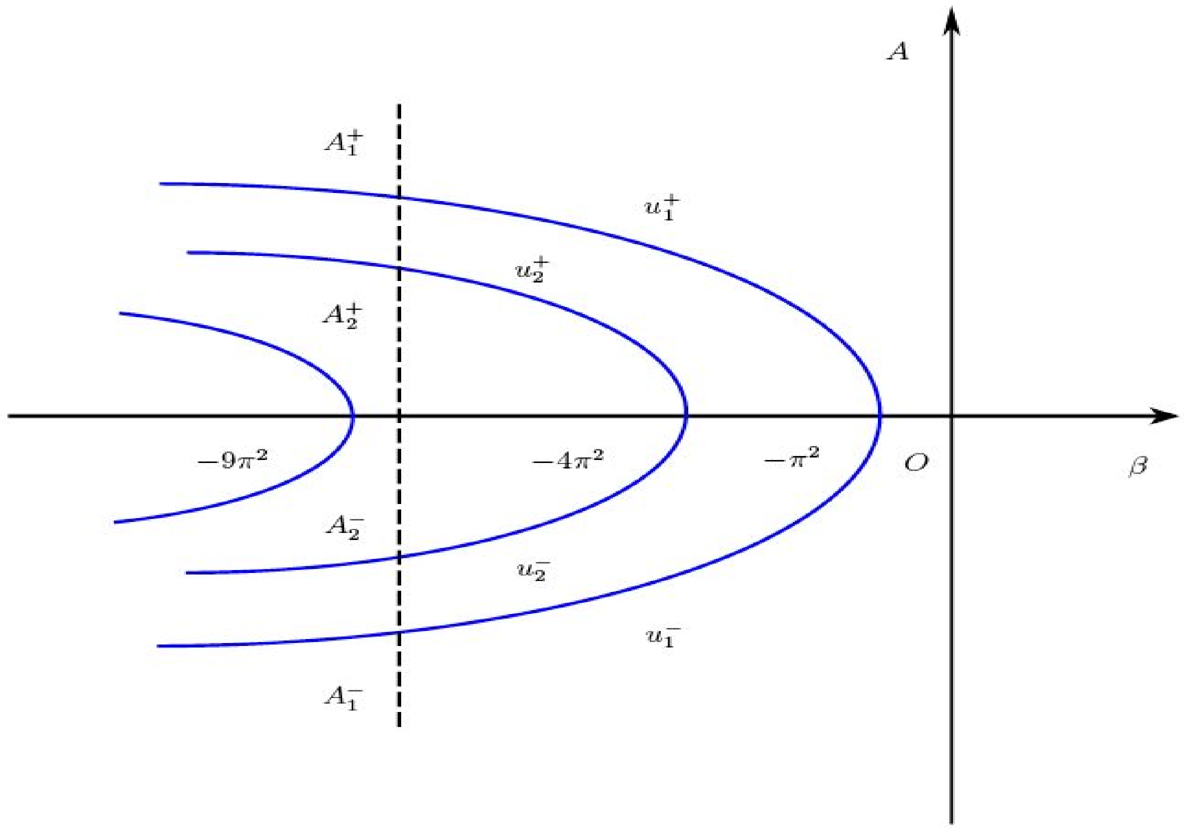
Let λ_1 be the first eigenvalue of A.

Theorem 4. If $\boxed{f + g = 0}$ and $\boxed{\beta > -\sqrt{\lambda_1}}$, then $\mathfrak{A} = \{z_g\} = \{(0, 0, \theta_g)\}$ (the unbuckled state) and

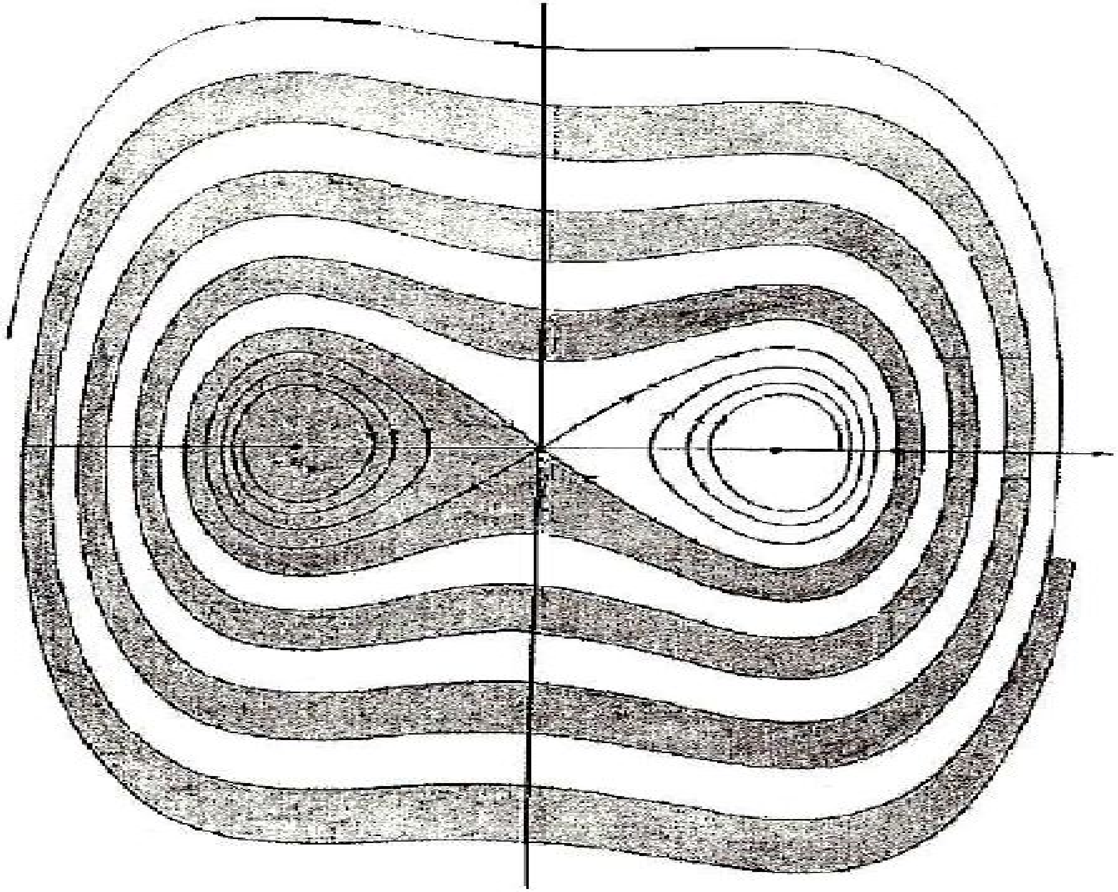
$$\boldsymbol{\delta}_{\mathcal{H}}(S(t)B, \mathfrak{A}) = \sup_{z \in B} \|S(t)z - z_g\|_{\mathcal{H}} \leq \mathcal{Q}(\|B\|_{\mathcal{H}})\mathrm{e}^{-\varkappa t},$$

for some $\varkappa > 0$ and some positive increasing function $\mathcal{Q}$. Both $\varkappa$ and $\mathcal{Q}$ can be explicitly computed.

If $\boxed{f + g = 0}$ and $\boxed{\beta = -\sqrt{\lambda_1}}$, then $\mathfrak{A} = \{z_g\}$, again, but the rate of attraction is not exponential.



β-$A(mplitude)$ plane: the case $\boxed{-\sqrt{\lambda_3} < \beta < -\sqrt{\lambda_2}}$



u - $\partial_t u$ plane: the case $\boxed{-\sqrt{\lambda_2} < \beta < -\sqrt{\lambda_1}}$