Controllability of transport equations in diffusive-dispersive limit

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Introduction

- We consider a 1-D transport equation

$$y_t - My_x = 0 \text{ in } [0, T] \times [0, 1],$$

with $M \in \mathbb{R} \setminus \{0\}$.

- Standard controllability problem : given $T > 0$, y_0 and y_1 in some function space, can we find a solution from y_0 at $t = 0$ to y_1 at $t = T$ by choosing ad hoc boundary conditions ?

- This equation is (trivially) controllable for $T > 1/|M|$ and not controllable for $T < 1/|M|$.

- Question. what can be said about the controllability of this system in a limit of vanishing dispersion or vanishing diffusion-dispersion ?

$$y_t - My_x - \varepsilon y_{xx} + \nu y_{xxx} = 0 \text{ as } \varepsilon, \nu \to 0?$$

A motivation

- Control of conservation laws in the context of (weak) entropy solutions

$$u_t + f(u)_x = 0, \ u : [0, T] \times [0, 1] \to \mathbb{R}, \ f : \mathbb{R} \to \mathbb{R},$$

  - cf. Ancona-Marson, Horsin : scalar convex conservation laws
  - See also Ancona-Coclite, Ancona-Marson, Bressan-Coclite, G. in the case of systems of conservation laws.

- Classical entropy solutions can be defined as weak solutions obtained by vanishing viscosity :

$$u^\varepsilon \to u \text{ as } \varepsilon \to 0^+ \text{ where } u^\varepsilon_t + f(u^\varepsilon)_x - \varepsilon u^\varepsilon_{xx} = 0.$$

- Question. Is it possible to obtain a uniform control for the viscous equation as $\varepsilon \to 0^+$?

Diffusive-dispersive limits

- In the same way, in certain physical situations (e.g. nonlinear elastodynamics with both viscosity and capillarity effects) it is interesting to consider diffusive-dispersive limits :

$$u_t + f(u)_x - \varepsilon u_{xx} + \nu u_{xxx} = 0 \text{ as } \varepsilon, \nu \to 0^+,$$

  which may converge to a weak solution different to the vanishing viscosity solution or to the same one, according to the situtation.

  - Cf. the theory of "nonclassical shock waves", in particular the book of LeFloch.
  - See also Lax-Levermore for the KdV → Burgers (purely dispersive) limit.

- Question. Is it possible to obtain a uniform control for the viscous-dispersive equation as $\varepsilon, \nu \to 0^+$?

- This question is open in general. Here we consider only the linear case.

The control system

- Consider the control system :

$$\begin{cases} y_t - My_x + \nu y_{xxx} - \varepsilon y_{xx} = 0 \text{ in } Q := (0,T) \times (0,1), \\ y_{|x=0} = v_1(t),\ y_{|x=1} = v_2(t),\ y_{x|x=1} = v_3(t) \text{ in } (0,T), \\ y_{|t=0} = y_0 \text{ in } (0,1), \end{cases} \quad (1)$$

- This system is well-posed in for sufficiently regular y_0, v_1, v_2 and v_3, cf. in particular Cattabriga, Bona-Sun-Zhang, Colliander-Kenig, Holmer, ... (see also Bona-Winter, Faminskii for the half-line)

- Other boundary conditions could be used such as $u_{|x=0}$, $u_{x|x=1}$ and $u_{xx|x=1}$ (Colin-Ghidaglia), see also Bubnov, ...

Questions

- Standard null-controllability problem. Given $T > 0$, is it possible to drive any y_0 to 0 at time T by using suitable controls v_1, v_2 and v_3 ? Is it still possible by using only v_1 and $(v_2, v_3) \equiv (0, 0)$?

- Uniform controllability problem. Given $T > 1/|M|$, is it possible to do so at a bounded cost as $\varepsilon, \nu \to 0^+$?

- Is it possible at least for $T \gtrsim 1/|M|$?

- Can we "estimate from below" the cost of the control when $T < 1/|M|$?

Previous studies : uniform controllability in the vanishing viscosity limit

- Coron-Guerrero : 1-D transport equation in the vanishing viscosity limit :
  $$y_t + My_x - \varepsilon y_{xx} = 0.$$
  $\rightarrow$ Cost of order $\mathcal{O}(e^{-1/\varepsilon})$ if $T \gtrsim 1/|M|$, of order $\mathcal{O}(e^{1/\varepsilon})$ if $T < 1/|M|$.

- Guerrero-Lebeau : N-D transport equation in the vanishing viscosity limit :
  $$y_t + M(t,x).\nabla y - \varepsilon \Delta y = 0.$$
  $\rightarrow$ Cost of order $\mathcal{O}(e^{-1/\varepsilon})$ if T is large enough and the characteristics all meet the control zone, of order $\mathcal{O}(e^{1/\varepsilon})$ for T small.

- G.-Guerrero : 1-D Burgers equation in the vanishing viscosity limit :
  $$y_t + yy_x - \varepsilon y_{xx} = 0.$$
  $\rightarrow$ One can reach a constant state $U \neq 0$ in time $\mathcal{O}(1/|U|)$ at a constant cost, for any initial condition in L^∞.

Previous studies : control of KdV equation

- Remark. One can transform the diffusive-dispersive equation in a purely dispersive equation : y satisfies

$$y_t - My_x + \nu y_{xxx} - \varepsilon y_{xx} = 0$$

if and only if

$$z = \exp(-\alpha x)y \text{ with } \alpha = \frac{\varepsilon}{3\nu}.$$

satisfies

$$z_t + \nu z_{xxx} - \left(\frac{\varepsilon^2}{3\nu} + M\right) z_x - \frac{\varepsilon}{3\nu}\left(M + \frac{2\varepsilon^2}{9\nu}\right) z = 0.$$

- In a "diffusive regime" ($\nu \to 0$ and $\varepsilon^2 \gg \nu$) this gives bad estimates...

Previous studies : controllability of KdV equation

- For fixed ν, this has been studied in particular in connection to the local controllability of the (nonlinear) KdV equation :

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0 & \text{in } (0,T)\times(0,1), \\ y_{|x=0} = v_1,\ y_{|x=1} = v_2,\ y_{x\,|x=1} = v_3 & \text{in } (0,T), \\ y_{|t=0} = y_0 & \text{in } (0,1), \end{cases}$$

  cf

  - With 3 controls or distributed control, cf. Russell-Zhang, Zhang, Banks, (local or global exact controllability) . . .
  - When $v_1 = v_2 = 0$, cf. Rosier, Coron-Crépeau, Cerpa, Cerpa-Crépeau,... (local exact controllability)
  - When $v_2 = v_3 = 0$ ("wavemaker"), cf. Rosier (null exact controllability),
  - When $v_3 = 0$, cf. G.-Guerrero (local exact controllability),
  - With a control $u(t)$ in the right hand side, cf. Chapouly (global exact controllability)
  - . . .

Results

Theorem (G.-Guerrero) : uniform controllability

There exists a positive constant K_0 such that for any **positive** constant M, there exist $c, C > 0$ such that for

- any $(\nu, \varepsilon) \in (0,1] \times [0,1]$,
- any $T \geq K_0/M$,
- any $y_0 \in L^2(0,1)$,

there exist a control $v_1 \in L^2(0,T)$ such that the solution of the system with $v_2 = v_3 = 0$ satisfies $y_{|t=T} = 0$ in $(0,1)$ and moreover the control is uniform in (ν, ε) in the sense that

$$\|v_1\|_{L^2} \leq \frac{C}{\sqrt{\nu}} \exp\left\{-\frac{c}{\max\{\nu^{1/2}, \varepsilon\}}\right\} \|y_0\|_{L^2}.$$

Results

Theorem (G.-Guerrero) : non uniform controllability

Consider $M \neq 0$ and $T > 0$ such that

$$T < \frac{1}{|M|}. \tag{2}$$

Then there are some constants $c > 0$ and $\ell \in \mathbb{N}$ (independent of $\varepsilon \in [0,1]$ and $\nu \in (0,1]$) and initial states $y_0 \in L^2(0,1)$ such that any control $v \in L^2(0,T)$ driving y_0 to 0 is estimated from below by

$$\|v\|_{L^2} \geq c\nu^{\ell} \exp\left\{ \frac{c}{\max\{\nu^{1/2}, \varepsilon\}} \right\} \|y_0\|_{L^2}.$$

Ideas of proof

1. Uniform controllability, purely dispersive case ($\varepsilon = 0$), using 3 controls

We consider a linear equation...

By the classical duality argument, we are led to establish an observability inequality for the adjoint system.

$$
\begin{cases}
L\varphi := -\varphi_t - \nu\varphi_{xxx} + M\varphi_x = 0 & \text{in } (0, T_0) \times (0, 1), \\
\varphi(t, 0) = \varphi(t, 1) = \varphi_x(t, 0) = 0 & \text{in } (0, T_0), \\
\varphi(T_0, x) = \varphi_0(x) & \text{in } (0, 1).
\end{cases}
$$

If one gets the following observability inequality

$$\int_0^1 |\varphi(0,x)|^2\,dx \le K(T_0,M,\nu)\int_0^{T_0} |\varphi_{xx|x=0}|^2\,dt.$$

then one can find controls v_1, $v_2 = v_3 = 0$ that drive the system to 0, with

$$\|v_1\|^2_{L^2(0,T_0)} \le \frac{K(T_0,M,\nu)}{\nu}\|y_0\|^2_{L^2(0,1)}.$$

Obtaining such an inequality relies on a Carleman estimate, cf. the ones of Fursikov-Imanuvilov (parabolic systems), Rosier (linear KdV). Set

$$\alpha(t,x) := \frac{\beta(x)}{t^{1/2}(T_0 - t)^{1/2}}.$$

with β a positive, increasing and concave polynomial of second degree.

Proposition

There exists $C > 0$ independent of T_0, ν and M such that for any φ solution of the dual system

$$\iint_{(0,T_0)\times(0,1)} \alpha e^{-2s\alpha}(|\varphi_{xx}|^2 + s^2\alpha^2|\varphi_x|^2 + s^4\alpha^4|\varphi|^2)\,dx\,dt \leq C\int_0^{T_0} \alpha_{|x=0} e^{-2s\alpha_{|x=0}} |\varphi_{xx|x=0}|^2\,dt,$$

for all $s \geq s_0 = C(T_0 + T_0^{1/2} + T_0|M|^{1/2}/\nu^{1/2})$.

- A close statement was proven by Rosier, with a weight of the form

$$\exp\left(\frac{s\psi(x)}{t(T_0-t)}\right),$$

which gives a different s_0.
- Ideas of proof of this inequality :
  - Set $\psi := e^{-s\alpha}\varphi$, where φ is a solution of the dual system $L\varphi = 0$.
  - Decompose $L(e^{s\alpha}\psi) = 0$ into

    $$L_1\psi + L_2\psi = L_3\psi,$$

    (with L_1 skew-symmetric, L_2 essentially symmetric, L_3 "unimportant terms")
  - Write (with $Q_0 := (0, T_0) \times (0, 1)$)

    $$\|L_1\psi\|^2_{L^2(Q_0)} + \|L_2\psi\|^2_{L^2(Q_0)} + 2\iint_{Q_0} L_1\psi\, L_2\psi\, dx\, dt = \|L_3\psi\|^2_{L^2(Q_0)}.$$

  - Develop the integral, do many integration by parts, absorb error terms by taking s large enough.

- Going back to the initial variable, we deduce a bound on the observability of the form :

$$\exp\left\{ C \frac{|M|^{1/2}}{\nu^{1/2}} \left(1 + \frac{1}{|M|\, T_0^{1/2}} \right) \right\}.$$

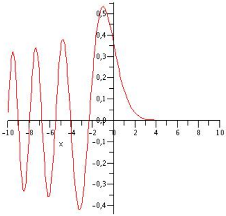
  This is huge ; one has to compensate this size of the observability constant.
- Is it possible to apply this part of the control on a very small initial condition ?

Preliminary step (before applying the above control).

- Extend smoothly y_0 by 0 on $\mathbb{R}$.
- Let the system evolve according to

$$y_t - My_x + \nu y_{xxx} = 0$$

$\Rightarrow$ explicit solution using the Airy function !



- Using basic properties of the Airy function and the fact that $M > 0$, we get

$$\|y(T_1,\cdot)\|_{L^\infty(0,1)} \lesssim \frac{\|y_0\|_\infty}{(\nu T_1)^{1/3}} \exp\left(-\frac{2}{3}\frac{(-MT_1-1)^{3/2}}{(3\nu T_1)^{1/2}}\right),$$

which compensates the size of the above observability inequality, provided that

$$T \geq \frac{K_0}{M}.$$

Ideas of proof, 2

2. Uniform controllability, general case ($\varepsilon \geq 0$), using 1 control

- First, one has to adapt the Carleman estimate. For the purely dispersive case, one uses : a weight of the form

  $$\alpha(t,x) := \frac{\beta(x)}{t^{1/2}(T_0 - t)^{1/2}},$$

  while in the parabolic case, it takes the form

  $$\alpha(t,x) := \frac{\beta(x)}{t(T_0 - t)}.$$

Set

$$\alpha(t,x) = \frac{\beta(x)}{t^\mu (T_0 - t)^\mu},$$

for $\mu \in [1/2, 1]$ and β as previously.

Proposition

There exists a positive constant C independent of T_0, $\nu > 0$, $\varepsilon \geq 0$ and $M \in \mathbb{R}$ such that, for any $\varphi_T \in L^2(0,1)$, we have

$$s \iint_{Q_0} \alpha e^{-2s\alpha} \Big(\nu^2 |\varphi_{xx}|^2 + (\nu^2 s^2 \alpha^2 + \varepsilon^2)|\varphi_x|^2 + (\nu^2 s^4 \alpha^4 + \varepsilon^2 s^2 \alpha^2)|\varphi|^2 \Big) \, dx \, dt$$
$$\leq C\nu \int_0^{T_0} (\nu s \alpha_{|x=0} + \varepsilon) e^{-2s\alpha_{|x=0}} |\varphi_{xx|x=0}|^2 \, dt,$$

for any $s \geq C T_0^\mu (T_0^\mu + (1 + T_0^\mu M^\mu)/(\nu^{1-\mu}\varepsilon^{2\mu-1}))$, where φ is the corresponding solution of the adjoint system.

- This gives a constant in the observability inequality of order
  - $$K \sim \exp\left\{\frac{C}{\nu^{1/2}}\right\},$$
    in the "dispersive regime" where $\nu \gtrsim \varepsilon^2$,
  - $$K \sim \left(\frac{\nu^2}{\varepsilon^2} + \frac{\nu}{\varepsilon}\right)\exp\left\{\frac{C}{\varepsilon}\right\}.$$
    in the "diffusive regime" where $\nu \lesssim \varepsilon^2$.
- Next, one has to obtain a "dissipation estimate" (here, at the level of the adjoint equation) to compensate these huge constants.

Exponential dissipation estimate

- A related work was done by Danchin for vortex patches
- Multiply the adjoint system by $\exp(r(M(T_1 - t) - x))\varphi$, integrate in x (r is a non-negative parameter).
- Here it is essential that the function $(t, x) \mapsto M(T_1 - t) - x$ solves the transport equation.
- After several integration by parts, one obtains

$$- \frac{d}{dt}\Big(\exp\{-(\nu r^3 + \varepsilon r^2)(T_1 - t)\} \int_0^1 \exp\{r(M(T_1 - t) - x)\} |\varphi(t, x)|^2 \, dx \Big) \leq 0.$$

- Integrating between t_1 and t_2, we get

$$\int_0^1 |\varphi(t_1, x)|^2 dx \leq \kappa \int_0^1 |\varphi(t_2, x)|^2 dx,$$

with

$$\kappa = \exp\{\nu(t_2 - t_1)r^3 + \varepsilon(t_2 - t_1)r^2 + (1 - M(t_2 - t_1))r\}.$$

- We optimize in r, and obtain

$$\int_0^1 |\varphi(t_1,x)|^2 dx \leq \kappa \int_0^1 |\varphi(t_2,x)|^2 dx,$$

with κ estimated by

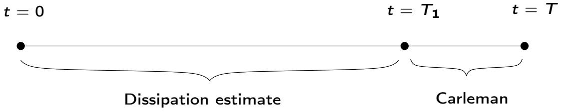
  - if $\varepsilon^2 \gtrsim \nu$:

$$\kappa \leq \exp\left\{ -c \frac{(M(t_2-t_1)-1)^2}{\varepsilon(t_2-t_1)} \right\},$$

  - if $\varepsilon^2 \lesssim \nu$:

$$\kappa \leq \exp\left\{ -c \frac{(M(t_2-t_1)-1)^{3/2}}{\nu^{1/2}(t_2-t_1)^{1/2}} \right\}.$$

- Again, provided that $t_2 - t_1 \geq K_0/|M|$, this can "absorb" the Carleman constant.



Ideas of proof, 3

3. Non uniform controllability when $T < 1/M$

- It is enough to estimate the observability constant from below : find a solution of the adjoint system

$$\begin{cases} -\varphi_t - \nu\varphi_{xxx} - \varepsilon\varphi_{xx} + M\varphi_x = 0, \\ \varphi_{|x=0} = \varphi_{|x=1} = \varphi_{x|x=0} = 0, \\ \varphi_{|t=T} = \varphi_T. \end{cases}$$

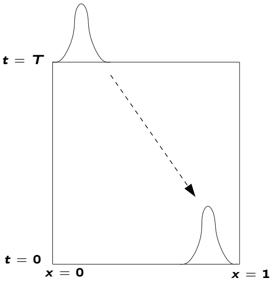
such that

$$\int_0^1 |\varphi(0,x)|^2 \, dx \geq c > 0,$$

and

$$\|\varphi_{xx|x=0}\|_{L^2(0,T)} \leq C \exp\Big(- c \frac{1}{\max(\nu^{1/2}, \varepsilon)}\Big) \|\varphi_T\|_{L^2(0,1)}.$$

- Consider φ_T supported close to the left side $\{0\}$, which would be transported as follows (if $M > 0$) :



- In the transport case ($\varepsilon = \nu = 0$), one clearly has

$$\int_0^1 |\varphi(0,x)|^2 \, dx \geq c > 0,$$

and

$$\varphi_{xx|x=0} = 0.$$

- Using the dissipation estimate and a regularizing effect of the equation (in a bounded domain !), one shows that for ε and ν small enough, this is still true up to a small error term.

- If $M < 0$, the same can be done by choosing φ_T with a support close to the right side $\{1\}$.

Other results, 1

- In the purely diffusive case, Coron and Guerrero obtained a uniform controllability result regardless of the sign of M. We can obtain the following, in the diffusive regime.

Theorem (G.-Guerrero)

Let $0 < \gamma \leq 1$. Then there exists K_0 (depending on γ), such that for any $M < 0$, any $T \geq K_0/|M|$, there are positive constants c and C (depending on T and γ) such that for any $(\nu, \varepsilon) \in (0,1] \times [0,1]$ satisfying

$$\varepsilon^2 \geq \gamma \nu |M|, \tag{3}$$

one can find a control driving y_0 to 0 and which can be estimated as follows :

$$\|v\|_{L^2} \leq \frac{C}{\sqrt{\nu |M|}} \exp\left\{-\frac{c|M|}{\varepsilon}\right\} \|y_0\|_{L^2}. \tag{4}$$

Other results, 2

- The dispersive term is even strong enough to manage a small diffusive term with the wrong sign :

  Theorem (G.-Guerrero)

  Suppose $\nu \in (0, 1]$ and ε is **negative** but satisfies $-\varepsilon < \kappa\nu$ (for some fixed $\kappa < 3/2$) :

  - the Cauchy problem is well-posed,
  - if moreover one has $M > 0$ and $-\varepsilon \leq \frac{3}{4}\sqrt{\nu M}$, then the uniform controllability holds as previously.

Open problems

- What happens for negative M in the dispersive regime ? (Recall the asymmetry of the Airy function)
- Can we recover the (nonlinear) KdV $\rightarrow$ Burgers convergence in a control setting ?
- Diffusive-dispersive limits for nonlinear nonconvex conservation laws ?
- Can one consider the case of systems ?