Partial reconstruction of a source term in a linear parabolic problem

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We consider an inverse problem of the form

$$\begin{cases} D_t\mathbf{u}(t,x,y) = A(t,x,D_x)\mathbf{u}(t,x,y) \\ +B(t,y,D_y)\mathbf{u}(t,x,y) + \mathbf{g}(t,x)f(t,x,y), \\ (t,x,y) \in [0,T] \times \mathbb{R}^m \times \mathbb{R}^n, \\ \mathbf{u}(0,x,y) = u_0(x,y), \quad (x,y) \in \mathbb{R}^m \times \mathbb{R}^n, \\ \mathbf{u}(t,x,0) = \phi(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^m, \end{cases} \tag{1}$$

with $\mathbf{u}$ and $\mathbf{g}$ unknown. A and B are elliptic operators of order $2p$ in the variables $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ respectively. The last equation in (1) should compensate the fact that g is unknown.

• Related literature:

- Pripepko, Orlovsky, Vasin (2000) :

$$\begin{cases} D_t\mathbf{u}(t,x) = L_x\mathbf{u}(t,x) + \mathbf{f}(x)h(t,x) + g(t,x), \\ (t,x) \in (0,T) \times \Omega, \\ B\mathbf{u}(t,x) = b(t,x), \quad (t,x) \in (0,T) \times \partial\Omega, \\ \mathbf{u}(0,x) = u_0(x), x \in \Omega, \\ l\mathbf{u}(x) = \chi(x), x \in \Omega, \end{cases} \tag{2}$$

with L_x second order elliptic, B suitable Dirichlet or first order boundary conditions, h and g given,

$$l\mathbf{u}(x) = \int_0^T \mathbf{u}(\tau,x)\omega(\tau)d\tau \tag{3}$$

or

$$l\mathbf{u}(x) = \mathbf{u}(T,x). \tag{4}$$

The problems (2)-(3) and (2)-(4) are treated also in the case of the linearized Navier-Stokes equation (Chapter 4). Considered also

$$\begin{cases} D_t\mathbf{u}(t,x) = L_x\mathbf{u}(t,x) + \mathbf{f}(t)h(t,x) + g(t,x), \\ (t,x) \in (0,T) \times \Omega, \\ B\mathbf{u}(t,x) = b(t,x), \quad (t,x) \in (0,T) \times \partial\Omega, \\ \mathbf{u}(0,x) = u_0(x), x \in \Omega, \\ l\mathbf{u}(t) = \chi(t), t \in (0,T), \end{cases} \tag{5}$$

with

$$lu(t) = \int_\Omega u(t,x)\omega(x)dx.$$

In Chapter 6 the abstract problem

$$\begin{cases} D_t\mathbf{u}(t,x) = A\mathbf{u}(t) + \Phi(t)\mathbf{p}(t) + F(t), \\ t \in [0,T], \\ \mathbf{u}((0) = u_0 \\ B\mathbf{u}(t) = \psi(t) \end{cases} \tag{6}$$

with A infinitesimal generator of semigroup in Banach space X, Y Banach space, $\Phi(t) \in \mathcal{L}(Y,X)$, $B\Phi(t)$ invertible for every t, $p : [0,T] \to Y$ unknown together with u, $B \in \mathcal{L}(X,Y)$. Assumed regularizing action of B with respect to A, that is, $\overline{BA} \in \mathcal{L}(X,Y)$. Not applicable in our case.

-Belov (2002) treated the case $p = 1$,

$$A(t, x, D_x) = A(x, D_x),$$

$$n = 1, B(t, y, D_y) = a(t)D_y^2 + b(t)D_y.$$

Main tool: Fourier transform in the y variable.

-Very recent paper by Anikonov-Lorenzi (2007):

$$\begin{cases} \mathbf{u}'(t) = A\mathbf{u}(t) + f(t)\mathbf{z}, \quad t \in [0, T], \\ \mathbf{u}(0) = u_1 \\ \int_0^T \mathbf{u}(t) d\mu(t) = u_2, \end{cases} \tag{7}$$

with $\mathbf{u}$ and $\mathbf{z}$ unknown, A infinitesimal generator of an analytic exponentially decreasing semigroup, f scalar valued, μ Borel measure in $[0, T]$. Results of existence and uniqueness of solution $(\mathbf{u}, \mathbf{z})$, together with representation formula.

• Now we want to state our main result. Assumptions:

(H1) $m, n \in \mathbb{N}$, $T \in \mathbb{R}^+$.

(H2) $p \in \mathbb{N}$; $A(t,x,D_x) = \sum_{|\alpha| \leq 2p} a_\alpha(t,x) D_x^\alpha$, *with* $a_\alpha \in C([0,T] \times \mathbb{R}^m)$ *(uniformly continuous and bounded),*

$$t \to a_\alpha(t,.) \in B([0,T]; C^\theta(\mathbb{R}^m)),$$

with $\theta \in (0,1)$; *there exists* $\nu \in \mathbb{R}^+$, *such that,* $\forall (t,x) \in [0,T] \times \mathbb{R}^m$, $\forall \xi \in \mathbb{R}^m$,

$$(-1)^p Re(\sum_{|\alpha|=2p} a_\alpha(t,x)\xi^\alpha) \leq -\nu |\xi|^{2p}. \qquad (8)$$

(H3) $B(t,y,D_y) = \sum_{|\beta| \leq 2p} b_\beta(t,y) D_y^\beta$, *with* $b_\beta \in C([0,T] \times \mathbb{R}^n)$, $t \to b_\beta(t,.) \in B([0,T]; C^\theta(\mathbb{R}^n))$; $\forall (t,y) \in [0,T] \times \mathbb{R}^n$, $\forall \eta \in \mathbb{R}^n$,

$$(-1)^p Re(\sum_{|\beta|=2p} b_\beta(t,y)\eta^\beta) \leq -\nu |\eta|^{2p}. \qquad (9)$$

(H4) $f \in C([0,T] \times \mathbb{R}^m \times \mathbb{R}^n)$, $t \to f(t,.,.) \in B([0,T];\ C^\theta(\mathbb{R}^m \times \mathbb{R}^n))$, *there exists* $\mu \in \mathbb{R}^+$, *such that*

$$|f(t,x,0)| \geq \mu, \quad \forall (t,x) \in [0,T] \times \mathbb{R}^m. \qquad (10)$$

(H5) $u_0 \in C^{2p+\theta}(\mathbb{R}^m \times \mathbb{R}^n)$.

(H6) $\phi \in C^1([0,T] \times \mathbb{R}^m)$ *(uniformly continuous and bounded together with the first order derivatives); the mapping*

$$t \to D_t\phi(t,.)$$

belongs to $B([0,T]; C^\theta(\mathbb{R}^m))$*; the mapping* $t \to \phi(t,.)$ *belongs to* $B([0,T]; C^{2p+\theta}(\mathbb{R}^m))$.

(H7) $u_0(x,0) = \phi(0,x)$ $\forall x \in \mathbb{R}^m$.

Main result:

Theorem 1 *Assume that the assumptions (H1)-(H7) are satisfied. Then (1) has a unique solution* (u, g)*, such that*

$$u \in C^1([0,T]; C(\mathbb{R}^{m+n})) \cap B([0,T]; C^{2p+\theta}(\mathbb{R}^{m+n})),$$

$$D_t u \in B([0,T]; C^{\theta}(\mathbb{R}^{m+n})), \tag{11}$$

$$g \in C([0,T]; C(\mathbb{R}^m)) \cap B([0,T]; C^{\theta}(\mathbb{R}^m)). \tag{12}$$

Outline of the proof:

Proceeding formally, we get from (1), taking $y = 0$:

$$\begin{gathered} D_t\phi(t,x) = A(t,x,D_x)\phi(t,x) \\ +B(t,0,D_y)u(t,x,0) + g(t,x)f(t,x,0), \\ (t,x) \in [0,+\infty) \times \mathbb{R}^n, \end{gathered} \tag{13}$$

so that, as $f(t,x,0) \neq 0 \ \forall (t,x) \in [0,+\infty) \times \mathbb{R}^n$, we obtain

$$g(t,x) = -f(t,x,0)^{-1}B(t,0,D_y)u(t,x,0)+F_1(t,x), \tag{14}$$

with

$$F_1(t,x) = f(t,x,0)^{-1}[D_t\phi(t,x)-A(t,x,D_x)\phi(t,x)]. \tag{15}$$

(completely known)

Replacing (14) in (1), we are reduced to the system

$$\begin{cases} D_t u(t,x,y) = A(t,x,D_x)u(t,x,y) \\ \quad +B(t,y,D_y)u(t,x,y) \\ \quad +c(t,x,y)B(t,0,D_y)u(t,x,0) + F(t,x,y), \\ \\ (t,x,y) \in [0,T] \times \mathbb{R}^m \times \mathbb{R}^n, \\ \\ u(0,x,y) = u_0(x,y), \quad (x,y) \in \mathbb{R}^m \times \mathbb{R}^n, \end{cases} \tag{16}$$

with

$$c(t,x,y) := -f(t,x,0)^{-1} f(t,x,y), \tag{17}$$

$$F(t,x,y) := f(t,x,0)^{-1}[D_t\phi(t,x) - A(t,x,D_x)\phi(t,x)]f(t,x,y). \tag{18}$$

• At first sight, it is unclear whether (16) is well posed, as the term

$$c(t,x,y)B(t,0,D_y)u(t,x,0)$$

does not seem a simple perturbation of the remaining part.

To see that the situation is not so bad, we start by considering the problem

$$\lambda v(y) - B(y, D_y)v(y) - c(y)B(0, D_y)v(0) = f(y), \\ y \in \mathbb{R}^n, \tag{19}$$

with the following assumptions:

(I1) $B(y, D_y) = \sum_{|\beta| \le 2p} b_\beta(y) D_y^\beta$, *with* $b_\beta \in C^\theta(\mathbb{R}^n)$ *(*$\theta \in (0, 1)$*); there exists* $\nu \in \mathbb{R}^+$*, such that,* $\forall y \in \mathbb{R}^n$*,* $\forall \eta \in \mathbb{R}^n$*,*

$$(-1)^p Re(\sum_{|\beta|=2p} b_\beta(y)\eta^\beta) \le -\nu|\eta|^{2p}. \tag{20}$$

(I2) $c \in C^\theta(\mathbb{R}^n)$*.*

We introduce the following operator B_0:

$$\begin{cases} D(B_0) & := \{u \in \bigcap_{1 \le q < \infty} W^{2p,q}_{loc}(\mathbb{R}^n) \cap C(\mathbb{R}^n) \\ & : B(., D_y)u \in C(\mathbb{R}^n)\}, \\ B_0 u(y) & := B(y, D_y)u(y), \quad y \in \mathbb{R}^n. \end{cases} \tag{21}$$

It is well known that B_0 is the infinitesimal generator of an analytic semigroup in $C(\mathbb{R}^n)$. Moreover, if $\alpha \in (0,1)$ and $2\alpha p \notin \mathbb{N}$,

$$(C(\mathbb{R}^n), D(B_0))_{\alpha,\infty} = C^{2p\alpha}(\mathbb{R}^n), \qquad (22)$$

with equivalent norms. These results can be extended to the operator B_c:

$$\begin{cases} D(B_c) := D(B_0), \\ B_c u(y) := B_0 u(y) + c(y) B_0 u(0), \quad y \in \mathbb{R}^n. \end{cases} \qquad (23)$$

We limit ourselves to show the following: that, if $\lambda > 0$, sufficiently large, $\lambda \in \rho(B_c)$ and

$$\|(\lambda - B_c)^{-1}\|_{\mathcal{L}(C(\mathbb{R}^n))} = O(\lambda^{-1}) \quad (\lambda \to \infty).$$

Let $\lambda \in \rho(B_0)$. We consider the equation

$$\lambda u - B_c u = f, \qquad (24)$$

with $f \in C(\mathbb{R}^n)$. (24) is equivalent to

$$u = (\lambda - B_0)^{-1} f + B_0 u(0)(\lambda - B_0)^{-1} c, \quad u \in D(B_0). \qquad (25)$$

(25) implies

$$\begin{aligned} B_0 u(0) = [B_0(\lambda - B_0)^{-1} f](0) \\ + B_0 u(0)[B_0(\lambda - B_0)^{-1} c](0). \end{aligned} \quad (26)$$

As B_0 generates a semigroup,

$$\|B_0(\lambda - B_0)^{-1} c\|_{C(\mathbb{R}^n)} = o(1) \quad (\lambda \to \infty) \quad (27)$$

(in fact, as $c \in C^\theta(\mathbb{R}^n)$, $\|B_0(\lambda - B_0)^{-1} c\|_{C(\mathbb{R}^n)} = O(\lambda^{-\theta/(2p)})$.

So, if λ is sufficiently large, from (26), we obtain

$$B_0 u(0) = \frac{[B_0(\lambda - B_0)^{-1} f](0)}{1 - [B_0(\lambda - B_0)^{-1} c](0)}, \quad (28)$$

and

$$|B_0 u(0)| \le C|B_0(\lambda - B_0)^{-1} f](0)| \le C\|f\|_{C(\mathbb{R}^n)}. \quad (29)$$

with C independent of λ and f. From (25), we easily obtain

$$\|u\|_{C(\mathbb{R}^n)} \le C\lambda^{-1}\|f\|_{C(\mathbb{R}^n)}. \quad (30)$$

The next step is the following

Theorem 2 *Assume that* $A(t,x,D_x)=A(x,D_x)$, $B(t,y,D_y)=B(y,D_y)$, $c\in C^{\theta}(\mathbb{R}^{m+n})$,

$$\|c\|_{C^{\theta}(\mathbb{R}^{m+n})}\leq A,$$

with $A\in\mathbb{R}^{+}$. *Consider the problem*

$$\begin{cases} \lambda u(x,y)-A(x,D_x)u(x,y)-B(y,D_y)u(x,y) \\ -c(x,y)B(0,D_y)u(x,0)=f(x,y), \\ x\in\mathbb{R}^m, y\in\mathbb{R}^n, \end{cases} \tag{31}$$

with $\lambda\in\mathbb{C}$, $f\in C^{\theta'}(\mathbb{R}^{m+n})$, *with* $0<\theta'\leq\theta$. *Then, there exist* $\phi\in(\pi/2,\pi)$, $R>0, M>0$ *(independent of* c *and* f*), such that, if* $|\lambda|\geq R$, $|Arg(\lambda)|\leq\phi$, *then (31) has a unique solution* u *belonging to* $C^{2p+\theta'}(\mathbb{R}^{m+n})$. *Moreover,*

$$\begin{aligned}|\lambda|\|u\|_{C^{\theta'}(\mathbb{R}^{m+n})}+\|u\|_{C^{2p+\theta'}(\mathbb{R}^{m+n})} \\ \leq M\|f\|_{C^{\theta'}(\mathbb{R}^{m+n})}\end{aligned} \tag{32}$$

A proof of Theorem 2 can be obtained by means of Theorem 2 and Da Prato-Grisvard's theory of sums of operators, allowing to consider the simpler case that $c(x, y) = c(y)$, in such a way that $A(x, D_x)$ and $u \to B(y, D_y)u + c(y)B(0, D_y)u(., 0)$ "commute". The more general case that c depends also on x can be obtain through a perturbation argument.

Using Theorem 2, one can prove the following

Theorem 3 *Consider the problem*

$$\begin{cases} D_t u(t,x,y) = A(t,x,D_x)u(t,x,y) \\ +B(t,y,D_y)u(t,x,y) + c(t,x,y)B(0,D_y)u(t,x,0) \\ +F(t,x,y), \quad (t,x,y) \in [0,T] \times \mathbb{R}^m \times \mathbb{R}^n, \\ \\ u(0,x,y) = u_0(x,y), \quad (x,y) \in \mathbb{R}^m \times \mathbb{R}^n, \end{cases} \tag{33}$$

with the assumptions that $F \in C([0,T] \times \mathbb{R}^{m+n})$, $\forall t \in [0,T]$ $F(t,.) \in C^\theta(\mathbb{R}^{m+n})$ *and* $t \to F(t,.) \in B([0,T];\ C^\theta(\mathbb{R}^{m+n}))$ *and* $u_0 \in C^{2+\theta}(\mathbb{R}^{m+n})$. *Then (33) has a unique solution* u *belonging to*

$$C^1([0,T];C(\mathbb{R}^{m+n})) \cap C([0,T];C^{2p}(\mathbb{R}^{m+n})) \\ \cap B([0,T];C^{2p+\theta}(\mathbb{R}^{m+n})),$$

with $D_t u \in B([0,T];C^{\theta'}(\mathbb{R}^{m+n}))$.

Theorem 1 is a straighforward consequence of Theorem 3.

With similar (simpler) arguments, one can show also the following fact, corresponding to the case $m = 0$:

Theorem 4 *Consider the problem*

$$\begin{cases} D_t\mathbf{u}(t,y) = B(t,y,D_y)\mathbf{u}(t,y) + \mathbf{g}(t)f(t,y), \\ (t,y) \in [0,T] \times \mathbb{R}^n, \\ \mathbf{u}(0,y) = u_0(y), \quad y \in \mathbb{R}^n, \\ \mathbf{u}(t,0) = \phi(t), \quad t \in [0,T] \end{cases} \tag{34}$$

with $f \in C([0,T]\times\mathbb{R}^n)$, $t \to f(t,.,.) \in B([0,T];$ $C^\theta(\mathbb{R}^n))$ *and such that there exist* $\mu \in \mathbb{R}^+$, *so that*

$$\min_{[0,T]} |f(t,0)| > 0, \tag{35}$$

$u_0 \in C^{2p+\theta}(\mathbb{R}^n)$. *Then (34) has a unique solution* (u,g), *such that*

$$u \in C^1([0,T];C(\mathbb{R}^n)) \cap B([0,T];C^{2p+\theta}(\mathbb{R}^n)),$$

$$D_t u \in B([0,T];C^\theta(\mathbb{R}^n)), \tag{36}$$

$$g \in C([0,T]). \tag{37}$$