Strong L^p-Solutions for certain Fluid-Solid Interaction Problems

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Formulation of the Problem

Fall of a rigid body in a Newtonian Fluid subject to Gravitation

Equations for the fluid

$$
\begin{aligned}
v_t + \mathrm{div}\, T(v,q) + v\cdot\nabla v &= g, && x\in\Omega(t), t>0\\
\mathrm{div}\, v &= 0, && x\in\Omega(t), t>0\\
v(t,x) &= \eta(t) + \omega(t)\times(x - x_c(t)), && x\in\Gamma(t), t>0\\
v(x,0) &= v_0(x)
\end{aligned}
$$

Equations for the rigid body: Conservation of momentum and angular momentum

$$
\begin{aligned}
m\frac{d\eta}{dt} &= F - \int_{\Gamma(t)} T(v,q)\cdot n(t)d\sigma, && t>0\\
\frac{d(J\omega)}{dt} &= M - \int_{\Gamma(t)} (x - x_c(t))\times T(v,q)\cdot n(t)d\sigma, && t>0
\end{aligned}
$$

$T(v,q)$: $2\mu D(v) - qId,$ where $D(v) = \frac{1}{2}(\nabla v + (\nabla v)^T)$

g : external force, here gravitation

η : velocity of center of mass x_c

ω : angular velocity of rigid body

$J(t)$: inertia tensor, $aJ(t)b := \int_{B(t)} \rho_B[a\times(x - x_c(t))][b\times(x - x_c(t))]$

Approaches known

- Weinberger '72, Serre '87, Galdi '97, Starovoitov '99, Silvestre '00, Desjardin- Esteban '01, Tucsnak '01, Galdi '03, Galdi-Silvestre '06
- Galdi '03: transformation to fixed exterior domain:

$$x = Q(t)y + x_c(t), \quad \xi(t) = Q^T(t)\eta(t), \quad \omega'(t) = Q^T(t)\omega(t)$$

yields new equations on fixed exterior domain Ω

$$\begin{array}{rcll}
w_t + (w-U)\nabla w + \omega' \times w & = & \operatorname{div} T(w,p) + G, & x \in \Omega, t > 0 \\
\operatorname{div} w & = & 0, & x \in \Omega, t > 0 \\
w(t,y) & = & U(t,y), & y \in \Gamma, t > 0 \\
m\frac{d\xi}{dt} + m\omega' \times \xi & = & mG - \int_\Gamma T(w,p)\cdot n d\sigma, & t > 0 \\
I\frac{d\omega'}{dt} + \omega' \times (I\omega') & = & -\int_\Gamma y \times T(w,p)\cdot n d\sigma, & t > 0 \\
\frac{dG}{dt} & = & G \times \omega', & t > 0
\end{array}$$

where

$$U(y,t) = \xi(t) + \omega'(t) \times y$$

Advantages-Disadavantages of this Transformation

- Galdi '03, Galdi-Silvestre '07:
  - steady state problem,
  - existence of periodic weak solution via Galerkin approximation
  - L^2-theory
- advantage: constant coefficient operaters
- disadvantage: Linear operator related to fluid is of the form

$$Au = P[\Delta u + (\omega \times x) \times \nabla u]$$

- Hishida '99: A generates a semigroup on $L^2_\sigma(\Omega)$ which is however not analytic
- maximal regularity technique is impossible

Different Approach

Idea: Choose transformation to fixed domain which follows rigid body in a neighborhood of the rigid body, but is equal to identity far away.

- mathematically: set $m(t) := \omega(t) \times x$, let $\omega, \eta \in W^{1,p}(0,T)$
- consider first

$$\begin{aligned} \partial_t X_0(t,y) &= m(t)(X_0(t,y) - x_c(t)) + \eta(t) \\ X_0(0,y) &= y \end{aligned}$$

  with solution $X_0(t,y) = Q(t)y + x_c(t)$
- choose cutoff-function $\varphi = 1$ close to body, 0 elsewehere
- set

$$b(t,x) := \varphi(x - x_c(t)) m(x - x_c(t)) - B(\nabla\varphi(\cdot - x_c(t)) m(t))(\cdot - x_c(t)) + \eta(t)$$

  where B is Bogovskii's operator
- $\operatorname{div} b = 0$ by construction

Transformed Coordinates

- Consider ODE

$$\begin{aligned} \partial_t X(t,y) &= b(t, X(t,y)), & t>0, y\in\mathbb{R}^3 \\ X(0,y) &= y, & t>0, y\in\mathbb{R}^3 \end{aligned}$$

  with solution $X(t,y)$. Then:
  - $X(t,\cdot)$ is diffeomorphism from $\Omega(t)$ onto Ω with inverse $Y(t,\cdot)$
  - $\det J_X = \det J_y = 1$
- transformed coordinates

$$\begin{aligned} u(t,y) &= J_Y(t,X)v(t,X) \\ \pi(t,y) &= q(t,X) \\ \omega'(t) &= Q^T(t)\omega(t) \\ \xi(t) &= Q^T(t)\eta(t) \\ G(t,y) &= J_Y(t,X)g \end{aligned}$$

- inertia tensor $I = Q^T(t)J(t)Q(t)$ is independent of t

Equivalent Formulation of the Problem

$$
\begin{array}{rcll}
u_t - \Delta u + \nabla(\pi - gy) & = & G - g - \mathcal{N}(u) + (\mathcal{L} - \Delta)u - \mathcal{M}u, & t \in (0,T), x \in \Omega \\
& & +(\nabla - \mathcal{G})(\pi - gy) - f & \\
\mathrm{div}\ u & = & 0, & t \in (0,T), x \in \Omega \\
u(t,y) & = & \xi(t) + \omega'(t) \times y, & t \in (0,T), x \in \Gamma \\
m\frac{d\xi}{dt} & = & mG(\cdot,0) - m(\omega' \times \xi) - \int_\Gamma T(u,\pi)n, & t \in (0,T) \\
I\frac{d\omega'}{dt} & = & -\omega' \times (I\omega') - \int_\Gamma y \times T(u,\pi)n, & t \in (0,T)
\end{array}
$$

and initial conditions $\xi(0) = \omega'(0) = u(0) = 0$, where

- $\mathcal{L}$ is transformed Stokes operator
- $\mathcal{N}$ transformed nonlinearity, $\mathcal{M}$ transformed time derivative, ... given by

$$
(\mathcal{L}u)_i = \sum \partial_j(g^{jk}\partial_k u_i) + 2\sum g^{kl}\Gamma^i_{jk}\partial_l u_j + \sum \partial_k(g^{kl}\Gamma^i_{kl}) + \sum g^{kl}\Gamma^m_{jl}\Gamma^i_{km}
$$

- Advantage: problem is truly parabolic
- Disadvantage: all terms have coefficients depending on x and t

Strategy

- Consider above problem as inhomogeneous Stokes problem coupled with two ODEs
- do fixed point argument in space of maximal L^p-regularity

Spaces of maximal regularity

- parabolic systems: Let $p \neq \frac{3}{2}, 3$

$$\begin{aligned} u_t - Au &= f, \quad t > 0, x \in \Omega \\ B_j u &= g_j \quad , t > 0, x \in \partial\Omega \\ u(0) &= u_0 \end{aligned}$$

have a strong solution $u \in W^1_p(0,T,L^q(\Omega)) \cap L^p(0,T,W^{2,q}(\Omega))$

$$\Updownarrow$$

$$\begin{aligned} f &\in L^p(0,T,L^q(\Omega)) \\ g_j &\in F^{k_j}_{pq}(0,T,L^q(\partial\Omega)) \cap L^p(0,T,B^{2k_j}_{qq}(\partial\Omega)), k_j = (2-m_j-1/q)/2 \\ u_0 &\in B^{2(1-1/p)}_{qp}(\Omega) + \text{compability conditions} \end{aligned}$$

- Solonnikov '64: $p = q$, $g = 0$,
- Weis '01: characterization by $\mathcal{R}$-bounds, $p \neq q$
- Denk, H., Prüss '07: $p \neq q$, $g \neq 0$

Inhomogeneous Stokes System

- Inhomogeneous Stokes system: let $p \neq \frac{3}{2}, 3$

$$\begin{aligned} u_t - \Delta u + \nabla p &= f, \quad t > 0, x \in \Omega \\ \text{div } u &= g, \quad t > 0, x \in \Omega \\ u &= h, \quad , t > 0, x \in \partial\Omega \\ u(0) &= u_0 \end{aligned}$$

has a strong solution

$$(u, p) \in W^1_p(0, T, L^p(\Omega)) \cap L^p(0, T, W^{2,p}(\Omega)) \times L^p(0, T, \widehat{W}^{1,p}(\Omega))$$

$$\Updownarrow$$

$$\begin{aligned} f &\in L^p(0, T, L^p(\Omega)) \\ g &\in L^p(0, T, W^{1,p}(\Omega)) \cap W^{1,p}(0, T, H_p^{-1}(\Omega)) \\ h &\in \{v \in W^{1/2-1/2p,p}(0, T, L^p(\Gamma)) \cap L^p(0, T, W^{2-1/p}(\Gamma)) : v(0) = 0\} \\ u_0 &\in W^{2(1-1/p),p}(\Omega), \text{div } u_0 = g(0) + \text{compability conditions} \end{aligned}$$

- Solonnikov '77: $g = h = 0, p \neq q$
- Bothe-Prüss '07: $g \neq 0, h \neq 0, p = q$
- seems to be still open: general case, $p \neq q$

Fixed point spaces

Set

- $X^T_{p,0} = \{u \in W^1_p(0,T,L^p(\Omega)) \cap L^p(0,T,W^{2,p}(\Omega)) \cap L^p_\sigma(\Omega) : u(0) = 0\}$
- $Y^T_p = L^p(0,T,\widehat{W}^{1,p}(\Omega))$
- $Z^T_{p,0} = \{u \in W^{1/2-1/2p,p}(0,T,L^p(\Gamma)) \cap L^p(0,T,W^{2-1/p}(\Gamma)) : u(0) = 0\}$
- $W^{1,p}_R(0,T) = \{u \in W^{1,p}(0,T) : u(0) = 0, \|u\|_{1,p} < R\}$

as well as

$$V^T_R = \{(v,q,\omega,\xi) \in X^T_{p,0} \times Y^T_p \times W^{1,p}_R \times W^{1,p}_R : \|v\|_{X^T_p} + \|q\|_{Y^T_p} < R\}$$

and consider the mapping

Fixed point argument

Consider the mapping

$$\Phi_R^T : V_R^T \to V_R^T, \quad \begin{pmatrix} v \\ q \\ \omega' \\ \xi' \end{pmatrix} \mapsto \begin{pmatrix} w \\ p \\ \omega \\ \xi \end{pmatrix}$$

where (w, p, ω, ξ) solves

$$\begin{aligned}
w_t - \Delta w + \nabla p &= G - g - \mathcal{N}(v) + (\mathcal{L} - \Delta)v - \mathcal{M}v, & t \in (0,T), x \in \Omega \\
& \quad +(\nabla - \mathcal{G})q - f & \\
\text{div } w &= 0, & t \in (0,T), x \in \Omega \\
w(t,y) &= \xi'(t) + \omega'(t) \times y, & t \in (0,T), x \in \Gamma \\
m\frac{d\xi}{dt} + \int_\Gamma T(w,p)n &= G(\cdot, 0) - g - m(\omega' \times \xi'), & t \in (0,T) \\
I\frac{d\omega}{dt} + \int_\Gamma y \times T(w,p)n, &= -\omega' \times (I\omega') & t \in (0,T)
\end{aligned}$$

and show that Φ_R^T is a contraction in V_R^T

- first three equations: use results for inhomogeneous Stokes system
- last two equations: use $\int_\Gamma T(w_1 - w_2)n$ is well-defined

Typical Estimates

Let $(v_1, q_1, \omega_1', \xi_1'), (v_2, q_2, \omega_2', \xi_2') \in V_R^T$. Then

$$
\begin{aligned}
\|(\mathcal{L}_1 - \Delta)v_1 - (\mathcal{L}_2 - \Delta)v_2\|_{pp} &\leq CT\|v_1 - v_2\|_{X_p^T} + CR[\|\omega_1' - \omega_2'\|_{1,p} + \|\xi_1' - \xi_2'\|_1 \\
\|\mathcal{N}_1 v_1 - \mathcal{N}_2 v_2\|_{pp} &\leq CR[\|v_1 - v_2\|_{X_p^T} + \|\omega_1' - \omega_2'\|_{1,p} + \|\xi_1' - \xi_2'\|_{1,p}]
\end{aligned}
$$

by making use of Mixed Derivative Theorem:
Let $\mu \in 0, 1, T_0 > 0$ and for s, r assume that $\frac{2-\mu}{2} + \frac{n}{2r} \geq \frac{n+2}{2p} - \frac{1}{s}$.

$\Rightarrow$ Then for all $T \in (0, T_0)$, the embedding

$$X_p^T \hookrightarrow L^s(0, T, W^{\mu,r}(\Omega))$$

is continuous.

Main Result

Theorem:

Let $p \geq 5/3$. Then there exists a unique local strong solution to fluid-rigid body problem in the space of maximal regularity

Advantage of this approach

- allows to treat also fluid-structure interaction problems for Non-Newtonian fluids.

Non-Newtonian Fluids

Replace above equations for the fluid by

$$\begin{aligned} v_t + \text{div } S_0(v,q) + v \cdot \nabla v &= g, & x \in \Omega(t), t > 0 \\ \text{div } v &= 0, & x \in \Omega(t), t > 0 \\ v(t,x) &= \eta(t) + \omega(t) \times (x - x_c(t)), & x \in \Gamma(t), t > 0 \\ v(x,0) &= v_0(x) \end{aligned}$$

with defomormation tensor $S_0^{(v)}$ of the form

- $S_0^{(v)}(x,t) = 2\mu(|\mathcal{E}_0^{(v)}(x,t)|_2^2)\mathcal{E}_0^{(v)}(x,t)$
- $\mathcal{E}_0^{(v)} = \frac{1}{2}(\nabla v + (\nabla v)^T)$
- $|\mathcal{E}_0^{(v)}|_2^2 = \sum_{i,j=1}^n (\epsilon_0^v)_{ij}^2$, Hilbert-Schmidt norm of $\mathcal{E}$
- $\mu \in C^2(\mathbb{R}_+, \mathbb{R}_+)$ with $\mu(s) > 0, \qquad \mu(s) + 2s\mu'(s) > 0, \quad s \geq 0$
- in particular: powerlike-fluids $\mu(s) = \mu_0(1+s)^{(d-2)/2}$ $\mu_0 > 0, d > 1$
- $1 < d < 2$: shear-thinning
- $d > 2$: shear-thickening

Basic Idea

- Basic Idea: write equations as

$$\begin{array}{rcll} u_t + \mathcal{A}_0(u)u + \nabla(\pi - gy) & = & G - g - \mathcal{N}(u) + (\mathcal{A}_0(u) - \mathcal{A}(u))u, & \\ & & -\mathcal{M}u + (\nabla - \mathcal{G})(\pi - gy) - f & x \in \Omega \\ \mathrm{div}\, u & = & 0, & x \in \Omega \\ u(t,y) & = & \xi(t) + \omega'(t) \times y, & x \in \Gamma \\ m\frac{d\xi}{dt} & = & mG(\cdot,0) - m(\omega' \times \xi) - \int_\Gamma T(u,\pi)n, & t \in (0,T) \\ I\frac{d\omega'}{dt} & = & -\omega' \times (I\omega') - \int_\Gamma y \times T(u,\pi)n, & t \in (0,T) \end{array}$$

- key result due to Bothe-Prüss ’07:
  Maximal regularity result for inhomogenous Stokes equation remains true if Δ is replaced by second order normally elliptic operator.
- existence of a local strong solution for $p > 5$