Increased stability in the continuation of solutions to the Helmholtz equation

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Motivated by control theory and prospecting by acoustical and electromagnetic waves we consider the Cauchy problem

$$(\Delta + a_0^2 k^2)u = f \text{ in } \Omega, \tag{0.1}$$

$$u = u_0,\ \partial_\nu u = u_1 \text{ on } \Gamma, \tag{0.2}$$

Ω is a domain in $\mathbf{R}^n$ and $\Gamma \subset \partial\Omega$. Due to Fritz John [8] in general case one can expect only quite weak logarithmic stability. However in important examples (e.g. nearfield acoustical holography [3]) it was observed that stability and resolution are increasing with k.

1 Increased stability estimates

Let $\Omega \subset \{0 < x_n < h, |x'| < r\}$ with Lipschitz $\partial\Omega$, $\bar{\Omega} \subset \{x_n < h\}$ and $\Gamma = \partial\Omega \cap \{0 < x_n < h\}$. Let $\Omega(d) = \Omega \cap \{d < x_n\}$. C are constants which depend on Ω, Γ. $\|u\|_{(l)}(\Omega)$ is the norm in the Sobolev space $H^l(\Omega)$ and $\|u\|(\Omega) = \|u\|_{(0)}(\Omega)$. We let $M_1 = \|u\|_{(1)}(\Omega)$, $F = \|f\|(\Omega) + \|u\|(\Gamma) + \|\nabla u\|(\Gamma)$ and $F(k,d) = \|f\|(\Omega) + (d^{-0.5}k + d^{-1.5})\|u\|(\Gamma) + \|\nabla u\|(\Gamma)$.

We assume that $1 \leq k, F(k,d) < M_1, d < 2r$.

Theorem 1.1 *[7], [4] Let* $a_0 \in C^1(\overline{\Omega}), 0 < a_0$ *on* $\overline{\Omega}$ *and*

$$0 < a_0 + \nabla a_0 \cdot x + \beta_n \partial_n a_0, \ 0 \leq \partial_n a_0 \ on \ \overline{\Omega} \tag{1.1}$$

for some $\beta_n > 0$.

Then for any ε *there are* $C, C(\varepsilon), \lambda(d) \in (0,1)$ *such that*

$$\|u\|(\Omega(d)) \leq C(F + \varepsilon\|u\|_{(1)}(\Omega) + C(\varepsilon)\frac{M_1^{1-\lambda}F(k,d)^{\lambda} + F}{d^2 k})$$

for all u *solving (0.1),(0.2).*

If $a_0 = 1$, *then*

$$\|u\|(\Omega(d)) \leq C(F + \frac{M^{1-\lambda}F(k,d)^{\lambda}}{d^{2-2\lambda}k}), \tag{1.2}$$

where

$$\lambda = \frac{2r^2 d + \frac{3}{8}d^3}{4r^2 h + h^2 d + \frac{1}{4}d^2 h + \frac{3}{8}d^3 + 3r^2 d}$$

Theorem 1.2 *[2] There exists* C *such that*

$$\|u\|^2(\Omega(0)) \leq CM_1^2(\varepsilon_1^2 + \frac{1}{(-ln\varepsilon_1 + k)^{\frac{1}{8}}}), \ \varepsilon_1 = \frac{F}{M_1},$$

for any solution u *to (0.1),(0.2).*

The John's counterexample [8] shows that in case when $\Gamma = \{x : |x| = 1\}$ and $\Omega = \{x : 1 < |x| < r_1\}$ in $\mathbf{R}^2$ the conditional logarithmic stability bound is the optimal bound which is uniform with respect to k.

Our proof of Theorem 1.1 is based on the following

Theorem 1.3 *There is a constant C such that for any solution u to the Cauchy problem (0.1), (0.2)*

$$\|u\|_{(1)}(\Omega(d)) \leq CF(k,d)^{\lambda}(\frac{M}{d^2})^{1-\lambda} \tag{1.3}$$

(λ is in Theorem 1.1).

2 Outlines of proofs

In the following, $V(\xi, x_n)$ denotes the (partial) Fourier transformation $\mathcal{F}v(\xi, x_n)$ of a function $v(x)$ and $\Omega^*(d) = \mathbf{R}^{n-1} \times (d, h)$. In the low frequency zone the equation (0.1) is x_n-hyperbolic, hence

Lemma 2.1 *Let $a_n \in C^1([0,h])$ and depend only on x_n. Let $v \in C^2(\bar{\Omega}^*)$ solve the initial value problem*

$$(\Delta + a_n^2 k^2)v = \partial_1 f_1 + ... + \partial_n f_n + k f_{n+1} + k^2 f_0 \ in \ \Omega^*(d),$$

$$v = 0 \ on \ \Omega^*(h_1)$$

for some $h_1 < h$, $f_j \in C^{\infty}(\bar{\Omega}^*(d))$, $f_j = 0$ on $\Omega^*(h_1)$, and

$$V(\xi, x_n) = 0 \ when \ \frac{a_n^2(x_n)}{2} k^2 < |\xi|^2 \tag{2.1}$$

Then there is constant C depending only on h, $sup|\partial_n a_n|$, $supa_n^{-1}$ over $(0,h)$ such that

$$\|v\|(\Omega^*(d)) \leq$$

$$C(\|f_1\|(\Omega^*(d)) + ... + \|f_{n+1}\|(\Omega^*(d)) + \|f_0\|(\Omega^*(d)) + \|\partial_n f_0\|^2(\Omega^*(d)).$$

Due to the Parseval's identity it suffices to consider the initial value problems for

$$\partial_n^2 V_j + (a_n^2 k^2 - |\xi|^2) V_j = -i\xi_j F_j \text{ on } (d,h), j = 1, ..., n-1,$$

multiply by $\partial_n \bar{V}_j e^{\tau x_n}$ and integrate over $(0,h)$.

Let

$$w(x;\tau) = \int_{-1}^{1} exp(2\tau e^{\sigma(|x-\beta|^2 - \theta^2 t^2)}) dt, \ \beta = (0, ..., 0, \beta_n).$$

Lemma 2.2 *Let the condition (1.1) be satisfied.*

Then there is constant C such that

$$\int_{\Omega_1} ((\tau^3 + \tau k^2)|u|^2 + \tau |\nabla u|^2) w(,\tau) \leq$$

$$C(\int_{\Omega_1} |(\Delta + a_0^2 k^2) u|^2 w(,\tau) + \int_{\partial\Omega_1} ((\tau^3 + \tau k^2)|u|^2 + \tau |\nabla u|^2) w(,\tau))$$

for all functions $u \in H^2(\Omega_1)$ and all $\tau > C$.

It is known [5] that under the condition (1.1) there are positive σ, θ depending on Ω, a_0, β such that with $\varphi(x,t) = e^{\sigma(|x-\beta|^2 - \theta^2 t^2)}$ we have the Carleman (energy, with the weight $e^{\tau\varphi}$) estimate

$$\int_{\Omega \times (-T,T)} (\tau^3 |U|^2 + \tau |\nabla U|^2 + \tau |\partial_t U|^2) e^{2\tau\varphi} \leq$$

$$C(\int_{\Omega \times (-T,T)} |(\Delta - a_0^2 \partial_t^2) U|^2 e^{2\tau\varphi} + bdry integrals. \qquad (2.2)$$

We will apply (2.2) to the function

$$U(x,t) = u(x) e^{ikt}.$$

When $a_0 = 1$ there is a better bound. We denote $l(x;\beta) = |x + \beta|$.

Lemma 2.3 *Let* Ω_1 *be a bounded Lipschitz domain in* $\mathbf{R^n}$. *Let* $L = supL(x;\beta)$ *over* $x \in \Omega_1$. *Let* $w(x) = exp(\tau(x_1^2 + ... + x_{n-1}^2 + (x_n+\beta)^2)$.

Then for some constant C *we have*

$$32\tau^3\|wlu\|^2(\Omega_1) + 5\tau\|w\nabla u\|^2(\Omega_1) \leq \|w(\Delta+k^2)u\|^2(\Omega_1) +$$

$$C((\tau^3(L^3+1)+\tau(k^2L+1))\|wu\|^2(\partial\Omega_1)+\tau(L+1)\|w\nabla u\|^2(\partial\Omega_1))$$

for all functions $u \in H^2(\Omega_1)$ *and all* $\tau > 0$.

Due to the substitution $u = w^{-1}v$, the final bound follows from

$$64\tau^3\|lv\|^2(\Omega_1) + 16\tau\|\nabla v\|^2 \leq$$

$$\|\Delta v - 4\tau(x+\beta e_n)\cdot\nabla v + (4\tau^2|x+\beta e_n|^2 - 2\tau n + k^2)v\|^2(\Omega_1) +$$

$$C((\tau^3(L^3+1)+\tau(k^2L+1))\|v\|^2(\partial\Omega_1)+\tau(L+1)\|\nabla v\|^2(\partial\Omega_1)).$$

Obviously,

$$(\Delta v - 4\tau(x+\beta e_n)\cdot\nabla v + (4\tau^2|x+\beta e_n|^2 - 2\tau n + k^2)v)^2 \geq$$

$$(\Delta v - 4\tau(x+\beta e_n)\cdot\nabla v + (4\tau^2|x+\beta e_n|^2 - 2\tau n + k^2)v)^2 -$$

$$(\Delta v + 4\tau(x+\beta e_n)\cdot\nabla v + (4\tau^2|x+\beta e_n|^2 + 2\tau n + k^2)v)^2 =$$

$$-16\tau(\Delta v)(x+\beta e_n)\cdot\nabla v - 8\tau n v\Delta v -$$

$$16\tau(x+\beta e_n)\cdot\nabla v(4\tau^2|x+\beta e_n|^2+k^2)v - 8\tau n(4\tau^2|x+\beta e_n|^2+k^2)v^2.$$

Now the proof can be completed by integration by parts.

Theorem 1.3 follows from Theorem 1.1 in a standard way [5]

Proof of Theorem 1.1.

Since Γ is Lipschitz, by known extension theorems there is a function u^* such that $u = u^*, \nabla u = \nabla u^*$ on Γ and

$$\|u^*\|_{(1)}(\Omega^*(0)) \leq C(\|u\|(\Gamma) + \|\nabla u\|(\Gamma)) \leq CF,$$

where we used the definition of F and the notation $\Omega^*(d)$ from Lemma 2.1. Let $u_1 = u - u^*$ on Ω and $u_1 = 0$ on $\Omega^*(0) \setminus \Omega$. It suffices to obtain the bound (1.2) for u_1 instead of u. Observe that (in the weak sense)

$$\Delta u_1 + k^2 u_1 = f^* - k^2 u^* \text{ in } \Omega^*(0),$$

where f^* is the linear continuous functional on $H_{(1)}(\Omega^*(0))$ defined as

$$f^*(w) = -\int_{\partial\Omega} \partial_\nu u w + \int_\Omega (f w + \nabla u \cdot \nabla w).$$

Due to known trace theorems

$$\|f^*\|_{(-1)}(\Omega^*) \leq C\|u^*\|_{(1)}(\Omega^*) \leq CF.$$

By mollyfying we replace elements of negative Sobolev spaces by smooth functions. We split $u_1 = v + u_2$. Let a cut off function $\chi_k(\xi') = 1$ when $|\xi| \leq \frac{k}{2}$ and zero for other $\xi' \in \mathbf{R}^{n-1}$. We let $v = \mathcal{F}^{-1}\chi_k \mathcal{F} u_1$ and $u_2 = u_1 - v$. We have

$$(\Delta + k^2)v = \partial_1 f_1 + ... + \partial_n f_n + f_{n+1} + k^2 f_0 \text{ on } \Omega^*(d),$$

$$\|f_j\|(\Omega^*(d)) \leq CF.$$

By Lemma 2.1

$$\|v\|(\Omega^*(d)) \leq CF.$$

Due to the definition of u_2 and to the elementary properties of the Fourier transformation,

$$\|u_2\|_{(1)}(\Omega^*(d)) \leq \|u_1\|_{(1)}(\Omega^*(d)) \leq \|u\|_{(1)}(\Omega) + \|u^*\|_{(1)}(\Omega), \quad (2.3)$$

$$\|u_2\|(\Omega^*(d)) \leq \frac{2}{k}\|u_2\|_{(1)}(\Omega^*(d)). \quad (2.4)$$

From (2.4), (2.3) we have

$$\|u_2\|(\Omega) \leq \frac{2}{k}(\|u\|_{(1)}(\Omega) + CF).$$

From this bound and from (1.3) we obtain the needed bound (1.2) for u_1 and complete the proof.

3 Numerical evidence

The setup for the exterior problem consists of two concentric semispheres and a semi circle given by

$$\begin{aligned}
\Gamma &= \{\|x\| = r_0, 0 < \phi < \pi, 0 < \theta < \pi\} \\
\Gamma_1 &= \{\|x\| = r_1, 0 < \phi < \pi, 0 < \theta < \pi\} \\
\Gamma_2 &= \{\|x\| = r_2, \phi = \frac{\pi}{2}, 0 \leq \theta \leq \frac{\pi}{2}\}
\end{aligned}$$

where ϕ, θ are polar angles, $r_0 = 2$, $r_1 = 1$ and $r_2 = \frac{1}{2}$. Five acoustical sources are placed on the semicircle Γ_2, their amplitudes and positions are given by Table 1.

We discretize Γ and Γ_1 by considering n equal angles between ϕ_1 and ϕ_2 and between θ_1 and θ_2. We obtain n^2 points on Γ

Amplitude	Position		
$A_1 = 1$	$(0,$	$0,$	$\frac{1}{2})$
$A_2 = 4$	$(0,$	$\frac{-1}{2\sqrt{2}},$	$\frac{1}{2\sqrt{2}})$
$A_3 = 5$	$(0,$	$-\frac{1}{2},$	$0)$
$A_4 = 2$	$(0,$	$\frac{-1}{\sqrt{2}2},$	$\frac{-1}{2\sqrt{2}})$
$A_5 = 3$	$(0,$	$0,$	$-\frac{1}{2})$

Table 1: Amplitudes and positions of acoustical sources

Frequency (k)	Error of reconstruction(in %)
2.0	782.00
4.0	544.44
8.0	67.13
16.0	2.79

Table 2: Errors of reconstruction at various frequencies for the exterior problem

and n^2 points on Γ_1. For this experiment $n = 10$. The Cauchy data are on Γ and we recontruct u on Γ_1.

The acoustic pressure and its normal (radial) derivative on Γ are given by

$$u(x) = \sum_{j=1}^{5} A_j \Phi(x, y_j), \;\; \Phi(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}. \tag{3.1}$$

Using (3.1) we can generate the Cauchy data on Γ by adding some 1% uniformly distributed random noise. Since u is a radiating solution

$$u(x) \approx \sum_{n=0}^{N} \sum_{m=-n}^{n} a_{n,m} h_n^{(1)}(|x|) Y_n^m \left(\frac{x}{|x|} \right) \tag{3.2}$$

We chosen $N = 9, 10$. We find the coefficients $a_{n,m}$ by matching the series expansion of the solution with the Cauchy data calculated from (3.1) on Γ. This is achieved by forming a linear algebraic system $Ax = b$ where x is a vector of coefficients to be determined. The solution to this system is obtained by forming the normal equations $A^*Ax = A^*b$ and by applying conjugate gradient technique.

Also, we would like to compare with the interior problem. The experimental setup for the interior problem consists of two

Frequency (k)	Error of reconstruction(in %)
8.0	67.13
16.0	117

Table 3: Errors of reconstruction at various frequencies for the interior problem

concentric hemispheres and a semi-circle given by Γ, Γ_1 and Γ_2 which is same as the setup for the exterior problem but with $r_0 = \frac{1}{2}$, $r_1 = 1$ and $r_2 = 2$. As before the Cauchy data is prescribed on the discretized surface Γ_0. This Cauchy data is matched with the approximate series expansion of the solution to the interior problem which is given by (3.2) with Bessel's functions j_n instead of Hankel's functions.

4 Inverse problem for the Schr ödinger equation

Let Ω be a domain in $\mathbf{R}^3$ with Lipschitz boundary. We consider the Schrödinger equation

$$-\Delta u - k^2 u + cu = 0 \text{ in } \Omega \tag{4.1}$$

with the Dirichlet boundary data

$$u = g \text{ on } \partial\Omega. \tag{4.2}$$

We will assume that the (complex valued) potential $c \in L_\infty(\Omega)$. We define the Dirichlet-to-Neumann map

$$\Lambda_c g = \partial_\nu u \text{ on } \partial\Omega,\ g \in H^{\frac{1}{2}}(\partial\Omega). \tag{4.3}$$

It is well-known that Λ_c is the continuous linear operator from $H^{\frac{1}{2}}(\Gamma)$ into $H^{-\frac{1}{2}}(\Gamma)$. We denote its norm by $||\Lambda_c||$. Uniqueness of c is due to groundbreaking result of Sylvester and Uhlmann

(1987), logarithmic stability was proven by Alessandrini [1] and its optimality by Mandache [9].

We assume that $\Omega \subset B(0;1)$.

Theorem 4.1 *Let*

$$||c_j||_\infty(\Omega) \le M,\ ||c_j||_{1,\infty}(\Omega) \le M_1,\ j = 1,2. \tag{4.4}$$

and $\varepsilon = \|\Lambda_2 - \Lambda_1\|, E = -log\varepsilon.$

If

$$\sqrt{k + \frac{1}{16}} \le E\frac{1}{4},\ 2 \le E,\ 2C_0^2 M < \frac{E^2}{2} - \frac{E}{4} - k + 2k^2 + 4, \tag{4.5}$$

then there is constant C *such that*

$$||c_2 - c_1||_2 \le CM^4((E+k)^{-\frac{1}{2}} + \varepsilon^{2-\sqrt{2}}E^5) + \frac{4M_1^2}{E + 4k + 1}. \tag{4.6}$$

If

$$\frac{1}{8}(E)^2 \le k < \varepsilon^{-\frac{9}{10}},\ 2C_0^2 M < k^2 + 2. \tag{4.7}$$

then there is constant C *such that*

$$||c_2 - c_1||_2 \le C_0(M^4((E^3 + k^{\frac{1}{10}})(k^2+2)^{-1} +$$

$$\varepsilon(\varepsilon^{-\frac{2}{10}} + M^4)(E^3 + k^{\frac{1}{10}})) + \frac{M_1^2}{E^2 + k^{\frac{1}{10}} + 1}. \tag{4.8}$$

5 Conclusion

1) Increased stability should be more dramatic when the data are given at a larger distance from Γ_2, when singularities of the solution are distributed over Γ_2, and for large k.

2) The increased stability for the Helmholtz equation is linked and to the problem of the exact controllabity for the wave equation in a subdomain by the data on a (arbibrariry small) part of the lateral boundary. The exact controllability for the wave equation from a "large" part of the boundary is relatively well understood [5].

3) The next natural step is to obtain similar estimates for the elliptic equations $-\Delta u + k^{-1} b \cdot \nabla u = 0$ and corresponding parabolic equations (large drift and small diffusion), the inverse scattering problems by obstacles and by the medium.

4) The high frequencies k are not included in Theorem 3.1, but they might be by using scattering theory as in [10]. The challenging question concerns the equation $-\Delta u - k^2 cu = 0$.

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