An inverse problem for the heat equation in a degenerate free boundary domain

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Introduction

Inverse problems for the equation

$$u_t = a(t)u_{xx} + b(x,t)u_x + c(x,t)u + f(x,t), \quad 0 < x < h(t), 0 < t < T$$

with unknown coefficient $a = a(t)$ were considered from different points of view in the case

★ $a(t) > 0,\ t \in [0,T];\ h(t) \equiv h = \text{const}$ is known,

in the monograph

M.Ivanchov (2003) Inverse Problems for Equations of Parabolic Type. Lviv: VNTL Publishers.

Inverse problems for degenerate parabolic equations when

★ $a(t) > 0,\ t \in (0,T],\ a(0) = 0;\ h(t) \equiv h = \text{const}$ is known,

were studied in the papers

N.Saldina (2006) An inverse problem for a weakly degenerate parabolic equation. Math. Methods and Phys.-Mech. Fields, V.49, no 3. P.7-17 (Ukrainian);

M.Ivanchov, N.Saldina (2006) An inverse problem for a strongly degenerate heat equation. J. Inv. Ill-Posed Probl., V.14, no 5. P.465-480;

M.Ivanchov, N.Saldina (2006) An inverse problem for a strongly power degenerate parabolic equation. Ukr. Math. J., V.58, no 11. P.1487-1500 (Ukrainian).

Inverse problems for parabolic equations in free boundary domains

★ $a(t) > 0, \; t \in [0, T]; \; h = h(t)$ is unknown,

were investigated in the papers

I.Baranska (2005) An inverse problem for a parabolic equation in a free boundary domain. Math. Methods and Phys.-Mech. Fields, V.48, no 2. P.32-42 (Ukrainian);

I.Baranska, M.Ivanchov (2007) An inverse problem for the two-dimensional heat equation in a free boundary domain. Ukr. Math. Bull., V.4, no 4. P.457-484 (Ukrainian).

In the papers

N.Hryntsiv (2007) An inverse problem for a degenerate parabolic equation in a free boundary domain. Math. Methods and Phys.-Mech. Fields, V.48, no 2. P.32-42 (Ukrainian);

N.Hryntsiv (2007) An inverse problem for a strongly degenerate parabolic equation in a free boundary domain. Visnyk Lviv. Univer. Ser. Mech.-Math. V.64. P.84-97 (Ukrainian)

was considered the case where

★ $a(t) > 0,\ t \in (0,T],\ a(0) = 0,\ h = h(t)$ is unknown.

A problem in a free boundary domain degenerating at the initial moment

★ $a(t) > 0,\ t \in [0,T]$ is known, $h = h(t) > 0,\ t \in (0,T],\ h(0) = 0$ is unknown, was studied in the paper

M.Ivanchov (2007) Heat conduction problem with a free boundary degenerating at the initial moment. Math. Methods and Phys.-Mech. Fields, V.50, no 3. P.82-87 (Ukrainian).

In the present work we consider an inverse problem for the heat equation in the domain which degenerates at the initial moment.

1. Statement of the problem

In the domain $\Omega_T \equiv \{(x,t) : 0 < x < t^{\alpha}h(t),\ 0 < t < T\}$ with unknown part of boundary $h = h(t)$ we consider the following inverse problem of finding functions $(a(t), u(x,t))$:

$$u_t = a(t)u_{xx} + f(x,t), \quad (x,t) \in \Omega_T, \tag{1}$$

$$u(0,t) = \mu_1(t), \quad u(t^\alpha h(t), t) = \mu_2(t), \quad 0 \le t \le T, \tag{2}$$

$$a(t)u_x(0,t) = \mu_3(t), \quad 0 \le t \le T, \tag{3}$$

$$\int\limits_0^{t^\alpha h(t)} u(x,t)dx = \mu_4(t), \quad 0 \le t \le T. \tag{4}$$

We define a solution of the problem (1)-(4) as a triple of functions $(a, h, u) \in C([0,T]) \times C^1([0,T]) \times C^{2,1}(\Omega_T) \cap C^{1,0}(\overline{\Omega}_T), a(t) > 0, h(t) > 0, t \in [0,T], \alpha > 0$ verifying the conditions (1)-(4).

Change of the variables. Introduce the new variables

$$y = \frac{x}{h(t)}, \quad \sigma = t^\alpha. \tag{5}$$

and notations:

$u(yh(\sigma^{1/\alpha}),\sigma^{1/\alpha}) \equiv v(y,\sigma), \quad a(\sigma^{1/\alpha}) \equiv b(\sigma), \quad h(\sigma^{1/\alpha}) \equiv g(\sigma),$ $\mu_i(\sigma^{1/\alpha}) \equiv \nu_i(\sigma),\ i=\overline{1,4} \quad f(yh(\sigma^{1/\alpha}),\sigma^{1/\alpha}) \equiv F(y,\sigma), \quad Q_{T_1} \equiv \{(y,\sigma :$ $0<y<\sigma, 0<\sigma<T_1)\},\ T_1 = T^{\alpha}.$

Reduction to the equivalent problem. In new variables the problem (1)-(4) is reduced to the equivalent problem

$$v_\sigma = \frac{\sigma^{\frac{1-\alpha}{\alpha}} b(\sigma)}{\alpha g^2(\sigma)} v_{yy} + \frac{y g'(\sigma)}{g(\sigma)} v_y + \frac{1}{\alpha}\sigma^{\frac{1-\alpha}{\alpha}} F(y,\sigma), \quad (y,\sigma)\in Q_{T_1}, \tag{6}$$

$$v(0,\sigma) = \nu_1(\sigma), \quad v(\sigma,\sigma) = \nu_2(\sigma), \quad 0 \le \sigma \le T_1, \tag{7}$$

$$b(\sigma) v_y(0,\sigma) = g(\sigma)\nu_3(\sigma), \quad 0 \le \sigma \le T_1, \tag{8}$$

$$g(\sigma)\int\limits_0^{\sigma} v(y,\sigma)dy = \nu_4(\sigma), \quad 0 \le \sigma \le T_1 \tag{9}$$

with unknowns $(g(\sigma), b(\sigma), v(y,\sigma))$.

Remark. (6)-(9) is an inverse problem for the degenerate parabolic equation. We consider the case of the weak degeneration when $\frac{1}{2} < \alpha < 1$.

2. Construction of solution of the direct problem

Consider the following direct problem:

$$u_t = a(t)u_{xx} + f(x,t), \quad 0 < x < t,\ 0 < t < T, \tag{10}$$

$$u(0,t) = \mu_1(t), \quad u(t,t) = \mu_2(t), \quad 0 \le t \le T. \tag{11}$$

We will construct the solution of the problem (10), (11) with the aid of the Green function.

Determination of the Green function. To construct the Green function we use the representation analogous to the parametrix method:

$$G(x,t,\xi,\tau) = G_0(x,t,\xi,\tau) + \int_0^t d\sigma \int_0^\sigma G_0(x,t,\eta,\sigma)\Phi(\eta,\sigma,\xi,\tau)d\eta, \quad (12)$$

where

$$G_0(x,t,\xi,\tau) = \frac{1}{2\sqrt{\pi(\theta(t)-\theta(\tau))}} \sum_{n=-\infty}^{n=\infty} \left(\exp\left(-\frac{(x-\xi+2nt)^2}{4(\theta(t)-\theta(\tau))}\right) \right.$$

$$\left. - \exp\left(-\frac{(x+\xi+2nt)^2}{4(\theta(t)-\theta(\tau))}\right)\right), \quad \theta(t) = \int_0^t a(\tau)d\tau,$$

and the function $\Phi(x,t,\xi,\tau)$ is a solution of the integral equation

$$\Phi(x,t,\xi,\tau) = -LG_0(x,t,\xi,\tau) - \int_0^t d\sigma \int_0^\sigma LG_0(x,t,\eta,\sigma)\Phi(\eta,\sigma,\xi,\tau)d\eta, \tag{13}$$

where $Lu = u_t - a(t)u_{xx}$.

Solution of the problem (10), (11). Using the Green function we obtain the solution of the problem (10), (11):

$$u(x,t) = \int_0^t G_\xi(x,t,0,\tau)a(\tau)\mu_1(\tau)d\tau - \int_0^t G_\xi(x,t,\tau,\tau)a(\tau)\mu_2(\tau)d\tau$$

$$+ \int_0^t d\tau \int_0^\tau G(x,t,\xi,\tau)f(\xi,\tau)d\xi. \tag{14}$$

3. Reduction of the problem (6)-(9) to an equivalent system of equations

We reduce the direct problem (6), (7) to the equivalent systems of integral equations

$$v(y,\sigma) = \nu_1(\sigma) + \frac{y}{\sigma}(\nu_2(\sigma) - \nu_1(\sigma)) + \int_0^\sigma d\tau \int_0^\tau G(y,\sigma,\xi,\tau)\Big(\frac{1}{\alpha}\tau^{\frac{1-\alpha}{\alpha}}F(\xi,\tau)$$

$$-\nu_1'(\tau) - \frac{\xi}{\tau}(\nu_2'(\tau) - \nu_1'(\tau)) + \frac{\xi}{\tau^2}(\nu_2(\tau) - \nu_1(\tau)) + \frac{\xi p(\tau)}{\tau g(\tau)}w(\xi,\tau)\Big)d\xi, \tag{15}$$

$$w(y,\sigma) = \frac{\nu_2(\sigma) - \nu_1(\sigma)}{\sigma} + \int_0^\sigma d\tau \int_0^\tau G_y(y,\sigma,\xi,\tau)\Big(\frac{1}{\alpha}\tau^{\frac{1-\alpha}{\alpha}}F(\xi,\tau) - \nu_1'(\tau)$$

$$-\frac{\xi}{\tau}(\nu_2'(\tau) - \nu_1'(\tau)) + \frac{\xi}{\tau^2}(\nu_2(\tau) - \nu_1(\tau)) + \frac{\xi p(\tau)}{\tau g(\tau)}w(\xi,\tau)\Big)d\xi, \tag{16}$$

where $G(y,\sigma,\xi,\tau)$ is the Green function for the equation

$$v_\sigma = \frac{\sigma^{\frac{1-\alpha}{\alpha}} b(\sigma)}{\alpha g^2(\sigma)} v_{yy} + F(y,\sigma)$$

with conditions (7). We derive others equations from the conditions (8), (9):

$$b(\sigma) = \frac{\nu_3(\sigma) g(\sigma)}{w(0,\sigma)}, \quad 0 \le \sigma \le T_1, \tag{17}$$

$$g(\sigma) = \frac{\nu_4(\sigma)}{\int\limits_0^\sigma v(y,\sigma)dy}, \quad 0 \le \sigma \le T_1, \tag{18}$$

$$p(\sigma) = \frac{1}{\nu_2(\sigma)} \Bigg(\nu_4'(\sigma) - \frac{1}{\alpha}\sigma^{\frac{1-\alpha}{\alpha}} \nu_3(\sigma) - \nu_2(\sigma) g(\sigma) - \frac{\sigma^{\frac{1-\alpha}{\alpha}} b(\sigma)}{\alpha g^2(\sigma)} w(\sigma,\sigma)$$
$$- \frac{1}{\alpha}\sigma^{\frac{1-\alpha}{\alpha}} \int\limits_0^\sigma F_y(y,\sigma)dy \Bigg), \quad 0 \le \sigma \le T_1, \tag{19}$$

where $w(y,\sigma) = v_y(y,\sigma), p(\sigma) = \sigma g'(\sigma)$.

4. Existence of solution to the system (15)-(19)

Assumptions. We suppose that the following assumptions are hold:

$(A1)$ $\nu_i \in C^1([0,T_1]),\ i = 1,2,4,\quad \nu_3 \in C([0,T_1]),\quad F \in C^{1,0}(\overline{Q}_{T_1})$;

$(A2)$ $\nu_i(\sigma) > 0,\ \sigma \in [0,T_1],\ i = \overline{1,4},\ \ F(y,\sigma) \geq 0,\ (y,\sigma) \in \overline{Q}_{T_1},\ \ \nu_1(0) = \nu_2(0),\quad \nu_1(\sigma) < \nu_2(\sigma),\ \ \sigma \in (0,T_1]$;

$(A3)$ there exist the limits

$$\lim_{\sigma\to 0+} \frac{\nu_4(\sigma)}{\sigma} > 0, \quad \lim_{\sigma\to 0+} \frac{\nu_2(\sigma) - \nu_1(\sigma)}{\sigma} > 0.$$

Estimates. To apply Schauder fixed-point theorem we need the estimates of the solutions of the system (15)-(19). From the maximum principle, we obtain

$$v(y,\sigma) \geq \min\{\min_{[0,T_1]} \nu_1(\sigma), \min_{[0,T_1]} \nu_2(\sigma)\} \equiv M_0 > 0, \quad (y,\sigma) \in \overline{Q}_{T_1}. \quad (20)$$

This estimate allows us to evaluate $g(y,\sigma)$ from above:

$$g(y,\sigma) \leq \frac{1}{M_0} \max_{[0,T_1]} \frac{\nu_4(\sigma)}{\sigma} \equiv M_1 > 0, \quad \sigma \in [0,T_1]. \quad (21)$$

Having the estimation of the domain we apply the maximum principle once more:

$$v(y,\sigma) \leq M_2 < \infty, \quad (y,\sigma) \in \overline{Q}_{T_1} \quad (22)$$

and obtain the estimate of $g(y,\sigma)$ from below:

$$g(y,\sigma) \geq \frac{1}{M_2} \min_{[0,T_1]} \frac{\nu_4(\sigma)}{\sigma} \equiv M_3 > 0, \quad \sigma \in [0,T_1]. \quad (23)$$

It follows from the assumptions $(A2)$ that there exists such number $T_2, 0 < T_2 \leq T_1$, that the following estimate is valid:

$$w(y,\sigma) \geq \frac{1}{2} \min_{[0,T_1]} \frac{\nu_2(\sigma) - \nu_1(\sigma)}{\sigma} \equiv M_4 > 0, \quad (y,\sigma) \in \overline{Q}_{T_2}. \qquad (24)$$

From this we derive

$$b(\sigma) \leq M_5 < \infty, \quad \sigma \in [0, T_2]. \qquad (25)$$

Notations. Let introduce the notations: $W(\sigma) \equiv \max_{0 \leq y \leq \sigma} |w(y,\sigma)|,\ b_{\min}(\sigma) \equiv \min_{0 \leq \tau \leq \sigma} b(\tau),\ W_1(\sigma) \equiv 1 + W(\sigma).$

From (16) and (19) we deduce

$$W(\sigma) \leq C_1 + C_2 \int_0^\sigma \frac{d\tau}{\sqrt{\theta(\sigma) - \theta(\tau)}} + C_3 \int_0^\sigma \frac{|p(\tau)| W(\tau) d\tau}{\sqrt{\theta(\sigma) - \theta(\tau)}}, \qquad (26)$$

$$|p(\sigma)| \leq C_4 + C_5 \sigma^{\frac{1-\alpha}{\alpha}} b(\sigma) W(\sigma). \tag{27}$$

From the definition of $\theta(\sigma)$ we have

$$\theta(\sigma) - \theta(\tau) = \int_0^\sigma \frac{\omega^{\frac{1-\alpha}{\alpha}} b(\omega) d\omega}{\alpha g^2(\omega)}.$$

Taking into account this inequality and (27) we put (26) under the form

$$W_1(\sigma) \leq C_6 + \frac{C_7}{\sqrt{b_{\min}(\sigma)}} \int_0^\sigma \frac{W_1^3(\tau) d\tau}{\sqrt{\sigma - \tau}}. \tag{28}$$

To go on, we find from (28)

$$\int_0^\sigma \frac{W_1^3(\tau) d\tau}{\sqrt{\sigma - \tau}} \leq C_8 + C_9 \int_0^\sigma W_1^9(\tau) d\tau. \tag{29}$$

Finally, we arrive to the nonlinear inequality

$$W_1(\sigma) \leq C_{10} + \frac{C_{11}}{\sqrt{b_{\min}(\sigma)}} + \frac{C_{12}}{b_{\min}^2(\sigma)} \int_0^\sigma W_1^9(\tau)d\tau, \quad \sigma \in [0, T_2]. \tag{30}$$

Applying the method of Gronwall's inequality we conclude that there exists such number $T_3, 0 < T_3 \leq T_2$, that the following estimate is valid:

$$W_1(\sigma) \leq C_{10} + \frac{C_{13}}{\sqrt{b_{\min}(\sigma)}} + \frac{C_{14}}{b_{\min}^2(\sigma)} \int_0^\sigma \left(1 + \frac{1}{\sqrt{b_{\min}(\tau)}}\right) \times b_{\min}^{-16}(\tau) \exp\left(C_{14} \int_\tau^\sigma b_{\min}^{-18}(\omega)d\omega\right) d\tau, \quad \sigma \in [0, T_3]. \tag{31}$$

We find from (17):

$$b_{\min}(\sigma) \geq \frac{C_{15}}{W_1(\sigma)}, \quad \sigma \in [0, T_3]. \tag{32}$$

Substituting (31) into (32) we obtain

$$b_{\text{min}}(\sigma) \geq M_6 > 0, \quad \sigma \in [0, T_4], \tag{33}$$

for some number $T_4, 0 < T_4 \leq T_3$. Then from (30) we have

$$W_1(\sigma) \leq M_6 < \infty, \ \sigma \in [0, T_4], \text{ or } |w(y, \sigma)| \leq M_6 < \infty, \ (y, \sigma) \in \overline{Q}_{T_4}. \tag{34}$$

After obtaining the estimates of solutions to the system (14)-(19) we can apply the Schauder fixed-point theorem by the usual way.

Theorem 1.

Suppose that the following assumptions are fulfilled:

(B1) $\mu_i \in C^1([0, T]), \ i = 1, 2, 4; \ \mu_3 \in C([0, T]); \ f \in C^{1,0}(\overline{\Omega}_T);$

(B2) $\mu_i(t) > 0, \ i = 1, 2, 3, \ t \in [0, T], \ \mu_4(t) > 0, \ t \in (0, T], \ \mu_2(t) > \mu_1(t), \ t \in (0, T], \ \mu_2(0) = \mu_1(0); \ f(x, t) \geq 0, \ (x, t) \in \overline{\Omega}_T;$

(B3) there exist the limits $\lim_{t\to 0+}\dfrac{\mu_2(t)-\mu_1(t)}{t^\alpha}>0, \quad \lim_{t\to 0+}\dfrac{\mu_4(t)}{t^\alpha}>0.$

Then it may be indicated such a number $T_4, 0 < T_4 \le T$, which depends on the given data, that there exists a solution to the problem (1)-(4) defined for $x\in[0,t], t\in[0,T_4]$.

5. Uniqueness of solution to the problem (1)-(4)

Theorem 2.

Suppose that the following assumptions are fulfilled:

(B4) $f\in C^{1,0}(\overline{\Omega}_T)$;

(B5) $\mu_i(t)\neq 0,\ i=2,3,\ t\in[0,T],\ \mu_4(t)\neq 0,\ t\in(0,T]$;

(B6) there exists the limit $\lim\limits_{t \to 0+} \dfrac{\mu_4(t)}{t^{\alpha}} \neq 0.$

Then the solution to the problem (1)-(4) is unique.

Proof. Let $g_i(\sigma),\ b_i(\sigma),\ v_i(y,\sigma),\ i = 1, 2,$ be two solutions to the problem (6)-(9). Enter the following notations:

$v \equiv v_1(y,\sigma) - v_2(y,\sigma),\ \ g(\sigma) \equiv g_1(\sigma) - g_2(\sigma),\ \ m_i(\sigma) \equiv \dfrac{b_i(\sigma)}{g_i^2(\sigma)}, i = 1, 2,\ \ m(\sigma) \equiv m_1(\sigma) - m_2(\sigma),\ \ q_i(\sigma) \equiv \dfrac{\sigma g_i'(\sigma)}{g_i(\sigma)},\ \ q(\sigma) \equiv q_1(\sigma) - q_2(\sigma).$

From (6)-(9) we deduce that the functions (v, g, m, q) are a solution of the problem

$$v_\sigma = \frac{1}{\alpha}\alpha^{\frac{1-\alpha}{\alpha}} m_1(\sigma)v_{yy} + \frac{y}{\sigma}q_1(\sigma)v_y + \frac{1}{\alpha}m(\sigma)v_{2yy}(y,\sigma) + \frac{y}{\sigma}q(\sigma)v_{2y}(y,\sigma)$$

$$+\frac{1}{\alpha}\sigma^{\frac{1-\alpha}{\alpha}}(f(yg_1(\sigma),\sigma^{1/\alpha}) - f(yg_2(\sigma),\sigma^{1/\alpha})), \quad (y,\sigma)\in Q_{T_1}, \tag{35}$$

$$v(0,\sigma) = v(\sigma,\sigma) = 0, \quad \sigma \in [0,T_1], \tag{36}$$

$$m_1(\sigma)v_y(0,\sigma) = m(\sigma)v_{2y}(0,\sigma) + \nu_3(\sigma)\left(\frac{1}{g_1(\sigma)} - \frac{1}{g_2(\sigma)}\right), \quad \sigma\in[0,T_1], \tag{37}$$

$$g_1(\sigma)\int_0^\sigma v(y,\sigma)dy = -g(\sigma)\int_0^\sigma v_2(y,\sigma)dy, \quad \sigma\in[0,T_1]. \tag{38}$$

The problem (35)-(38) is equivalent to the following system of equations:

$$v(y,\sigma) = \int_0^\sigma d\tau \int_0^\tau G(y,\sigma,\xi,\tau)\left(\frac{\xi}{\tau}q_1(\tau)w(\xi,\tau) + \frac{1}{\alpha}m(\tau)v_{2\xi\xi}(\xi,\tau) + \frac{\xi}{\tau}q(\tau)\right.$$

$$\times v_{2\xi}(\xi,\tau) + \frac{1}{\alpha}\tau^{\frac{1-\alpha}{\alpha}}(f(\xi g_1(\tau),\tau^{1/\alpha}) - f(\xi g_2(\tau),\tau^{1/\alpha}))\Big)d\xi, \quad (y,\sigma) \in \overline{Q}_{T_1}, \tag{39}$$

$$w(y,\sigma) = \int_0^\sigma d\tau \int_0^\tau G_y(y,\sigma,\xi,\tau)\Big(\frac{\xi}{\tau}q_1(\tau)w(\xi,\tau) + \frac{1}{\alpha}m(\tau)v_{2\xi\xi}(\xi,\tau) + \frac{\xi}{\tau}q(\tau)$$

$$\times v_{2\xi}(\xi,\tau) + \frac{1}{\alpha}\tau^{\frac{1-\alpha}{\alpha}}(f(\xi g_1(\tau),\tau^{1/\alpha}) - f(\xi g_2(\tau),\tau^{1/\alpha}))\Big)d\xi, \quad (y,\sigma) \in \overline{Q}_{T_1}, \tag{40}$$

$$m(\sigma) = \frac{1}{v_{2y}(0,\sigma)}\left(m_1(\sigma)w(0,\sigma) - \nu_3(\sigma)\left(\frac{1}{g_1(\sigma)} - \frac{1}{g_2(\sigma)}\right)\right), \quad \sigma \in [0,T_1], \tag{41}$$

$$g(\sigma) = \frac{-g_1(\sigma)\int_0^\sigma v(y,\sigma)dy}{\int_0^\sigma v_2(y,\sigma)dy}, \quad \sigma \in [0,T_1], \tag{42}$$

$$q(\sigma)=\frac{1}{\nu_2(\sigma)}\left(\nu_4'(\sigma)-\frac{1}{\alpha}\sigma^{\frac{1-\alpha}{\alpha}}\nu_3(\sigma)\right)\left(\frac{1}{g_1(\sigma)}-\frac{1}{g_2(\sigma)}\right)-\frac{\sigma^{\frac{1-\alpha}{\alpha}}}{\alpha\nu_2(\sigma)}$$
$$\times\left(\frac{m(\sigma)w_1(\sigma,\sigma)}{g_1(\sigma)}+\frac{m_2(\sigma)w(\sigma,\sigma)}{g_2(\sigma)}+m_2(\sigma)w_1(\sigma,\sigma)\left(\frac{1}{g_1(\sigma)}-\frac{1}{g_2(\sigma)}\right)\right)$$
$$-\frac{\sigma^{\frac{1-\alpha}{\alpha}}}{\alpha\nu_2(\sigma)}\int\limits_0^\sigma(f(yg_1(\sigma),\sigma^{1/\alpha})-f(yg_2(\sigma),\sigma^{1/\alpha}))dy,\quad \sigma\in[0,T_1],\qquad (43)$$

where $G(y,\sigma,\xi,\tau)$ is the Green function for the equation

$$v_\sigma=\frac{1}{\alpha}\alpha^{\frac{1-\alpha}{\alpha}}m_1(\sigma)v_{yy}.$$

Taking into account the equalities

$$f(yg_1(\sigma),\sigma^{1/\alpha})-f(yg_2(\sigma),\sigma^{1/\alpha})=yg(\sigma)\int\limits_0^1\frac{\partial}{\partial z}f(z,\sigma^{1/\alpha})\Big|_{z=yg_2(\sigma)+y\theta g(\sigma)}d\theta,$$
$$\frac{1}{g_1(\sigma)}-\frac{1}{g_2(\sigma)}=-\frac{g(\sigma)}{g_1(\sigma)g_2(\sigma)},\qquad (44)$$

we deduce from (39)-(44)

$$v(y,\sigma) = w(y,\sigma) = 0,\ (y,\sigma) \in \overline{Q}_{T_1}, \quad b(\sigma) = g(\sigma) = q(\sigma) = 0,\ \sigma \in [0, T_1].$$

$\square$