A positivity principle for parabolic integro-differential equations and final overdetermination

Jaan Janno and *Kairi Kasemets*

We consider the following parabolic problem (direct problem):

$$\beta u_t = Au - m * Au + \chi \quad \text{in} \quad Q = \Omega \times (0,T), \tag{1}$$

$$\begin{aligned} &u = u_0 \ \text{ in } \Omega \times \{0\}\,, \\ &Bu = b \ \text{ in } S = \Gamma \times (0,T) \end{aligned} \tag{2}$$

where $\Omega \subset \mathbb{R}^n$ – bounded, open with sufficiently smooth boundary Γ,

$$\beta(x),\ \chi(x,t),\ u_0(x),\ b(x,t) \ \text{ and } \ m(t) \ \ - \text{ given functions}$$

and either

$$Bu = u \qquad (\textit{case}\ \mathrm{I}) \tag{3}$$

or

$$\begin{aligned} Bu(x,t) = {} & \omega(x)\cdot\nabla_x u(x,t) \\ & - \textstyle\int_0^t m(t-\tau)\,\omega(x)\cdot\nabla_x u(x,\tau)d\tau \qquad (\textit{case}\ \mathrm{II}) \end{aligned} \tag{4}$$

with $\omega(x)\cdot N(x) > 0$, $N(x)$ - outer normal of Γ at x. Moreover,

$$A = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} a_j(x)\frac{\partial}{\partial x_j} + a(x,t) \tag{5}$$

and $*$ stands for the time convolution, i.e.

$$v * w \ = \int_0^{\cdot} v(\cdot - \tau)w(\tau)d\tau.$$

We assume the relations

$$-\sum_{i,j=1}^{N} a_{ij}\,\lambda_i\lambda_j \geq \epsilon|\lambda|^2 \quad \text{for any} \;\; \lambda \in \mathbb{R}^n \qquad \text{with some} \quad \epsilon \in (0,\infty) \tag{6}$$

and

$$\beta \geq \beta_0 > 0 \quad \text{with some} \quad \beta_0 \in (0,\infty) \tag{7}$$

be valid in Ω.

Inverse problems:

IP1: Let the source term be of the following form:

$$\chi(x,t) \; = \; z(x)\phi(x,t) + \chi_0(x,t). \tag{8}$$

Given $m, \beta, a_{ij}, a_j, a, u_0, b, \phi, \chi_0$ and a function $u_T(x)$, $x \in \Omega$,

find z and u so that the direct problem (1), (10), the relation (8) and the final condition

$$u \; = \; u_T \;\; \text{in} \; \Omega \times \{T\} \tag{9}$$

hold.

IP2: Let $a_t = 0$. Given $m, \beta, a_{ij}, a_j, u_0, b, \chi$ and a function $u_T(x)$, $x \in \Omega$,

find a and u so that the direct problem (1), (10) and final condition (9) hold.

IP3: Given $m, a_{ij}, a_j, a, u_0, b, \chi$ and a function $u_T(x)$, $x \in \Omega$,

find β and u so that the direct problem (1), (10) and final conditon (9) hold.

Let U be a finite-dimensional manifold and $f, g \in L^1(U)$. We write

$$f \geq g \ \text{ in } U \quad \text{if } f(x) \geq g(x) \text{ a.e. } x \in U,$$

$$f > g \ \text{ in } U \quad \text{if } \forall U_1\colon \overline{U}_1 \subseteq U \ \exists \varepsilon_{U_1} \in \mathbb{R}, \varepsilon_{U_1} > 0\colon \ f \geq g + \varepsilon_{U_1} \ \text{ in } U_1.$$

It is not difficult to prove that

$$f \geq g, f \neq g \ \text{ in } U \quad \Rightarrow \quad \exists U_2 \subseteq U : \operatorname{meas} U_2 \neq 0, \ f > g \text{ in } U_2.$$

Positivity principle. Sign inertia.

$$\beta u_t = Au - m * Au + \chi \quad \text{in} \quad Q = \Omega \times (0,T),$$
$$u = u_0 \ \text{ in } \Omega \times \{0\}\,,$$
$$Bu = b \ \text{ in } S = \Gamma \times (0,T)$$

We introduce the resolvent kernel k satisfying the equation

$$k(t) - \int_0^t m(t-\tau)k(\tau)d\tau = m(t), \ \ t \in (0,T). \tag{10}$$

Denote

$$f \ = \ \chi + k * \chi, \tag{11}$$

$$\begin{aligned} &g = b \quad \text{in case I}\,, \\ &g = b + k * b \quad \text{in case II}. \end{aligned} \tag{12}$$

Theorem 1. *Assume* (6), (7), $\beta, a_{ij}, a_j \in C(\overline{\Omega})$, $a \in C(\overline{Q})$ *and*

$$k \in W_1^1(0,T)\,, \quad k \geq 0\,, \quad k' \leq 0. \tag{13}$$

Let $u \in W_p^{2,1}(Q)$ *with some* $p \in (1,\infty)$ *solve the direct problem* (1), (10) *and*

$$u_0 \geq 0, \ \ g \geq 0, \ \ f \geq 0.$$

Then the following assertions are valid:

(i) $u \geq 0$;

(ii) *if, in addition,* $\beta, a_{ij}, a_j \in C^l(\Omega)$, $a \in C^{l,\frac{l}{2}}(Q)$ *with some* $l \in (0,1)$ *and either* $f \neq 0$ *or* $g \neq 0$, *then*

$u(\cdot,T) > 0$ *in* Ω *in case* I *and* $u(\cdot,T) > 0$ *in* $\overline{\Omega}$ *in case* II.

Sufficient conditions for

$$k \in W_1^1(0,T)\,, \quad k \geq 0\,, \quad k' \leq 0.$$

in terms of original kernel m are

$$m \in W_1^1(0,T), \;\; m \geq 0\,, \quad m'(t) \leq -m(0)m(t).$$

For instance, the exponential kernels $m(t) = \sum_{i=1}^{N} \alpha_i e^{-\beta_i t}$ satisfy these conditions provided

$$\beta_i \geq \alpha_i \geq 0.$$

Example for the assumption

$$f = \chi + k * \chi \geq 0.$$

(χ is the original source term.)

Let $k \geq 0$, $k \neq 0$.

Choosing

$$\chi = 1 \text{ in } \Omega \times (0, T-\delta) \quad \text{and} \quad \chi = -\epsilon < 0 \text{ in } \Omega \times (T-\delta, T),$$

where the numbers $\epsilon > 0$ and $\delta > 0$ are sufficiently small, so that

$$\int_0^{T-\delta} k(t-\tau)d\tau \geq \epsilon \left(1 + \int_{T-\delta}^{t} k(t-\tau)d\tau\right) \quad \text{for any} \quad t \in (T-\delta, T)$$

then we have

$$f = \chi + k * \chi \geq 0.$$

Sketch of proof of Thm 1 *assertion* (i).

$$\beta u_t = Au - m * Au + \chi \quad \text{in} \quad Q = \Omega \times (0,T),$$

$$u = u_0 \ \text{ in } \Omega \times \{0\}\,,$$

$$Bu = b \ \text{ in } S = \Gamma \times (0,T)$$

Apply operator $I+k*$ to the parabolic equation and the boundary condition in case II:

$$\beta u_t + k * u_t = Au + \underbrace{\chi + k * \chi}_{f} \quad \text{in} \quad Q,$$

$$u = u_0 \ \text{ in } \Omega \times \{0\}\,,$$

$$B_1 u = g \ \text{ in } S\,.$$

Here $B_1 = I$ in case I, $B_1 = \omega \cdot \nabla$ in case II.

Integrate by parts in the term $k * u_t$:

$$\beta u_t + k' * u + k(0)u - ku_0 = Au + f \quad \text{in} \quad Q,$$

$$u = u_0 \ \text{ in } \Omega \times \{0\}\,,$$

$$B_1 u = g \ \text{ in } S$$

Transform the terms with k to the right-hand side:

$$\beta u_t = (A - k(0))u - k' * u + ku_0 + f \quad \text{in} \quad Q,$$

$$u = u_0 \ \text{ in } \Omega \times \{0\}\,,$$

$$B_1 u = g \ \text{ in } S$$

We represent u as the limit

$$u = \lim_{n \to \infty} u^n$$

where $u^0 = 0$ and u^n, $n = 1, 2, \ldots$ solve the problems

$$\beta u_t^n = (A - k(0))u^n - k' * u^{n-1} + ku_0 + f \quad \text{in} \quad Q,$$

$$u^n = u_0 \ \text{ in } \Omega \times \{0\}\,,$$

$$B_1 u^n = g \ \text{ in } S$$

Assuming $f \geq 0$, $u_0 \geq 0$, $g \geq 0$, $k \geq 0$, $k' \leq 0$ and $u^{n-1} \geq 0$,
well-known extremum principle for parabolic equations implies $u^n \geq 0$.

Thus, we have the implication

$$u^{n-1} \geq 0 \quad \Rightarrow \quad u^n \geq 0.$$

Since $u^0 = 0$, we obtain $u^n \geq 0$ for $n = 1, 2, \ldots$.

This proves

$$u = \lim u^n \geq 0.$$

Results for IP1

IP1 is equivalent to the following problem for (z, u):

$$\begin{aligned} &\beta(u_t + k * u_t) = Au + zr + f_0 \ \text{ in } Q\,, \\ &u = u_0 \ \text{ in } \Omega \times \{0\}\,, \\ &B_1 u = g \ \text{ in } S, \end{aligned} \tag{14}$$

$$u \;=\; u_T \ \text{ in } \Omega \times \{T\}, \tag{15}$$

where $B_1 = I$ in case I, $B_1 = \omega \cdot \nabla$ in case II, as before, and

$$r \;=\; \phi + k * \phi\,, \quad f_0 = \chi_0 + k * \chi_0. \tag{16}$$

Theorem 2. *Let* (6), (7) hold, $k \in W_1^1(0,T)$, $k \geq 0$, $k' \leq 0$ and $\beta, a_{ij}, a_j \in C^l(\Omega)$, $a \in C^{l,\frac{l}{2}}(Q)$, $a_t \in L^p(Q)$ with some $l \in (0,1), p \in (1,\infty)$. Moreover, let $a_t \geq 0$, $r \in C^{l,\frac{l}{2}}(Q), r_t \in L^p(Q)$,

$$r \geq 0\,, \quad r_t + k * r_t - \theta r \geq 0 \tag{17}$$

$$\begin{aligned} &\text{and for any } \ U \subseteq \Omega, \ \text{meas}\, U > 0, \ \text{ there holds} \\ &r_t + k * r_t - \theta r \neq 0 \ \text{ in } U \times (0,T). \end{aligned} \tag{18}$$

Here

$$\theta = \sup_{x \in \Omega} \frac{a(x,T)}{\beta(x)}.$$

If $(z,u) \in C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ solves (14), (15) and $f_0, u_0, g, u_T = 0$ then $z = 0$, $u = 0$.

To deal with the existence and stability we have to impose additional assumptions on r:

$$\begin{aligned}
&r \geq \delta \ \text{ in } \overline{\Omega} \times (T-\delta, T) \quad \text{with some } \delta \in (0, \tfrac{T}{2}) \text{ and} \\
&\text{either } \ r \geq \delta \ \text{ in } \overline{\Omega} \times (0,\delta) \ \ (\textit{case } (1)) \ \text{ or } \ r = 0 \ \text{ in } \overline{\Omega} \times (0,\delta) \ \ (\textit{case } (2)).
\end{aligned} \tag{19}$$

In case I & (1) it is possible to reformulate IP1 so that the unknown z is zero at the boundary Γ.

Theorem 3. *Let* (6), (7) *hold and* $\beta, a_{ij}, a_j \in C^l(\Omega)$, $a \in C^{l,\frac{l}{2}}(Q)$, $a_t \in L^p(Q)$, *with some* $l \in (0,1), p \in (1,\infty)$ *and* $a_t \geq 0$.
In addition, let $f_0 \in C^{l,\frac{l}{2}}(Q)$, $u_0 \in C^{2+l}(\Omega)$, $g \in C^{2+l-\nu,1+\frac{l}{2}-\frac{\nu}{2}}(S)$, $u_T \in C^{2+l}(\Omega)$ *and the consistency conditions*

$$\begin{aligned}
&u_0 = g\,, \quad \beta g_t = A u_0 + f_0 \quad \textit{in case } \mathrm{I} \quad \textit{in } \Gamma \times \{0\}, \\
&\omega \cdot \nabla_x u_0 = g \quad \textit{in case } \mathrm{II} \quad \textit{in } \Gamma \times \{0\} \\
&u_T = g \ \ \textit{in case } \mathrm{I}, \quad \omega \cdot \nabla_x u_T = g \quad \textit{in case } \mathrm{II} \quad \textit{in } \Gamma \times \{T\}
\end{aligned} \tag{20}$$

be satisfied.

Moreover, let r *satisfy the assumptions listed in Theorem* 2 *and* (19) *and* $k \in W^1_{\frac{2}{2-l}}(0,T)$, $k \geq 0$, $k' \leq 0$.

In case I & (1) *we assume* $Au_T = 0$ *in* $\Gamma \times \{T\}$, *too.*

Then the inverse problem (14), (15) *has a unique solution* (z,u) *in the space* $C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ *and in case* I & (1) *there holds* $z = 0$ *in* Γ. *The solution satisfies the following stability estimate:*

$$\begin{aligned}
&\|z\|_l + \|u\|_{2+l,1+\frac{l}{2}} \\
&\quad \leq \Lambda(\beta, a_{ij}, a_j, a, k, r) \left\{ \|f_0\|_{l,\frac{l}{2}} + \|u_0\|_{2+l} + \|g\|_{2+l-\nu,1+\frac{l}{2}-\frac{\nu}{2}} + \|u_T\|_{2+l} \right\}
\end{aligned} \tag{21}$$

with some constant Λ *depending on the quantities shown in brackets.*

Here ν is the order of the boundary operator B, i.e $\nu = 0$ in case I and $\nu = 1$ in case II.

Results for IP2

IP2 is equivalent to the following problem for (a, u):

$$\begin{aligned} \beta(u_t + k * u_t) &= A_0 u + au + f \ \text{ in } Q\,, \\ u = u_0 \ \text{ in } \Omega \times \{0\}\,, &\quad B_1 u = g \ \text{ in } S\,, \end{aligned} \tag{22}$$

$$u = u_T \ \text{ in } \Omega \times \{T\}\,, \tag{23}$$

where f, B_1 and g are given as before and

$A_0 u = \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{j=1}^{n} a_j u_{x_j}$.

Let us define the following set of the coefficients a that depends on $\theta \in \mathbb{R}$:

$$\mathcal{A}^l_{\beta,\theta} \ = \ \{a \in C^l(\Omega) \ : \ \sup_{x\in\Omega} \frac{a(x)}{\beta(x)} \leq \theta\}.$$

Theorem 4. *Let (6), (7) hold, $\beta, a_{ij}, a_j \in C^l(\Omega)$ with some $l \in (0,1)$ and $\theta \in \mathbb{R}$. Then the following assertions are valid.*

(i) *If $k \in W_1^1(0,T)$, $k \geq 0$, $k' \leq 0$ and the problem (22), (23) has the solutions $(a_1, u_1) \in C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$, $(a_2, u_2) \in \mathcal{A}^l_{\beta,\theta} \times C^{2+l,1+\frac{l}{2}}(Q)$, where $u = u_1$ satisfies the conditions*

$$\begin{aligned} & u \geq 0\,, \quad u_t + k * u_t - \theta u \geq 0\,, \\ & \textit{for any } U \subseteq \Omega, \ \mathrm{meas}\, U > 0 \\ & \textit{there holds } \ u_t + k * u_t - \theta u \neq 0 \ \textit{ in } U \times (0,T), \end{aligned} \tag{24}$$

then $a_1 = a_2$ and $u_1 = u_2$.

(ii) *If $k \in W^1_{\frac{2}{2-l}}(0,T)$, $k \geq 0$, $k' \leq 0$ and (22), (23) has a solution $(a,u) \in \mathcal{A}^l_{\beta,\theta} \times C^{2+l,1+\frac{l}{2}}(Q)$ such that u fulfills (24),*

$$\begin{aligned} & u \geq \delta \ \textit{ in } \overline{\Omega} \times (T-\delta, T) \quad \text{and} \\ & u = 0 \ \textit{ in } \overline{\Omega} \times (0,\delta) \ \textit{with some } \delta \in (0, \tfrac{T}{2}), \end{aligned} \tag{25}$$

then for any $\widetilde{f}, \widetilde{u}_0, \widetilde{g}, \widetilde{u}_T$ such that

$$D := \|\widetilde{f} - f\|_{l,\frac{l}{2}} + \|\widetilde{u}_0 - u_0\|_{2+l} + \|\widetilde{g} - g\|_{2+l-\nu,1+\frac{l}{2}-\frac{\nu}{2}} + \|\widetilde{u}_T - u_T\|_{2+l} < \frac{1}{2\lambda^2},$$

$$\lambda = \Lambda(\beta, a_{ij}, a_j, a, k, u),$$

where

$$\begin{aligned} & \widetilde{u}_0 = \widetilde{g}\,, \ \ \beta\widetilde{g}_t = (A_0 + a)\widetilde{u}_0 + \widetilde{f} \quad \textit{in case } \mathrm{I} \quad \textit{in } \Gamma \times \{0\}, \\ & \omega \cdot \nabla_x \widetilde{u}_0 = \widetilde{g} \quad \textit{in case } \mathrm{II} \quad \textit{in } \Gamma \times \{0\}, \\ & \widetilde{u}_T = \widetilde{g} \quad \textit{in case } \mathrm{I}, \quad \omega \cdot \nabla_x \widetilde{u}_T = \widetilde{g} \quad \textit{in case } \mathrm{II} \quad \textit{in } \Gamma \times \{T\}, \end{aligned}$$

the problem (22), (23) with f_0, u_0, g, u_T replaced by $\widetilde{f}_0, \widetilde{u}_0, \widetilde{g}, \widetilde{u}_T$, has a unique solution $(\widetilde{a}, \widetilde{u})$ in the ball

$$\mathcal{U} \ = \ \Big\{ (\widetilde{a}, \widetilde{u}) \ : \ \|\widetilde{a} - a\|_l + \|\widetilde{u} - u\|_{2+l,1+\frac{l}{2}} \leq \frac{1}{\lambda}\Big(1 - \sqrt{1 - 2\lambda^2 D}\Big) \Big\}.$$

(iii) *If* $k \in W_1^1(0,T)$, $k \ge 0$, $k' \le 0$, $a \in \mathcal{A}^l_{\beta,\theta}$, $u_0 \in C^{2+l}(\Omega)$, $A_0 u_0 \in W_p^{2-\frac{2}{p}}(\Omega)$, $f \in C^{l,\frac{l}{2}}(Q)$, $f_t \in L^p(Q)$, $g \in C^{2+l-\nu,1+\frac{l}{2}-\frac{\nu}{2}}(S)$, $g_t \in W_p^{2-\nu-\frac{1}{p},1-\frac{\nu}{2}-\frac{1}{2p}}(S)$ *with some* $p \in (1,\infty)$,

$$\begin{aligned}
&u_0 = g,\ \beta g_t + A_0 u_0 + a u_0 = f \quad \textit{in case } \mathrm{I} \quad \textit{in } \Gamma \times \{0\},\\
&\omega \cdot \nabla_x u_0 = g \quad \textit{in case } \mathrm{II} \quad \textit{in } \Gamma \times \{0\},\\
&u_0 \ge 0,\ f \ge 0,\ g \ge 0,\ f_t + k * f_t - \theta f \ge 0,\ g_t + k * g_t - \theta g \ge 0,\\
&f_t + k * f_t - \theta f \ne 0 \quad \textit{or} \quad g_t + k * g_t - \theta g \ne 0
\end{aligned}$$

and

$$(\theta\beta - a)u_0 \le A_0 u_0 + f(\cdot,0)$$

then the solution u *of the direct problem* (22) *belongs to* $C^{2+l,1+\frac{l}{2}}(Q)$ *and satisfies* (24).

If, in addition, $f(\cdot,t) = 0$ *and* $g(\cdot,t) = 0$ *for* $t \in (0,\delta_0)$ *with some* $\delta_0 \in (0,\frac{T}{2})$, $u_0 = 0$ *and* $g > 0$ *in* $\Gamma \times \{T\}$ *in case I, then* u *satisfies* (25), *too.*

We remark that in case $u_0 = 0$ the assumptions of (iii) do not contain the unknown a, except for the condition $a \in \mathcal{A}^l_{\beta,\theta}$.

Results for IP3

IP3 is equivalent to the following problem for (β, u):

$$
\begin{aligned}
&\beta(u_t + k * u_t) = Au + f \ \text{ in } Q\,, \\
&u = u_0 \ \text{ in } \Omega \times \{0\}\,, \ \ B_1 u = g \ \text{ in } S\,,
\end{aligned} \tag{26}
$$

$$
u = u_T \ \text{ in } \Omega \times \{T\}\,. \tag{27}
$$

Let us introduce the following set for the coefficients β that depends on $\beta_0 > 0$:

$$
\mathcal{B}^l_{\beta_0} \ = \ \{\beta \in C^l(\Omega) \ : \ \inf_{x\in\Omega} \beta(x) \geq \beta_0\}
$$

and define $\theta_{\beta_0} = \max\big\{0; \frac{1}{\beta_0} \sup_{x\in\Omega} a(x,T)\big\}$.

Theorem 5. *Let* (6) *hold,* $a_{ij}, a_j \in C^l(\Omega)$*,* $a \in C^{l,\frac{l}{2}}(Q)$*,* $a_t \in L^p(Q)$ *with some* $l \in (0,1)$*,* $p \in (1,\infty)$*,* $a_t \geq 0$ *and* $\beta_0 > 0$*. Then the following assertions are valid.*

(i) *If* $k \in W^1_1(0,T)$*,* $k \geq 0$*,* $k' \leq 0$ *and the problem* (26)*,* (27) *has the solutions* $(\beta_1, u_1) \in C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$*,* $(\beta_2, u_2) \in \mathcal{B}^l_{\beta_0} \times C^{2+l,1+\frac{l}{2}}(Q)$ *where* $u = u_1$ *satisfies the conditions*

$$
\begin{aligned}
&u_{tt} \in L^p(\Omega) \ \ \textit{and} \\
&u_t + k * u_t \geq 0\,, \\
&\hat{u} := (u_t + k * u_t)_t + k * (u_t + k * u_t)_t - \theta_{\beta_0}(u_t + k * u_t) \geq 0\,, \\
&\textit{for any } \ U \subseteq \Omega, \ \text{meas}\, U > 0 \ \ \textit{there holds } \ \hat{u} \neq 0 \ \ \textit{in } U \times (0,T),
\end{aligned} \tag{28}
$$

then $\beta_1 = \beta_2$ *and* $u_1 = u_2$*.*

(ii) *If* $k \in W^1_{\frac{2}{2-l}}(0,T)$, $k \geq 0$, $k' \leq 0$ *and the problem* (26)*,* (27) *has a solution* $(\beta, u) \in \mathcal{B}^l_{\beta_0} \times C^{2+l,1+\frac{l}{2}}(Q)$ *such that* u *fulfills* (28)*,*

$$\begin{aligned} &u_t + k * u_t \geq \delta \ \ \textit{in } \overline{\Omega} \times (T-\delta, T) \quad \text{and} \\ &u_t = 0 \ \ \textit{in } \overline{\Omega} \times (0,\delta) \ \textit{with some } \delta \in (0, \tfrac{T}{2}), \end{aligned} \tag{29}$$

then for any $\widetilde{f}, \widetilde{u}_0, \widetilde{g}, \widetilde{u}_T$ *such that*

$$D < \frac{1}{2\bar{\lambda}^2(1+\|k\|)}\,, \quad \bar{\lambda} = \Lambda(\beta, a_{ij}, a_j, a, k, u_t + k * u_t), \ \|k\| = \|k\|_{C[0,T]},$$

with D *defined in Theorem* 4*,*

$$\begin{aligned} &\widetilde{u}_0 = \widetilde{g}\,, \ \ \beta\widetilde{g}_t = A\widetilde{u}_0 + \widetilde{f} \quad \textit{in case } \mathrm{I} \quad \textit{in } \Gamma \times \{0\}, \\ &\omega \cdot \nabla_x \widetilde{u}_0 = \widetilde{g} \quad \textit{in case } \mathrm{II} \quad \textit{in } \Gamma \times \{0\}, \\ &\widetilde{u}_T = \widetilde{g} \quad \textit{in case } \mathrm{I}, \quad \omega \cdot \nabla_x \widetilde{u}_T = \widetilde{g} \quad \textit{in case } \mathrm{II} \quad \textit{in } \Gamma \times \{T\}, \end{aligned}$$

the problem (26)*,* (27) *with* f_0, u_0, g, u_T *replaced by* $\widetilde{f}_0, \widetilde{u}_0, \widetilde{g}, \widetilde{u}_T$*, has a unique solution* $(\widetilde{\beta}, \widetilde{u})$ *in the ball*

$$\bar{\mathcal{U}} = \Big\{ (\widetilde{\beta}, \widetilde{u}) \, : \, \|\widetilde{\beta} - \beta\|_l + \|\widetilde{u} - u\|_{2+l,1+\frac{l}{2}} \leq \frac{1}{\bar{\lambda}(1+\|k\|)} \left(1 - \sqrt{1 - 2\bar{\lambda}^2(1+\|k\|)D} \right) \Big\}.$$

(iii) *If* $k \in W_1^1(0,T)$, $k \geq 0$, $k' \leq 0$, $\beta \in \mathcal{B}^l_{\beta_0}$, $a_t = 0$, $u_0 \in C^{2+l}(\Omega)$, $A(0)u_0 \in W_p^{2-\frac{2}{p}}(\Omega)$, $f \in C^{l,\frac{l}{2}}(Q)$, $f_t, f_{tt} \in L^p(Q)$, $f_t(\cdot,0) \in W_p^{2-\frac{2}{p}}(\Omega)$, $g \in C^{2+l-\nu,1+\frac{l}{2}-\frac{\nu}{2}}(S)$, $g_t, g_{tt} \in W_p^{2-\nu-\frac{1}{p},1-\frac{\nu}{2}-\frac{1}{2p}}(S)$,

$$\begin{aligned}
&r_f := f_t + k * f_t \geq 0,\ r_g := g_t + k * g_t \geq 0,\\
&\hat{r}_f := r_{f,t} + k * r_{f,t} - \theta_{\beta_0} r_f \geq 0,\\
&\hat{r}_g := r_{g,t} + k * r_{g,t} - \theta_{\beta_0} r_g \geq 0,\\
&\hat{r}_f \neq 0 \ \textit{or}\ \hat{r}_g \neq 0,
\end{aligned}$$

$$\begin{aligned}
&u_0 = g,\ \beta g_t + A_0 u_0 + a u_0 = f \quad \textit{in case}\ \mathrm{I} \quad \textit{in}\ \Gamma \times \{0\},\\
&\omega \cdot \nabla_x u_0 = g \quad \textit{in case}\ \mathrm{II} \quad \textit{in}\ \Gamma \times \{0\}
\end{aligned}$$

and the relations

$$\begin{aligned}
&\frac{1}{\beta}(A(0)u_0 + f(\cdot,0)) \in W_p^2(\Omega),\ A\Big[\frac{1}{\beta}(A(0)u_0 + f(\cdot,0))\Big] \in W_p^{2-\frac{2}{p}}(\Omega),\\
&A(0)u_0 + f(\cdot,0) \geq 0,\\
&A\Big[\frac{1}{\beta}(A(0)u_0 + f(\cdot,0))\Big] - \theta_{\beta_0} A(0)u_0 + f_t(\cdot,0) - \theta_{\beta_0} f(\cdot,0) \geq 0
\end{aligned}$$

hold, then the solution u of the direct problem (26) *belongs to* $C^{2+l,1+\frac{l}{2}}(Q)$ *and satisfies* (28).

If, in addition,

$$\begin{aligned}
&f_t(\cdot,t) = 0,\ g_t(\cdot,t) = 0 \ \textit{for}\ t \in (0,\delta_0)\ \textit{with some}\ \delta_0 \in (0,\tfrac{T}{2}),\\
&A(0)u_0 + f(\cdot,0) = 0
\end{aligned}$$

and $r_g > 0$ in $\Gamma \times \{T\}$ in case I, then u satisfies (29)*, too.*

We remark that in case $u_0 = 0$ and $f(\cdot,0) = 0$ then the assumptions of (iii) do not contain the unknown β, except for $\beta \in \mathcal{B}^l_{\beta_0}$.