Some uniqueness results for parameter identification in nonlinear hyperbolic PDEs

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Overview

- Motivation: reversibly nonlinear material behaviour in piezoelectricity
- Identifiability by reformulation as a Volterra integral equation
- Identifiability by closeness to an identifiable problem
- Identifiability by ε-expansion

Piezoelectric Transducers

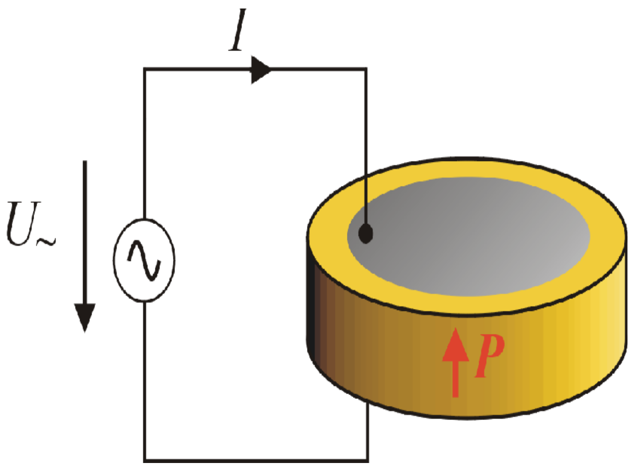
Direct effect: apply mechanical force $\longrightarrow$ measure electric voltage

Indirect effect: impress electric voltage $\longrightarrow$ observe mechanical displacement

Application Areas:

- Ultrasound (medical imaging & therapy)
- Force- and acceleration Sensors
- Actor injection valves (common–rail Diesel engines)
- SAW (surface-acoustic-wave) sensors
- . . .

Piezoelectric PDEs:



$$\begin{array}{rcll} \rho\frac{\partial^2\vec{d}}{\partial t^2} - DIV\left(\mathbf{c}^E DIV^T\vec{d} + \mathbf{e}^T\text{grad}\phi\right) & = & 0 & \text{in } \Omega \\ -\text{div}\left(\mathbf{e}DIV^T\vec{d} - \varepsilon^S\text{grad}\phi\right) & = & 0 & \text{in } \Omega \end{array}$$

Boundary conditions:

$$\begin{array}{rcll} N^T\sigma & = & 0 & \text{on } \partial\Omega \\ \phi & = & 0 & \text{on } \Gamma_g \\ \phi & = & \phi^e & \text{on } \Gamma_e \\ \vec{D}\cdot\vec{n} & = & 0 & \text{on } \Gamma \end{array}$$

Γ_e ... loaded electrode Γ_g ... grounded electrode

$\Gamma = \partial\Omega \setminus (\Gamma_g \cup \Gamma_e)$ ϕ^e ... impressend voltage

Well-posedness:

[Miara '01], [Akamatsu & Nakamura '02], [Sändig& Geis & Mishuris '04], [Nicaise & Mercier, '06], [B.K. & Lahmer & Mohr, '06]

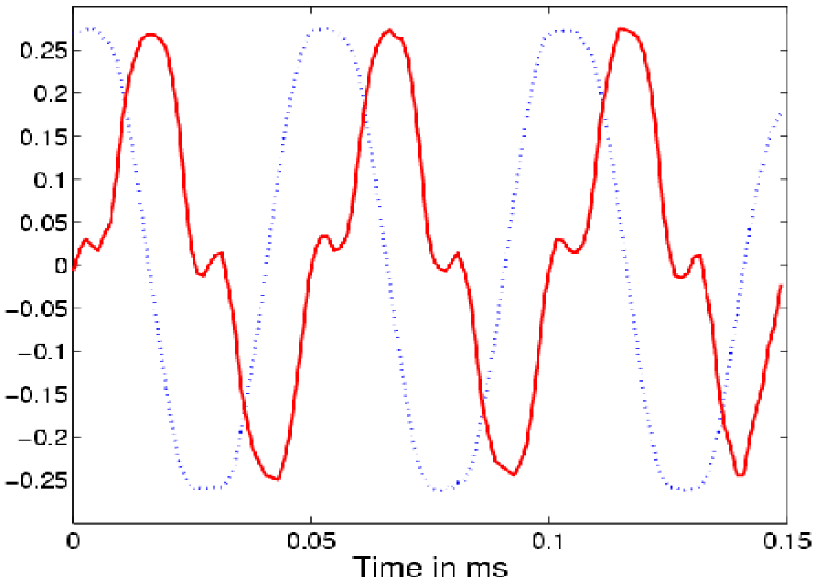
Simulation of piezoelectric transducers requires knowledge of material tensors $\mathbf{c}^E, \mathbf{e}, \varepsilon^S$

Nonlinear Material Behaviour: Higher Harmonics

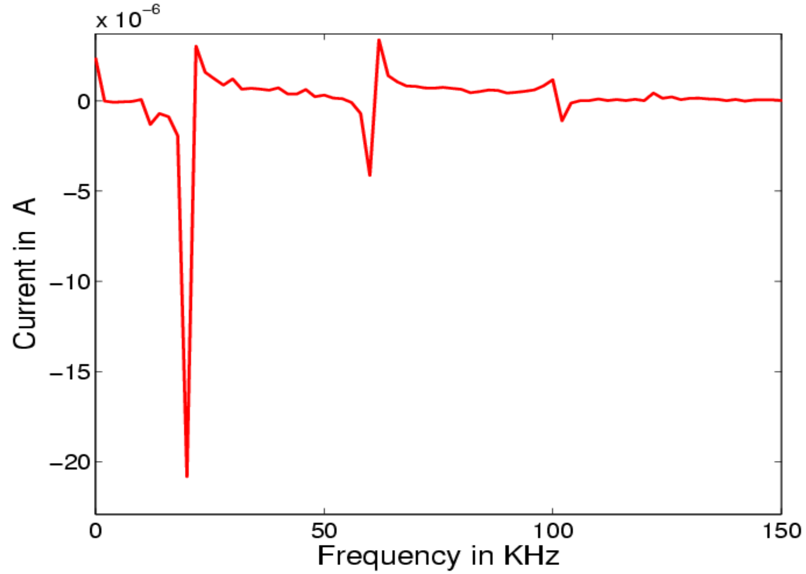
Current response for harmonic voltage excitation (electric field $E \sim 200KV/m$):

— current measurement

$\cdots$ voltage excitation (scaled)



Fourier transformed:



Nonlinear Piezoelectric PDEs

Large excitations (actuator applications):

$$\begin{aligned} \rho\frac{\partial^2\vec{d}}{\partial t^2} - DIV\Big(\mathbf{c}^E(S)DIV^T\vec{d} + \mathbf{e}(S,E)^T\mathrm{grad}\phi\Big) &= 0 \\ -\mathrm{div}\Big(\mathbf{e}(S,E)DIV^T\vec{d} - \varepsilon^S(E)\mathrm{grad}\phi\Big) &= 0 \end{aligned}$$

$S = |DIV^T\vec{d}| \qquad E = |\mathrm{grad}\phi|$

Identification of the (typically smooth) curves $\mathbf{c}^E$, $\mathbf{e}$, ε^S is an infinite dimensional (unstable) problem.

Appropriate measurement setup

$\leadsto$ elimination of ϕ, nonlinearity of only one curve, reduction to one space dimension

Identifiability: A Model Problem

PDE:	$u_{tt} - (c(u_x))_x = 0 \quad$ in $(0,T) \times (0,1)$
boundary conditions:	$u(\cdot,0) = 0 \qquad c(u_x(\cdot,1)) = g$
initial conditions:	$u(0,\cdot) = u_0 \qquad u_t(0,\cdots) = u_1$
measurements:	$m = u(\cdot,1)$
searched for parameter curve:	$\lambda \to c(\lambda)$

Well-posedness of forward problem:

$c \in C^3$, $c(0) = 0$, $c' \geq \underline{\gamma} > 0$, initial and boundary data smooth and compatible

$\Rightarrow$ existence and uniqueness of $C^{2,3}$ solution u

$\Rightarrow$ Exact data $m = u(\cdot,1)$ are C^2-smooth.

Instability

PDE:	$u_{tt} - (c(u_x))_x = 0 \quad \text{in } (0,T) \times (0,1)$
boundary conditions:	$u(\cdot,0) = 0 \qquad c(u_x(\cdot,1)) = g$
initial conditions:	$u(0,\cdot) = u_0 \qquad u_t(0,\cdots) = u_1$
measurements:	$m = u(\cdot,1)$
searched for parameter curve:	$\lambda \to c(\lambda)$

Exact data $m = u(\cdot,1)$ are C^2-smooth.

Measured data are only L^∞-smooth
(pointwise measurement error — derivatives cannot be measured)

$\Rightarrow$ Ill-posedness of identification problem (instability) $\to$ regularization methods
Stability for the inverse problem in weaker norms still possible.

Uniqueness

PDE:	$u_{tt} - (c(u_x))_x = 0 \quad \text{in } (0,T) \times (0,1)$
boundary conditions:	$u(\cdot, 0) = 0 \qquad c(u_x(\cdot, 1)) = g$
initial conditions:	$u(0, \cdot) = u_0 \qquad u_t(0, \cdots) = u_1$
measurements:	$m = u(\cdot, 1)$
searched for parameter curve:	$\lambda \to c(\lambda)$

If for two curves $\widetilde{c}$, c the measurements of the corresponding PDE solutions $\widetilde{u}$, u on the boundary $\widetilde{m} = \widetilde{u}(1, \cdot)$, $m = u(1, \cdot)$ coincide, then $\widetilde{c}$ and c must be identical

Uniqueness ?

PDE:	$u_{tt} - (c(u_x))_x = 0$ in $(0,T) \times (0,1)$
boundary conditions:	$u(\cdot,0) = 0 \qquad c(u_x(\cdot,1)) = g$
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If for two curves $\widetilde{c}$, c the measurements of the corresponding PDE solutions $\widetilde{u}$, u on the boundary $\widetilde{m} = \widetilde{u}(1,\cdot)$, $m = u(1,\cdot)$ coincide, must $\widetilde{c}$ and c be identical?

Identifiability by reformulation as a Volterra integral equation (I)

$v := \widetilde{u} - u$ solves

$$\boxed{\begin{array}{l} v_{tt} - (a\ v_x\ +\ (\widetilde{c} - c)(u_x))_x = 0 \text{ in } (0,T) \times (0,1) \\ \quad v(\cdot,0) = 0 \quad a(\cdot,1)v_x(\cdot,1) + (\widetilde{c} - c)(u_x(\cdot,1)) = 0 \\ \quad v(0,\cdot) = 0 \quad v_t(0,\cdot) = 0 \end{array}}$$

where $a(t,x) = \int_0^1 \tilde{c}'(\widetilde{u}_x(t,x) + \theta(u_x(t,x) - \widetilde{u}_x(t,x)))d\theta$

Proposition [BK '04] $a(t,x) \equiv \bar{a}$,
$g(0) = 0$, $g' > 0$, T and $|\widetilde{c} - c|_{C^3}$ sufficiently small.

$$\left| \pm\sqrt{\bar{a}}\, u_{xx}(t,x) + u_{xt}(t,x) \right| \geq \kappa > 0 \quad \text{in } (0,T) \times (0,1)$$

Then, with $\bar{\lambda} := c^{-1}(g(T))) > 0$,

$$\|\widetilde{c} - c\|_{L^2(0,\bar{\lambda})} \leq C\|v(\cdot,1)\|_{H^1(0,T)}$$

Identifiability by reformulation as a Volterra integral equation (II)

$v := \widetilde{u} - u$ solves
$$\boxed{\begin{array}{l} v_{tt} - (a\ v_x \ + \ (\widetilde{c} - c)(u_x))_x = 0 \text{ in } (0,T) \times (0,1) \\ \quad v(\cdot,0) = 0 \quad a(\cdot,1)v_x(\cdot,1) + (\widetilde{c} - c)(u_x(\cdot,1)) = 0 \\ \quad v(0,\cdot) = 0 \quad v_t(0,\cdot) = 0 \end{array}}$$

where $a(t,x) = \int_0^1 \tilde{c}'(\widetilde{u}_x(t,x) + \theta(u_x(t,x) - \widetilde{u}_x(t,x)))d\theta$

Conjecture [BK '04]
$g(0) = 0$, $g' > 0$, T and $|\widetilde{c} - c|_{C^3}$ sufficiently small.

$$|\frac{d}{dt}u_x(t,x(t))| \geq \kappa > 0 \quad \text{ in } (0,T) \times (0,1)$$

Then with $\bar{\lambda} := c^{-1}(g(T))) > 0$,

$$\|c - \widetilde{c}\|_{L^2(0,\bar{\lambda})} \leq C\|m - \widetilde{m}\|_{H^1(0,T)}$$

Idea of Proof: integrate along characteristics $x(t)$ $\longrightarrow$ Volterra integral equation of the first kind.
(cf. [Isakov, 1998] for space dependent coefficients)

Identifiability by closeness to an identifiable problem (I)

PDE:	$u_{tt} - (c(u_x))_x = f \quad \text{in } (0,T) \times (0,1)$
boundary conditions:	$u(\cdot,0) = m_0 \qquad u(\cdot,1) = m_1$
initial conditions:	$u(0,\cdot) = u_0 \qquad u_t(0,\cdots) = u_1$
measurements:	$c(u_x(t,1)) = g$
searched for parameter curve:	$\lambda \to c(\lambda)$

If we would know $u_x(t,1)$ and if $t \mapsto u_x(1,t)$ strictly monotone, we could identify c on $\{u_x(1,t) \ : \ t \in (0,T)\}$ directly from the measurements.

Identifiability by closeness to an identifiable problem (II)

PDE:	$u_{tt} - (c(u_x))_x = f$ in $(0,T) \times (0,1)$
boundary conditions:	$u(\cdot,0) = m_0 \qquad u(\cdot,1) = m_1$
initial conditions:	$u(0,\cdot) = u_0 \qquad u_t(0,\cdots) = u_1$
measurements:	$c(u_x(t,1)) = g$
searched for parameter curve:	$\lambda \to c(\lambda)$

Auxiliary problem:

$$s_{tt} - (\bar{c}s_x)_x = f_x \text{ in } (0,T) \times (0,1)$$
$$(\bar{c}s_x)(\cdot,j) = m_j''(t) - f(\cdot,j) \quad , \; j = 0,1,$$
$$s(0,\cdot) = u_0' \qquad s_t(0,\cdot) = u_1'$$

where $\bar{c}(x) = c_0'(u_0'(x))\,, \quad c_0$ close to $\tilde{c}$

$$\Rightarrow \quad s(1,t) \approx u_x(1,t) \text{ for } t \text{ small}$$

Identifiability by closeness to an identifiable problem (III)

$$\mathcal{D} \subseteq C^3(\mathbf{R}) \text{ such that } \forall \widetilde{c}, c \in \mathcal{D} : \|\widetilde{c} - c\|_{W^{2,\infty}} \leq K \|\widetilde{c} - c\|_{L^\infty}$$

e.g., $c, \tilde{c}$ bandlimited with bound on bandwitdth.

Theorem([B.K.&Lorenzi'07])
Let $t \mapsto s(t,1)$ be continuous and strictly monotone and assume that T and $|c' - c_0'|_{L^\infty}$ are sufficiently small.
Then under some a priori regularity assumptions on u
$(u \in W^{1,1}(0,T;W^{2,4}(\Omega))) \cap W^{1,\infty}(0,T;W^{1,4}(\Omega)))$

$u_{tt} - (c(u_x))_x = f$
+BC+IC
$c(u_x(\cdot,1)) = g$
$s(\cdot,1) \approx u_x(\cdot,1)$

$$\|c - \widetilde{c}\|_{L^\infty} \leq C \Big\{ \|d_0 - \widetilde{d}_0\|_{H^2} + \|d_1 - \widetilde{d}_1\|_{H^1} + \|f - \widetilde{f}\|_{W^{1,1}(0,T;L^2)} + \|g - \widetilde{g}\|_{L^\infty} \Big\}.$$

holds for all $\widetilde{c}, c \in \mathcal{D}$.

Idea: identifiability criterion on initial data, closeness for short times.

Extendable to 3-d anisotropic PDE for finite dimensional c, [BK&Lorenzi'07]

Identifiability by closeness to an identifiable problem (IV)

3-d anisotropic PDE, finite dimensional c: $\underline{c}(y) = \sum_{k=1}^{n} \alpha_k \nabla_y c_k(y)$

PDE:	$u_{tt} - \mathrm{div}_x[\underline{c}(\nabla_x u)] = f, \quad \text{in } (0,T)\times\Omega$
boundary conditions:	$u = 0 \quad \text{on } \partial\Omega$
initial conditions:	$u(0,\cdot) = u_0 \qquad u_t(0,\cdots) = u_1$
measurements:	$\int_{\partial\Omega} \varphi_j \nu \cdot \underline{c}(\nabla_x u(T_j,\cdot))\, d\Gamma = \delta_j, \quad j = 1,\ldots n$
searched for coefficients:	$\alpha_1,\ldots,\alpha_n$

Theorem $\varphi_j \in (L^p(\partial\Omega))^*$, T sufficiently small, $\Omega \subseteq \mathbb{R}^d$ a C^2 domain

$$\det W \neq 0 \quad \text{where } W_{jk} = \int_{\partial\Omega} \varphi_j \nu \cdot \nabla_y c_k(\nabla_x u_0)\, d\Gamma$$

Then under some a priori regularity assumptions on u ($u \in W^{1,\infty}(0,T;W^{2,\infty}(\Omega))$)

$$\|\alpha - \widetilde{\alpha}\|_{l^\infty} \leq C\Big\{\|u_0 - \widetilde{u}_0\|_{H^2} + \|u_1 - \widetilde{u}_1\|_{H^1} + \|f - \widetilde{f}\|_{W^{1,1}(0,T;L^2)} + \|\delta - \widetilde{\delta}\|_{l^\infty}\Big\}.$$

Identifiability by ε-expansion (I)

PDE:	$u_{tt} - (c(u_x))_x = 0 \quad$ in $(0,T) \times (0,1)$
boundary conditions:	$u(\cdot,0) = \varepsilon f \qquad u(\cdot,1) = 0$
initial conditions:	$u(0,\cdot) = 0 \qquad u_t(0,\cdots) = 0$
measurements:	$c(u_x(\cdot,1)) = g$
searched for parameter curve:	$\lambda \longrightarrow c(\lambda)$

Polynomial $c \;\Rightarrow\; \varepsilon$- expansion of u

Do the same excitation f at different intensities εf, $\quad \varepsilon = \varepsilon_1, \ldots, \varepsilon_J$
$\rightsquigarrow$ identify polynomial coefficients of c.

Identifiability by ε-expansion (II)

Theorem ([Nakamura&Watanabe'07, B.K.&Nakamura&Watanabe'08])

$$c(\lambda) = \sum_{i=1}^{J} \gamma_i \lambda^i + R \qquad |R^{(p)}(\lambda)| \le C|\lambda|^{J+1-p}$$

$f \in C^{J+1}(0,T)$, supp$f \subset (0,T)$.
Then for $\varepsilon \le \varepsilon_0$ suff. small there exists a solution $u \in \bigcap_{j=0}^{J+1} C^j(0,T;H^{J+1-j}(\Omega))$ and

$$u = \sum_{j=1}^{J} \varepsilon^j u_j(t,x) + O(\varepsilon^{J+1}) \qquad \text{as } \varepsilon \longrightarrow 0$$

$u_{1tt} - (\gamma_1 u_{1x})_x = 0$ $u_1(\cdot,0) = f,\ u_1(\cdot,1) = 0$ $u_1(0,\cdot) = 0,\ u_{1t}(0,\cdots) = 0$	$u_{j_{tt}} - (\gamma_1 u_{j_x})_x = (P_j(u_{1x},\ldots,u_{j-1_x})_x$ $u_j(\cdot,0) = 0,\ u_j(\cdot,1) = 0$ $u_j(0,\cdot) = 0,\ u_{j_t}(0,\cdots) = 0$

$$P_j(\lambda_1,\ldots,\lambda_{j-1}) = \sum_{i=1}^{j} \gamma_i \sum_{\mathbf{i}\in\mathcal{I}(j-1,i,j)} \binom{i}{\mathbf{i}} \lambda_1^{i_1}\cdots\lambda_{j-1}^{i_{j-1}}, \quad \binom{i}{\mathbf{i}} = \frac{i!}{i_1!\cdot i_2!\cdot\cdots\cdot i_{j-1}!}$$

Identifiability by ε-expansion (II)

Theorem ([Nakamura&Watanabe'07, B.K.&Nakamura&Watanabe'08])

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$$u = \sum_{j=1}^{J} \varepsilon^j u_j(t,x) + O(\varepsilon^{J+1}) \qquad \text{as } \varepsilon \to 0$$

$u_{1tt} - (\gamma_1 u_{1x})_x = 0$ $u_1(\cdot,0) = f,\ u_1(\cdot,1) = 0$ $u_1(0,\cdot) = 0,\ u_{1t}(0,\cdots) = 0$	$u_{j_{tt}} - (\gamma_1 u_{j_x})_x = (P_j(u_{1x},\ldots,u_{j-1_x})_x$ $u_j(\cdot,0) = 0,\ u_j(\cdot,1) = 0$ $u_j(0,\cdot) = 0,\ u_{j_t}(0,\cdots) = 0$

extendable to space-dependent coefficients $c = c(x,\lambda)$

Identifiability by ε-expansion (III)

$$
\begin{array}{ll}
u_{tt} - (c(u_x))_x = 0 & \text{in } (0,T)\times(0,1) \\
u(\cdot,0) = \varepsilon f & u(\cdot,1) = 0 \\
u(0,\cdot) = 0 & u_t(0,\cdots) = 0
\end{array}
$$

excitations $\varepsilon_k f$, $k = 1, \dots J$
$\rightsquigarrow$ solutions u^{ε_k}, $k = 1, \dots J$
$\rightsquigarrow$ measurements $c(u_x^{\varepsilon_k}(\cdot,1)) = g^{\varepsilon_k}$

Theorem ([B.K.&Nakamura&Watanabe'08])

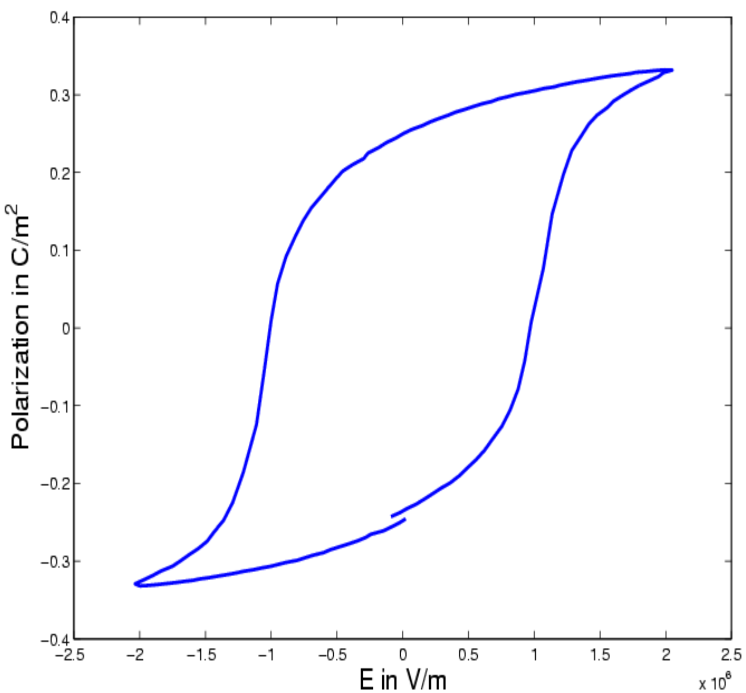
$$c(\lambda) = \sum_{i=1}^{J} \gamma_j \lambda^j + R \qquad |R^{(p)}(\lambda)| \le C|\lambda|^{J+1-p}$$

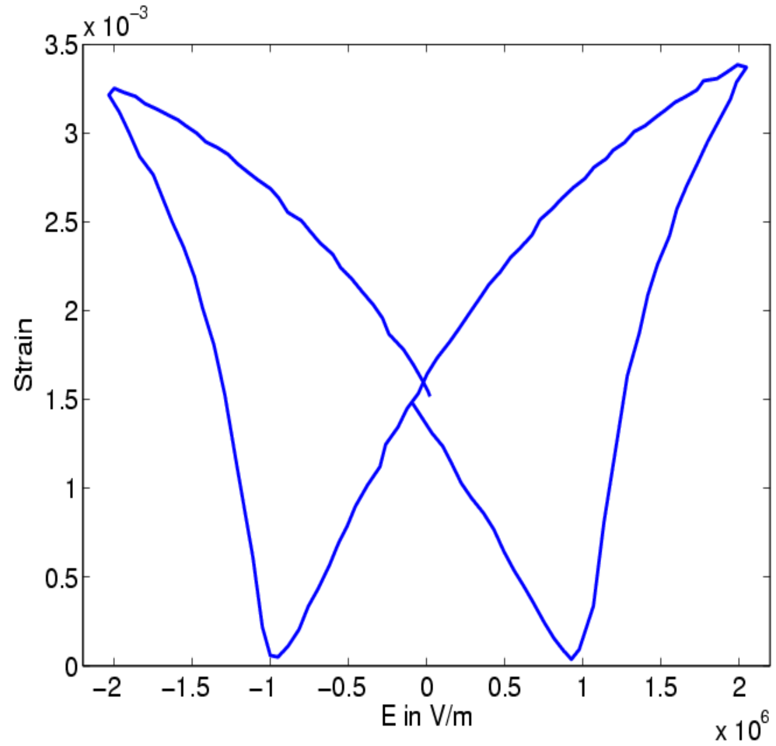
$f \in C^{J+1}(0,T)$, supp$f \subset (0,T)$, $\varepsilon_k = 2^{k-1}\varepsilon$
Then γ_j can be reconstructed up to an error $O(\varepsilon^{J+1-j})$, $j = 1, \dots J$
from measurements g^{ε_k} $k = 1, \dots J$.

Idea of proof: Multinomial Theorem $\Rightarrow$ $g^{\varepsilon_k}(t) = \sum_{j=1}^{J} \Gamma_j(\gamma_1, \dots, \gamma_j; t)\varepsilon_k^j + O(\varepsilon^{J+1})$
identify Γ_j by solving Vandermonde system; identify sucessively γ_j from Γ_j, $j = 1, \dots J$

Hysteresis in Piezoelectricity

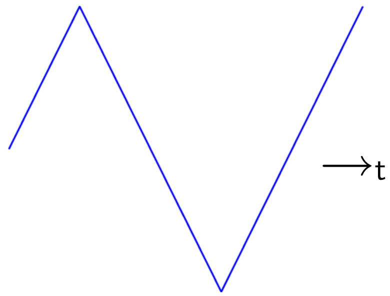
Measured polarization and strain at large electric field excitation ($E \sim 2MV/m$):



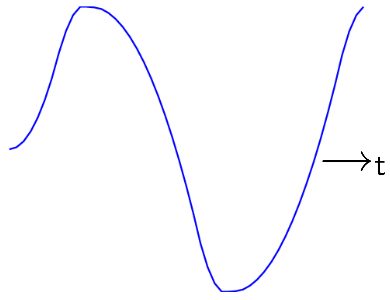


Hysteresis

input:

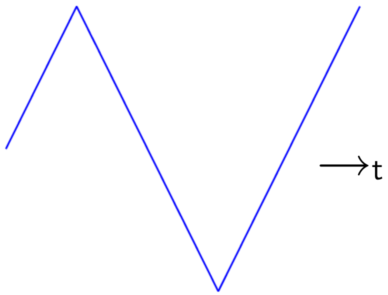


output:

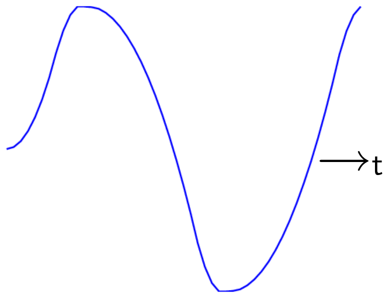


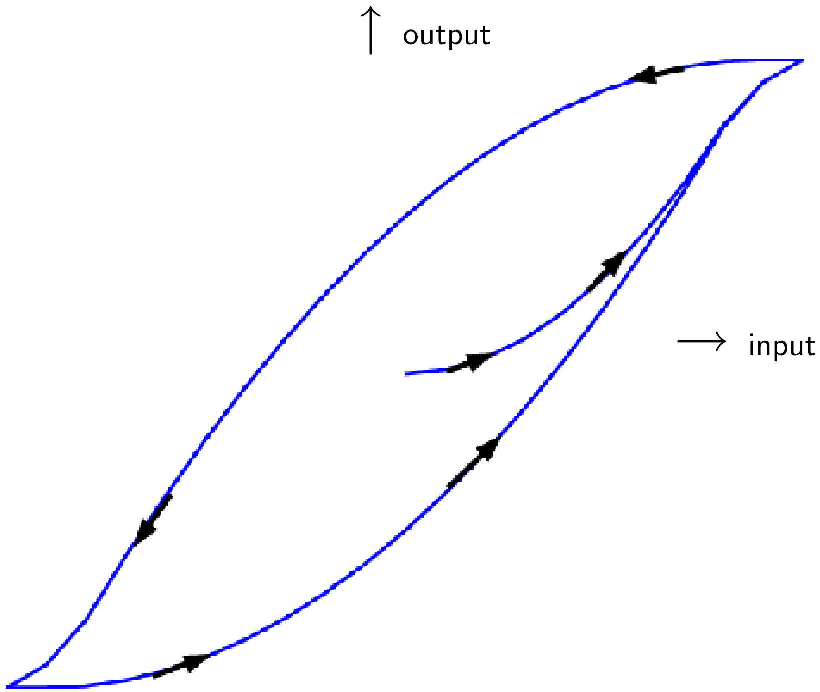
Hysteresis

input:



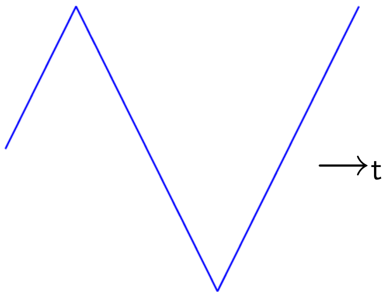
output:



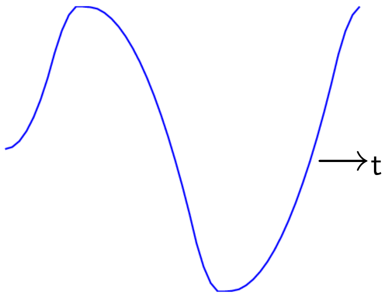


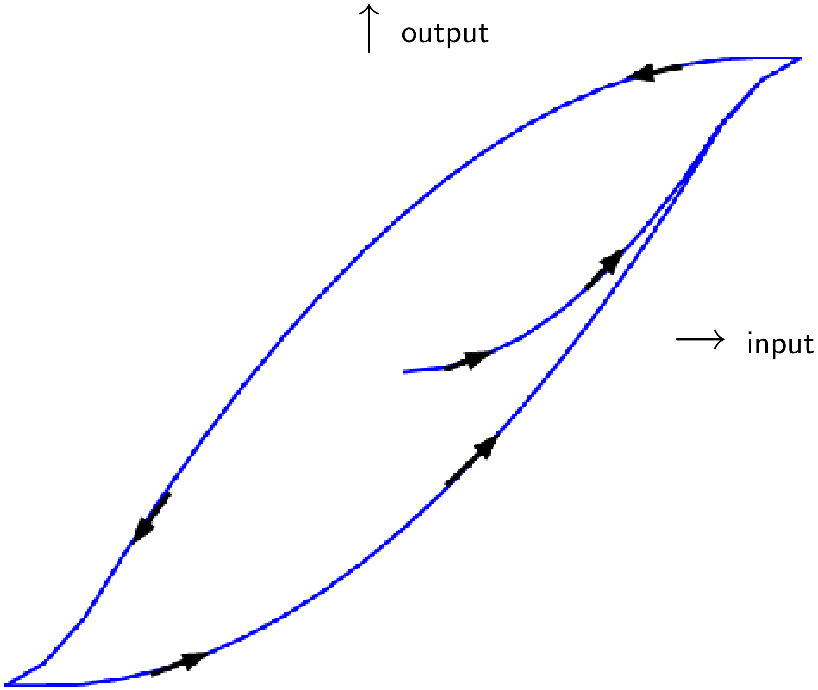
Hysteresis

input:



output:





- magnetics
- piezoelectricity
- plasticity
- . . .

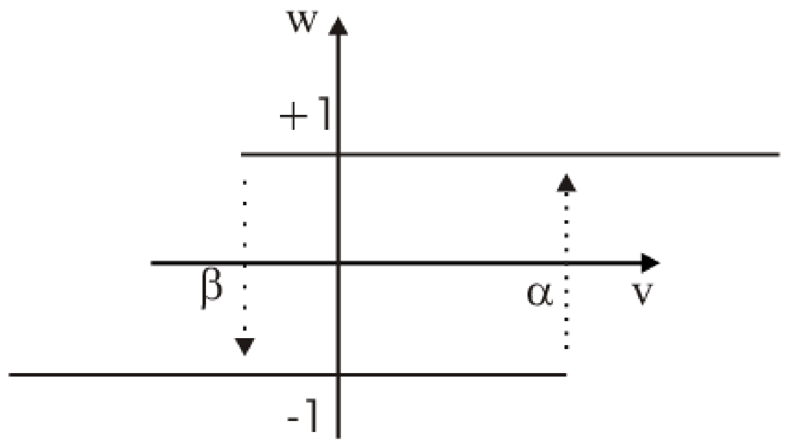
∗ memory

∗ Volterra property

∗ rate independence

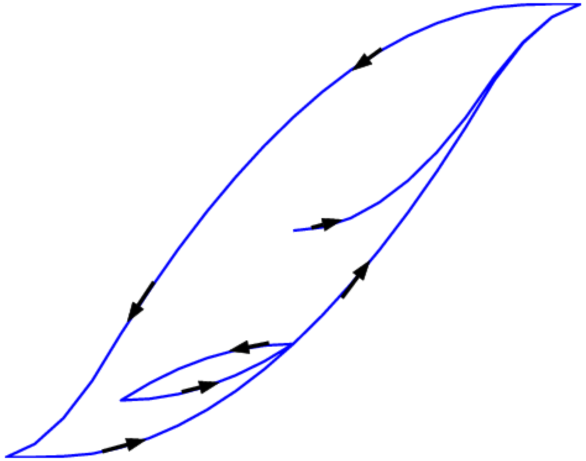
Krasnoselksii-Pokrovskii '83, Mayergoyz '91, Visintin '94, Krejčí '96, Brokate-Sprekels '96

A Simple Example: The Relay

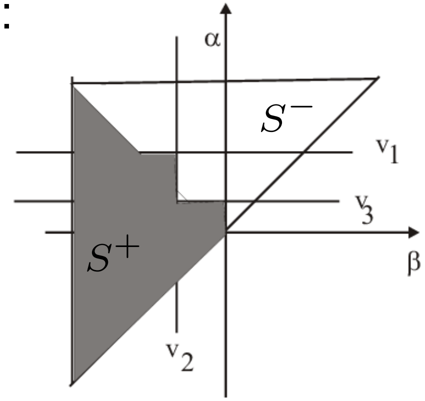


$$\begin{aligned}\mathcal{R}_{\beta,\alpha}[v](t) &= w(t) \\ &= \begin{cases} +1 & \text{if } v(t) > \alpha \text{ or } (w(t_i) = +1 \wedge v(t) > \beta) \\ -1 & \text{if } v(t) < \beta \text{ or } (w(t_i) = -1 \wedge v(t) < \alpha) \end{cases} \qquad t \in [t_i, t_{i+1}]\end{aligned}$$

A General Hysteresis Model: The Preisach Operator



weighted superposition of relays with Preisach weight function $\wp$ defined on the Preisach plane $S = S^+ \cup S^-$:



$$\mathcal{P}[v](t) = \iint_{\alpha,\beta\in S} \wp(\beta,\alpha)\mathcal{R}_{\beta,\alpha}[v](t)\, d(\alpha,\beta)$$

$$= \iint_{\alpha,\beta\in S^+(t)} \wp(\beta,\alpha)\, d(\alpha,\beta) - \iint_{\alpha,\beta\in S^-(t)} \wp(\beta,\alpha)\, d(\alpha,\beta)$$

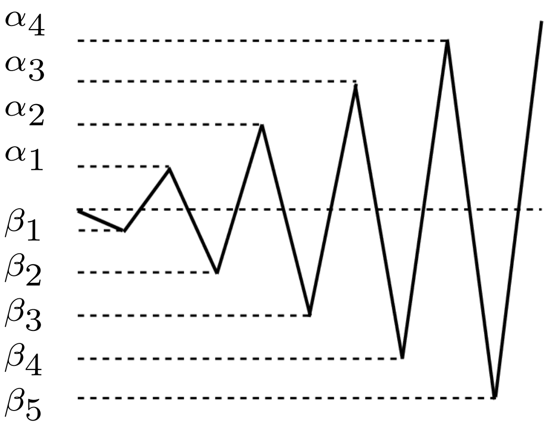
Hysteresis Identification from Input-Output model

Given input $(v(t))_{t\in[0,T]}$ measure output $(w(t))_{t\in[0,T]} = (\mathcal{P}[v](t))_{t\in[0,T]}$.

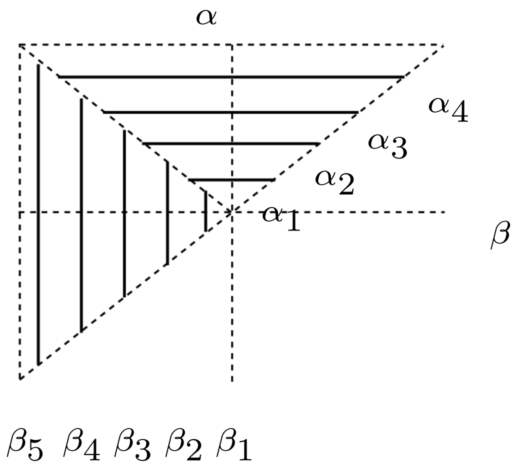
Identify $\mathcal{P}$ (i.e., $\wp$) from $\boxed{\mathcal{P}[v](t) = \iint_{\alpha,\beta\in S} \wp(\beta,\alpha)\mathcal{R}_{\beta,\alpha}[v](t)\, d(\alpha,\beta) = w(t)}$

$\rightsquigarrow$ linear integral equation.

Nonuniqueness, since $v : \underbrace{[0,T]}_{\subset \mathbb{R}^1} \to \mathbb{R}$ but $\wp : \underbrace{S}_{\subset \mathbb{R}^2} \to \mathbb{R}$!



$v^{\vec{\alpha}\vec{\beta}}$ yields $\mathcal{E}$ (with $\wp = \partial_1\partial_2\mathcal{E}$) on the lines $|$ and — in S:



$\rightsquigarrow$ identifiability from $\Lambda^{\mathcal{P}} : \{v^n\}_{n\in\mathbb{N}} \mapsto \{\mathcal{P}[v^n]\}_{n\in\mathbb{N}}$ with

$v^n = v^{\vec{\alpha^n}\vec{\beta^n}}$ and $\vec{\alpha^n} = \vec{\beta^n} = (\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n})$ [Hoffmann&Meyer '89]

Hysteresis in the Piezoelectric PDEs

1-d Piezoelectric PDEs:

$$\rho d_{tt} - \Big(c^E d_x + e(P)\phi_x \; + S(P)\Big)_x = 0$$
$$\Big(e(P)d_x - \varepsilon_0^S \phi_x \quad - P\Big)_x = 0$$

$$P = \tilde{\mathcal{P}}[-\phi_x]$$

d. . . mech. displacement
ϕ. . . electric potential
S . . . irreversible strain
P. . . irreversible polarization
ρ. . . mass density
c^E . . . elastic coefficient
ε^S . . . dielectric coeff.
e . . . coupling coeff.

elimination of ϕ:

$$\boxed{\rho d_{tt} - \mathcal{P}[d_x]_x = 0 \quad + \text{ inhom. BC}}$$

$\rightsquigarrow$ hyperbolic PDE with hysteresis; well-posedness: [Krejčí'93]

switching of electric dipoles $\longrightarrow$ Preisach model

Conclusions and Outlook

- Motivation: Piezoelectricity
  $\rightsquigarrow$ parameter identification in nonlinear hyperbolic PDE

- identifiability: three different approaches

$\longrightarrow$ identifiability of hysteresis operators in hyperbolic PDEs