Weierstrass Institute for Applied Analysis and Stochastics
in Forschungsverbund Berlin e.V.

Pavel Krejčí

Uniqueness in nonlinearly coupled PDE systems

(joint work with Lucia Panizzi)

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Mohrenstrasse 39, 10117 Berlin, Germany   +49 (30) 2 03 72 0   http://www.wias-berlin.de/~krejci   krejci@wias-berlin.de

Model problem: surface hardening of steel

In a domain $Q_T = \Omega \times (0,T)$, $\Omega \subset \mathbb{R}^3$ bounded and smooth, consider the system

$$\begin{aligned} \theta_t - \Delta\theta &= r(\theta, c) \\ c_t - \text{div}\,(D(\theta,c)\nabla c) &= 0 \end{aligned}$$

with boundary conditions on $\Sigma_T = \partial\Omega \times (0,T)$

$$\begin{aligned} \frac{\partial\theta}{\partial\nu} + h(x,\theta,\theta_\Gamma(x,t)) &= 0 \\ -D(\theta,c)\frac{\partial c}{\partial\nu} &= b(x,t) \end{aligned}$$

and initial conditions $\theta(x,0) = \theta^0(x)$, $c(x,0) = c^0(x)$.

$\theta^0 \geq \theta_* > 0$, $\theta_\Gamma \geq \theta_*$, $b \in L^2(\Sigma_T)$, $c^0 \in L^2(\Omega)$ ... given data

$r \geq 0$, $D \in [d_0, d_1]$, $h(x,\cdot,\cdot)$ ... (smooth) constitutive functions

Existence and regularity I.

Assumption on h :

$$\exists a > 0 : \quad \theta, \theta_\Gamma \geq \theta_* \quad \Longrightarrow \quad h(x, \theta, \theta_\Gamma)(\theta - \theta_\Gamma) \; \geq \; a(\theta - \theta_\Gamma)^2 \, .$$

Existence can be proved by Schauder argument.

Maximal expected regularity for c :

$$\nabla c \in L^2(Q_T)$$

$$c_t \in L^2(0, T; (H^1(\Omega))')$$

$$c \in L^\infty(0, T; L^2(\Omega))$$

Regularity for θ :

$$\theta_t, \Delta\theta \in L^2(Q_T)$$

$\theta \geq \theta_*$ (maximum principle)

$\theta \in L^\infty(Q_T)$ (Alikakos-Moser)

This is not enough for uniqueness !

Partial Kirchhoff transform

We introduce new functions

$$u = F(\theta, c) := \int_0^c D(\theta, c')\mathrm{d}c' \quad \Longleftrightarrow \quad c = G(\theta, u)$$

$$R(\theta, u) := r(\theta, G(\theta, u))\ , \quad H(\theta, u) := \partial_\theta F(\theta, G(\theta, u))\ , \quad U(x, t) := \int_0^t u(x, \tau)\mathrm{d}\tau$$

We assume that F, G, R, H are Lipschitz, and H is bounded.

In terms of θ, U, the system of equations has the form

$$\int_\Omega (\theta_t\, \varphi + \nabla\theta \cdot \nabla\varphi - R(\theta, U_t)\, \varphi)\, \mathrm{d}x + \int_{\partial\Omega} h(x, \theta, \theta_\Gamma(x, t))\, \varphi\, \mathrm{d}S \quad = \quad 0$$

$$\int_\Omega (G(\theta, U_t)_t\, \psi + \nabla U_t \cdot \nabla\psi - H(\theta, U_t)\nabla\theta \cdot \nabla\psi)\, \mathrm{d}x + \int_{\partial\Omega} b(x, t)\psi\, \mathrm{d}S \quad = \quad 0$$

for every test functions $\varphi, \psi \in H^1(\Omega)$.

Uniqueness I.

Consider two solutions (θ_1, U_1) , (θ_2, U_2) , and set $\bar\theta = \theta_1 - \theta_2$, $\bar U = U_1 - U_2$.

Test the difference of the equations for θ_1 , θ_2 by $\varphi = \bar\theta$;

Integrate the equations for U_1 , U_2 in time, and test the difference by $\psi = \bar U_t$.

On the left hand side, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|\bar\theta(t)|^2_{L^2(\Omega)} + |\nabla\bar U(t)|^2_{L^2(\Omega)}\right) + |\nabla\bar\theta(t)|^2_{L^2(\Omega)} + |\bar U_t(t)|^2_{L^2(\Omega)}$$

We omit the good terms on the right hand side that can be estimated by Gronwall's argument. The only bad term is the integral

$$\int_\Omega\left(\int_0^t (H(\theta_1,(U_1)_t)\nabla\theta_1 - H(\theta_2,(U_2)_t)\nabla\theta_2)\,\mathrm{d}\tau\right)\cdot\nabla\bar U_t(x,t)\,\mathrm{d}x$$

We have to integrate by parts in time !

Uniqueness II.

Two bad terms remain:

$$(1)\ \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\int_0^t (H(\theta_1,(U_1)_t)\nabla\theta_1 - H(\theta_2,(U_2)_t)\nabla\theta_2)\,\mathrm{d}\tau\right)\cdot\nabla\bar{U}(x,t)\,\mathrm{d}x$$

$$(2)\ \int_\Omega (H(\theta_1,(U_1)_t)\nabla\theta_1 - H(\theta_2,(U_2)_t)\nabla\theta_2)\cdot\nabla\bar{U}(x,t)\,\mathrm{d}x$$

We integrate in time. Omitting again the terms that can be handled via Gronwall, we obtain for the left hand side

$$|\bar{\theta}(t)|^2_{L^2(\Omega)} + |\nabla\bar{U}(t)|^2_{L^2(\Omega)} + \int_0^t \left(|\nabla\bar{\theta}(\tau)|^2_{L^2(\Omega)} + |\bar{U}_t(\tau)|^2_{L^2(\Omega)}\right)\mathrm{d}\tau$$

the upper bounds

$$(1)^*\ \int_\Omega |\nabla\bar{U}(x,t)|\left(\int_0^t (|\bar{\theta}|+|\bar{U}_t|)\,|\nabla\theta_1|(x,\tau)\,\mathrm{d}\tau\right)\mathrm{d}x$$

$$(2)^*\ \int_0^t\int_\Omega |\nabla\bar{U}|\,(|\bar{\theta}|+|\bar{U}_t|)\,|\nabla\theta_1|(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau$$

Regularity II.

A necessary and sufficient condition to continue the above uniqueness estimates reads

$$\nabla\theta \in L^2(0,T;L^\infty(\Omega))$$

It is proved by differentiating θ in tangential directions and using the boundary condition to check the regularity of the normal component.

- Boundary of Ω must be of class $C^{2,1}$
- Different regularity is obtained for the normal and for the tangential derivatives
- Anisotropic Sobolev embedding theorem

For a vector $\mathbf{p} = (p_1, \ldots, p_N)$ we define the space $L^{\mathbf{p}}(\mathbb{R}^N)$ as the subspace of $L^1(\mathbb{R}^N)$ of functions u with finite norm

$$\|u\|_{\mathbf{p}} = \left(\int_{\mathbb{R}} \left(\ldots \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |u(x)|^{p_1} \, \mathrm{d}x_1 \right)^{p_2/p_1} \mathrm{d}x_2 \ldots \right)^{p_N/p_{N-1}} \mathrm{d}x_N \right)^{1/p_N}$$

Regularity III.

For a matrix $\mathbf{P} = (P_{ij})_{i,j=1}^{N}$, $P_{ij} = 1/p_{ij}$, we define the anisotropic Sobolev space

$$W^{1,\mathbf{P}}(\mathbb{R}^N) = \left\{ u \in L^1(\mathbb{R}^N) : \frac{\partial u}{\partial x_i} \in L^{\mathbf{p}_i}(\mathbb{R}^N)\,, i = 1, \ldots, N \right\}, \quad \mathbf{p}_i = (p_{i1}, \ldots, p_{iN})$$

We denote by $\varrho(\mathbf{P})$ the spectral radius of $\mathbf{P}$, by $\mathbf{I}$ the identity $N \times N$ matrix, and by $\mathbf{1}$ the vector $\mathbf{1} = (1, 1, \ldots, 1)$.

Theorem *Let* $\varrho(\mathbf{P}) < 1$ *, and let*

$$(\mathbf{I} - \mathbf{P})^{-1}\mathbf{1} \;=\; \mathbf{b} \;=\; (b_1, \ldots, b_N)\,.$$

Then $W^{1,\mathbf{P}}(\mathbb{R}^N)$ *is compactly embedded in* $L^\infty(\mathbb{R}^N)$ *, and there exists a constant* $C > 0$ *such that each* $u \in W^{1,\mathbf{P}}(\mathbb{R}^N)$ *has for all* $x, z \in \mathbb{R}^N$ *the Hölder property*

$$|u(z) - u(x)| \;\le\; C\,\|u\|_{W^{1,\mathbf{P}}(\mathbb{R}^N)} \sum_{i=1}^{N} |z_i - x_i|^{1/b_i}\,.$$

Uniqueness III.

In the uniqueness estimates with left hand side

$$|\bar{\theta}(t)|^2_{L^2(\Omega)} + |\nabla \bar{U}(t)|^2_{L^2(\Omega)} + \int_0^t \left(|\nabla \bar{\theta}(\tau)|^2_{L^2(\Omega)} + |\bar{U}_t(\tau)|^2_{L^2(\Omega)} \right) \mathrm{d}\tau$$

and right hand side

$$(1)^* \int_\Omega |\nabla \bar{U}(x,t)| \left(\int_0^t \left(|\bar{\theta}| + |\bar{U}_t| \right) |\nabla \theta_1|(x,\tau)\, \mathrm{d}\tau \right) \mathrm{d}x$$

$$(2)^* \int_0^t \int_\Omega |\nabla \bar{U}| \left(|\bar{\theta}| + |\bar{U}_t| \right) |\nabla \theta_1|(x,\tau)\, \mathrm{d}x\, \mathrm{d}\tau$$

we first use the Cauchy-Schwarz inequality

Uniqueness III.

In the uniqueness estimates with left hand side

$$|\bar{\theta}(t)|^2_{L^2(\Omega)} + |\nabla \bar{U}(t)|^2_{L^2(\Omega)} + \int_0^t \Big(|\nabla \bar{\theta}(\tau)|^2_{L^2(\Omega)} + |\bar{U}_t(\tau)|^2_{L^2(\Omega)} \Big)\, \mathrm{d}\tau$$

and right hand side

$$(1)^* \ |\nabla \bar{U}(t)|_{L^2(\Omega)} \left(\int_\Omega \left(\int_0^t \big(|\bar{\theta}| + |\bar{U}_t|\big)\, |\nabla \theta_1|(x,\tau)\, \mathrm{d}\tau \right)^2 \mathrm{d}x \right)^{1/2},$$

$$(2)^* \ \int_0^t \int_\Omega |\nabla \bar{U}| \, \big(|\bar{\theta}| + |\bar{U}_t|\big)\, |\nabla \theta_1|(x,\tau)\, \mathrm{d}x \, \mathrm{d}\tau \,,$$

we first use the Cauchy-Schwarz inequality

Uniqueness III.

In the uniqueness estimates with left hand side

$$|\bar{\theta}(t)|^2_{L^2(\Omega)} + |\nabla \bar{U}(t)|^2_{L^2(\Omega)} + \int_0^t \Big(|\nabla \bar{\theta}(\tau)|^2_{L^2(\Omega)} + |\bar{U}_t(\tau)|^2_{L^2(\Omega)} \Big) \, \mathrm{d}\tau$$

and right hand side

$$(1)^* \int_\Omega \left(\int_0^t \left(|\bar{\theta}| + |\bar{U}_t| \right) |\nabla \theta_1|(x,\tau) \, \mathrm{d}\tau \right)^2 \mathrm{d}x \,,$$

$$(2)^* \int_0^t \int_\Omega |\nabla \bar{U}| \left(|\bar{\theta}| + |\bar{U}_t| \right) |\nabla \theta_1|(x,\tau) \, \mathrm{d}x \, \mathrm{d}\tau \,,$$

then the Young inequality

Uniqueness III.

In the uniqueness estimates with left hand side

$$|\bar{\theta}(t)|^2_{L^2(\Omega)} + |\nabla \bar{U}(t)|^2_{L^2(\Omega)} + \int_0^t \Big(|\nabla \bar{\theta}(\tau)|^2_{L^2(\Omega)} + |\bar{U}_t(\tau)|^2_{L^2(\Omega)} \Big) \, \mathrm{d}\tau$$

and right hand side

$$(1)^* \int_\Omega \left(\int_0^t \big(|\bar{\theta}| + |\bar{U}_t| \big) \, |\nabla \theta_1|(x,\tau) \, \mathrm{d}\tau \right)^2 \mathrm{d}x \,,$$

$$(2)^* \int_0^t \int_\Omega |\nabla \bar{U}| \, \big(|\bar{\theta}| + |\bar{U}_t| \big) \, |\nabla \theta_1|(x,\tau) \, \mathrm{d}x \, \mathrm{d}\tau \,,$$

we finally eliminate again the terms subject to Gronwall's argument.

Uniqueness III.

In the uniqueness estimates with left hand side

$$|\bar{\theta}(t)|^2_{L^2(\Omega)} + |\nabla \bar{U}(t)|^2_{L^2(\Omega)} + \int_0^t \left(|\nabla \bar{\theta}(\tau)|^2_{L^2(\Omega)} + |\bar{U}_t(\tau)|^2_{L^2(\Omega)} \right) \mathrm{d}\tau$$

and right hand side

$$(1)^* \int_\Omega \left(\int_0^t \left(|\bar{\theta}| + |\bar{U}_t|\right) |\nabla \theta_1|(x,\tau)\, \mathrm{d}\tau \right)^2 \mathrm{d}x\,,$$

$$(2)^* \int_0^t \int_\Omega |\nabla \bar{U}| \left(\qquad |\bar{U}_t| \right) |\nabla \theta_1|(x,\tau)\, \mathrm{d}x\, \mathrm{d}\tau\,,$$

we finally eliminate again the terms subject to Gronwall's argument.

Uniqueness III.

In the uniqueness estimates with left hand side

$$|\bar{\theta}(t)|^2_{L^2(\Omega)} + |\nabla \bar{U}(t)|^2_{L^2(\Omega)} + \int_0^t \Big(|\nabla \bar{\theta}(\tau)|^2_{L^2(\Omega)} + |\bar{U}_t(\tau)|^2_{L^2(\Omega)}\Big)\,\mathrm{d}\tau$$

and right hand side

$$(1)^* \int_\Omega \left(\int_0^t \left(|\bar{\theta}| + |\bar{U}_t|\right) |\nabla \theta_1|(x,\tau)\,\mathrm{d}\tau\right)^2 \mathrm{d}x\,,$$

we finally eliminate again the terms subject to Gronwall's argument.

Uniqueness III.

In the uniqueness estimates with left hand side

$$|\bar\theta(t)|^2_{L^2(\Omega)} + |\nabla\bar U(t)|^2_{L^2(\Omega)} + \int_0^t \left(|\nabla\bar\theta(\tau)|^2_{L^2(\Omega)} + |\bar U_t(\tau)|^2_{L^2(\Omega)} \right) \mathrm{d}\tau$$

and right hand side

$$(1)^* \int_\Omega \left(\int_0^t (\qquad |\bar U_t|) \, |\nabla\theta_1|(x,\tau) \, \mathrm{d}\tau \right)^2 \mathrm{d}x \, ,$$

the remaining term is estimated using Minkowski's inequality

$$\int_\Omega \left(\int_0^t |\bar U_t| \, |\nabla\theta_1| \, \mathrm{d}\tau \right)^2 \mathrm{d}x \le \left(\int_0^t \left(\int_\Omega |\bar U_t|^2 \, |\nabla\theta_1|^2 \mathrm{d}x \right)^{1/2} \mathrm{d}\tau \right)^2$$

Uniqueness IV.

The critical inequality has the form

$$\int_0^t |\bar{U}_t(\tau)|^2_{L^2(\Omega)}\,\mathrm{d}\tau \;\le\; \left(\int_0^t \gamma(\tau)\,|\bar{U}_t(\tau)|_{L^2(\Omega)}\,\mathrm{d}\tau\right)^2$$

with a function $\gamma \in L^2(0,T)$: $\boxed{\gamma(t) = C \text{ sup ess}_{x\in\Omega} |\nabla\theta_1(x,t)|}$.

Lemma (L^p Gronwall inequality)

Let $s \in L^p(0,T)$ *,* $\gamma \in L^{p'}(0,T)$ *,* $\alpha \in L^\infty(0,T)$ *be nonnegative functions such that*

$$\left(\int_0^t s^p(\tau)\,\mathrm{d}\tau\right)^{1/p} \le \alpha(t) + \int_0^t \gamma(\tau)\,s(\tau)\,\mathrm{d}\tau$$

for every $t \in [0,T]$ *, with a fixed* $p > 1$ *. Then there exists* $M_\gamma > 0$ *such that*

$$\left(\int_0^T s^p(\tau)\,\mathrm{d}\tau\right)^{1/p} \le M_\gamma \operatorname*{sup\,ess}_{[0,T]} \alpha\,.$$

Conclusions

- In coupled parabolic systems, lack of regularity in one equation can be compensated by a higher regularity in the other equation;
- Anisotropic Sobolev embeddings might be useful;
- Gronwall's inequality is not the only argument for uniqueness;
- These observations might be of interest also in other situations (Penrose-Fife with state dependent heat conductivity, say; In this case, the phase variable is the more regular one).