Discrete elliptic Carleman estimates
and
controllability of semi-discrete parabolic equations

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in collaboration with

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We consider the following linear parabolic problem

$$(S)\begin{cases}\partial_t y - \nabla\cdot\gamma\nabla y = 1_\omega\, v & \text{in } Q=(0,T)\times\Omega\\ y=0 & \text{on } \Sigma=(0,T)\times\partial\Omega\\ y(0)=y_0 & \text{in } \Omega\end{cases}$$

where $T>0$, Ω regular open subset of $\mathbb{R}^n$, $\omega\Subset\Omega$, $y_0\in L^2(\Omega)$.

Can we choose $v\in L^2(Q)$ s.t. $y(T)=0$?

The dual problem is given by

$$(S^*) \begin{cases} -\partial_t q - \nabla \cdot \gamma \nabla q = 0 & \text{in } Q \\ q = 0 & \text{on } \Sigma \\ q(T) = q_T & \text{in } \Omega \end{cases}$$

We say that (S^*) is observable at time T in ω if

$$\|q(0)\|^2_{L^2(\Omega)} \leq C_{\text{obs}} \iint_{(0,T)\times\omega} |q(t,x)|^2 \, dt \, dx.$$

Theorem

(S^*) observable in ω at time T $\Leftrightarrow$ (S) null-controllable

and there exists a control $v \in L^2(Q)$ s.t. $\|v\|_{L^2(Q)} \leq \sqrt{C_{\text{obs}}} \, \|y_0\|_{L^2(\Omega)}$

$$\begin{cases} \partial_t y - \nabla \cdot (\gamma \nabla y) = \chi_\omega v & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega, \end{cases}$$

Let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_k \ldots$ be the eigenvalues of $-\nabla \cdot (\gamma \nabla)$ and $(\phi_k)_{k \in \mathbb{N}}$ be orthonormal eigenfunctions in $L^2(\Omega)$ + Dirichlet.

Theorem (Lebeau-Robbiano, 95; Jerison-Lebeau, 96; Lebeau-Zuazua, 98)

There exists $C > 0$ *s.t. forall* $\mu > 0$

$$\sum_{\mu_i \leq \mu} |\alpha_i|^2 \leq C e^{C\sqrt{\mu}} \int_\omega \Big| \sum_{\mu_i \leq \mu} \alpha_i \phi_i(x) \Big|^2 \, dx, \qquad (\alpha_i)_i \subset \mathbb{C}$$

This allows to ‘explicitely’ construct a control by using the parabolic dissipation ⇒ observability *but* no time-dependent potential functions

Proof by local Carleman estimate for the augmented elliptic operator $L = -\partial_t^2 - \nabla \cdot \gamma \nabla$ on $(0, \mathcal{T}_*) \times \Omega$ for some $\mathcal{T}_* > 0$

Let $E_j = \mathrm{span}\{\phi_k;\ \mu_k \leq 2^{2j}\}$

Partial control in E_j on $(a, a+\mathcal{T})$, $\mathcal{T} > 0$:

$$\begin{cases} \partial_t y - \Delta y = \Pi_{E_j}(1_\omega v) & \text{in } (a, a+\mathcal{T}) \times \Omega, \\ y = 0 & \text{on } (a, a+\mathcal{T}) \times \partial\Omega, \\ y(a) = y_0 \in E_j & \text{in } \Omega, \end{cases}$$

There exists $v = V_j(y_0, a, \mathcal{T})$, s.t. $y(a+\mathcal{T}) = 0$, that satisfies

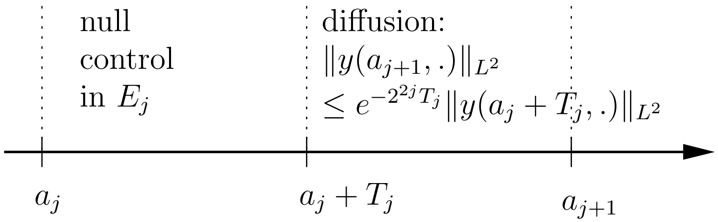
$$\|V_j(y_0, a, \mathcal{T})\|_{L^2((a,a+\mathcal{T})\times\Omega)} \leq C\mathcal{T}^{-\frac{1}{2}} e^{C2^j} \|y_0\|_{L^2(\Omega)}$$

Controllability: the method of Lebeau and Robbiano

We write $[0, T] = \bigcup_{j \in \mathbb{N}} [a_j, a_{j+1}]$

- $a_0 = 0$, $a_{j+1} = a_j + 2T_j$, for $j \in \mathbb{N}$
- $T_j = K 2^{-j\rho}$ with $\rho \in (0, 1)$
- K is such that $2 \sum_{j=0}^{\infty} T_j = T$

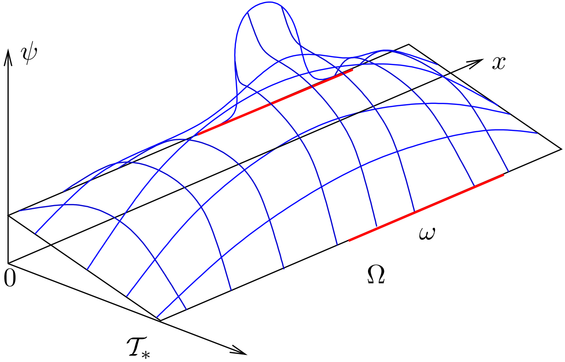
Control strategy:



The subsequent control is s.t. $\|v\|_{L^2((0,T)\times\Omega)} \leq C \|y_0\|_{L^2(\Omega)}$

Let $Q_* = (0, \mathcal{T}_*) \times \Omega$. Choosing a weight function ψ in $\mathscr{C}^2(\overline{Q_*}, \mathbb{R})$

$$|\nabla\psi| \geq c \text{ and } \psi > 0 \text{ in } Q_*, \quad \partial_{n_x}\psi(t,x) < 0 \text{ in } (0, \mathcal{T}_*) \times \partial\Omega,$$
$$\partial_t\psi \geq c \text{ on } \{0\} \times (\Omega \setminus \omega), \quad \nabla_x\psi = 0 \text{ and } \partial_t\psi \leq -c \text{ on } \{\mathcal{T}_*\} \times \Omega,$$



We set $\varphi = e^{\lambda\psi}$. $\qquad P = -\partial_t^2 - \nabla\cdot\gamma\nabla$

Theorem (Global Carleman estimate)

For $\lambda \geq 1$ sufficiently large, there exist $C > 0$ and $s_0 \geq 1$ such that

$$
\begin{aligned}
s^3\|e^{s\varphi}u\|^2_{L^2(Q_*)} + s\|e^{s\varphi}\nabla u\|^2_{L^2(Q_*)} + s\left|e^{s\varphi(0,.)}\partial_t u(0,.)\right|^2_{L^2(\Omega)} \\
+ se^{2s\varphi(\mathcal{T}_*)}|\partial_t u(\mathcal{T}_*,.)|^2_{L^2(\Omega)} + s^3 e^{2s\varphi(\mathcal{T}_*)}|u(\mathcal{T}_*,.)|^2_{L^2(\Omega)} \\
\leq C\Big(\|e^{s\varphi}Pu\|^2_{L^2(Q_*)} + se^{2s\varphi(\mathcal{T}_*)}|\nabla_x u(\mathcal{T}_*,.)|^2_{L^2(\Omega)} \\
+ s\left|e^{s\varphi(0,.)}\partial_t u(0,.)\right|^2_{L^2(\omega)}\Big),
\end{aligned}
$$

for $s \geq s_0$, and for all $u \in H^2(Q_*)$,
satisfying $u|_{\{0\}\times\Omega} = 0$, $u|_{(0,\mathcal{T}_*)\times\partial\Omega} = 0$.

Choose $u(t,x) = \sum_{\mu_j \le \mu} \alpha_j \frac{\sinh(\sqrt{\mu_j}\, t)}{\sqrt{\mu_j}} \phi_j(x)$

$Pu = 0$

absorbed
if $s^2 \geq C\mu$

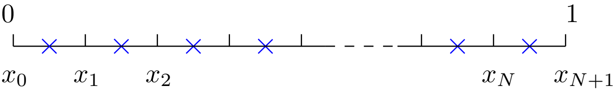
$$e^{2s\varphi(\mathcal{T}_*)} \overbrace{s^3 \underbrace{|u(\mathcal{T}_*,.)|^2_{L^2(\Omega)}}} \leq C e^{2s\varphi(\mathcal{T}_*)} \overbrace{s|\nabla_x u(\mathcal{T}_*,.)|^2_{L^2(\Omega)}} + Cs \underbrace{\left| e^{s\varphi(0,.)} \partial_t u(0,.) \right|^2_{L^2(\omega)}}$$

$$|u(\mathcal{T}_*,.)|^2_{L^2(\Omega)} \rightarrow \sum_{\mu_j \le \mu} |\alpha_j|^2$$

$$\left| e^{s\varphi(0,.)} \partial_t u(0,.) \right|^2_{L^2(\omega)} \rightarrow \underbrace{e^{2s \sup \varphi(0,.)}}_{\rightarrow\, e^{C\sqrt{\mu}}} \left| \sum_{\mu_j \le \mu} \alpha_j \phi_j \right|^2_{L^2(\omega)}$$

The semi-discrete problem

In one space dimension: $\Omega = (0,1)$



$$u = \sum_{i=1}^{N} \mathbf{1}_{[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]} u_i, \qquad v = \sum_{i=0}^{N} \mathbf{1}_{[x_i, x_{i+1}]} v_{i+\frac{1}{2}}.$$

$$(Du)_{i+\frac{1}{2}} = \frac{1}{h}(u_{i+1} - u_i), \qquad (\overline{D}v)_i = \frac{1}{h}(v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}})$$

We consider the following linear parabolic problem

$$(S)\begin{cases}\partial_t y - \overline{D}(\gamma D y) = 1_\omega\, v & \text{in } Q=(0,T)\times\Omega\\ y_0 = y_{N+1} = 0 & \\ y(0) = y_0 & \text{in } \Omega\end{cases}$$

Can we choose $v \in L^2(Q)$ s.t. $y(T)=0$?
With $\|v\|_{L^2(Q)} \leq C\|y_0\|_{L^2(\Omega)}$ uniformly in h ?

Existing results:

- Lopez-Zuazua, 98. Dimension 1, constant coefficient, uniform mesh size h.
  Explicit form of the eigenvectors. Sum of exponentials
- Zuazua, ICM 06. Dimension 2, counter-example to approximate controllability
  Localized eigenmodes at the end of the discrete spectrum
- Labbé-Trélat, 06. Abstract result for approximate h-controllability.

We set $\varphi = e^{\lambda\psi}$. $\qquad P = -\big(\xi_1\partial_t^2 + \overline{D}(\xi_{2d} D)\big)$

$$0 < \xi_{\min} \leq \xi_1, \xi_2 \leq \xi_{\max} < \infty, \qquad |D\xi_1|, |\overline{D}\xi_2| \leq L < \infty.$$

Theorem

For the parameter $\lambda \geq 1$ sufficiently large, there exist C, $s_0 \geq 1$, $h_0 > 0$, $\varepsilon_0 > 0$, all depending on ω, $\mathcal{T}_*$, $\xi_{\min}$, $\xi_{\max}$ and L, such that

$$s^3\|e^{s\varphi}u\|^2_{L^2(Q_*)} + s\|e^{s\varphi}\partial_t u\|^2_{L^2(Q_*)} + s\|e^{s\varphi_d}Du\|^2_{L^2(Q_*)} + s\left|e^{s\varphi(0,.)}\partial_t u(0,.)\right|^2_{L^2(\Omega)}$$
$$+ se^{2s\varphi(\mathcal{T}_*)}|\partial_t u(\mathcal{T}_*,.)|^2_{L^2(\Omega)} + s^3 e^{2s\varphi(\mathcal{T}_*)}|u(\mathcal{T}_*,.)|^2_{L^2(\Omega)}$$
$$\leq C\left(\|e^{s\varphi}Pu\|^2_{L^2(Q_*)} + se^{2s\varphi(\mathcal{T}_*)}|Du(\mathcal{T}_*,.)|^2_{L^2(\Omega)} + s\left|e^{s\varphi(0,.)}\partial_t u(0,.)\right|^2_{L^2(\omega)}\right),$$

for all $s \geq s_0$, $0 < h \leq h_0$ and $sh \leq \varepsilon_0$, and u satisfying $u|_{\{0\}\times\Omega} = 0$, $u|_{(0,\mathcal{T}_*)\times\partial\Omega} = 0$.

We set $\rho = e^{-s\varphi}$, $v = e^{s\varphi} u$ and $e^{s\varphi} P e^{-s\varphi} v = e^{s\varphi} f$.
We obtain $Av + Bv = g$

$$Av = \underbrace{\xi_1 \partial_t^2 v + \rho^{-1} \bar{\tilde{\rho}}\, \overline{D}(\xi_{2d} D v)}_{A_1 v} + \underbrace{\xi_1 \rho^{-1} (\partial_t^2 \rho)\, v + \xi_2 \rho^{-1} (\overline{D} D \rho)\, \bar{\tilde{v}}}_{A_2 v},$$

$$Bv = \underbrace{2\xi_1 \rho^{-1} (\partial_t \rho) \partial_t v + 2\rho^{-1} \overline{D\rho}\, \xi_2\, \overline{Dv}}_{B_1 v} \quad \underbrace{-2s(\Delta_\xi \varphi) v}_{B_2 v},$$

$$g = \rho^{-1} f - \frac{h}{4} \rho^{-1} \overline{D\rho} (\overline{D} \xi_{2d}) (\tau^+ D v - \tau^- D v) - \frac{h^2}{4} (\overline{D} \xi_{2d}) \rho^{-1} (\overline{D} D \rho) \overline{Dv}$$
$$- h\mathcal{O}(1) \rho^{-1} \overline{D\rho}\, \overline{Dv} - \big(\rho^{-1} (\overline{D} \xi_{2d}) \overline{D\rho} + h\mathcal{O}(1) \rho^{-1} (\overline{D} D \rho) \big)\, \bar{\tilde{v}} - 2s(\Delta_\xi \varphi) v$$

with $\Delta_\xi f = \xi_1 \partial_t^2 f + \xi_2 \partial_x^2 f$

$$\|Av\|^2_{L^2(Q_*)} + \|Bv\|^2_{L^2(Q_*)} + 2\mathrm{Re}(Av, Bv)_{L^2(Q_*)} = \|g\|^2_{L^2(Q_*)}.$$

Choose $u(t,x) = \sum_{\mu_j \le \mu} \alpha_j \frac{\sinh(\sqrt{\mu_j} t)}{\sqrt{\mu_j}} \phi_j(x)$

$Pu = 0$

absorbed
if $s^2 \geq C\mu$

$$e^{2s\varphi(\mathcal{T}_*)} \overbrace{s^3 \underbrace{|u(\mathcal{T}_*,.)|^2_{L^2(\Omega)}}_{\sum_{\mu_j \le \mu} |\alpha_j|^2}} \le C e^{2s\varphi(\mathcal{T}_*)} \overbrace{s|\nabla_x u(\mathcal{T}_*,.)|^2_{L^2(\Omega)}} + Cs \underbrace{\left| e^{s\varphi(0,.)} \partial_t u(0,.) \right|^2_{L^2(\omega)}}$$

$$\underbrace{e^{2s \sup \varphi(0,.)}}_{e^{C\sqrt{\mu}}} \left| \sum_{\mu_j \le \mu} \alpha_j \phi_j \right|^2_{L^2(\omega)}$$

In the proof we need $s^2 \geq C\mu$.
Yet we have $sh \leq \varepsilon_0$

The inequality can only be achieved for $\mu \leq \varepsilon_1/h^2 \sim \varepsilon_1 N^2$

Theorem

There exist $C > 0$, $\varepsilon_1 > 0$ and h_0 such that, for any mesh with $h \leq h_0$, for all $0 < \mu \leq \varepsilon_1/h^2$, we have

$$\sum_{\mu_k \leq \mu} |\alpha_k|^2 = \int_\Omega \Big| \sum_{\mu_k \leq \mu} \alpha_k \phi_k \Big|^2 \leq C e^{C\sqrt{\mu}} \int_\omega \Big| \sum_{\mu_k \leq \mu} \alpha_k \phi_k \Big|^2, \qquad \forall (\alpha_k)_{1 \leq k \leq N} \subset \mathbb{C}.$$

Remark

We have $Ck^2 \leq \mu_k \leq C'k^2, \quad 1 \leq k \leq N$

We are dealing with a constant portion of the discrete spectrum:

$$1 \leq k \leq \varepsilon_2 N$$

We set $\mu_{\max} = \varepsilon_1/h^2$ and $j_{\max} = \max\{j;\ 2^{2j} \leq \mu_{\max}\}$

$$E_j = \mathrm{Span}\{\phi_k;\ 1 \leq \mu_k \leq 2^{2j}\}, \quad j \in \mathbb{N},$$

Lemma

There exists $C \geq 0$ such that, for $j \leq j_{\max}$ and $\mathcal{T} > 0$, the semi-discrete solution q in $\mathscr{C}^\infty([0,\mathcal{T}], E_j)$ to the adjoint parabolic system

$$\begin{cases} -\partial_t q - \overline{D}(\gamma D q) = 0 & \text{in } (0,\mathcal{T}) \times \Omega, \\ q_0 = q_{N+1} = 0, \\ q(\mathcal{T}) = q_f \in E_j, \end{cases}$$

satisfies the following observability estimate

$$\|q(0)\|^2_{L^2(\Omega)} \leq \frac{C e^{C 2^j}}{\mathcal{T}} \int_0^{\mathcal{T}} \int_\omega |q(t)|^2 \, dt.$$

Uniform controllability of the lower part of the spectrum

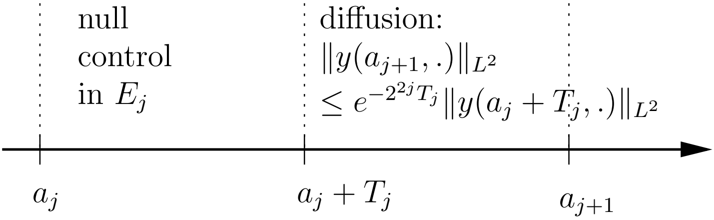
Partial control in E_j on $(a, a+\mathcal{T}_*)$, $j \leq j_{\max}$ and $\mathcal{T}_* > 0$:

$$\begin{cases} \partial_t y - \overline{D}(\gamma D y) = \Pi_{E_j}(1_\omega v) & \text{in } (a, a+\mathcal{T}_*) \times \Omega, \\ y_0 = y_{N+1} = 0, & \\ y(a) = y_0 \in E_j & \text{in } \Omega, \end{cases}$$

There exists $v = V_j(y_0, a, \mathcal{T}_*)$, s.t. $y(a+\mathcal{T}_*) = 0$, that satisfies

$$\|V_j(y_0, a, \mathcal{T}_*)\|_{L^2((a,a+\mathcal{T}_*)\times\Omega)} \leq C\mathcal{T}_*^{-\frac{1}{2}} e^{C2^j} \|y_0\|_{L^2(\Omega)}$$

Control strategy for $j \leq j_{\max}$:



Analysis of the L-R control strategy yields:

$$\|v\|_{L^2((0,T)\times\Omega)} \leq C\|y_0\|_{L^2(\Omega)},$$
$$\|y(a_{j+1})\|_{L^2(\Omega)} \leq Ce^{-C2^{j(2-\rho)}}\|y_0\|_{L^2(\Omega)}, \qquad 0 \leq j \leq j_{\max}.$$

with $\rho \in (0,1)$.

It follows that at time T we have:

$\Pi_{E_{j_{\max}}} y(T) = 0,$ and $\|y(T)\|_{L^2(\Omega)} \leq Ce^{-(C/h)^{(2-\rho)}}\|y_0\|_{L^2(\Omega)}$

The observability estimate we obtain is

$$\|q(0)\|^2_{L^2(\Omega)} \leq C_{\text{obs}} \iint\limits_{(0,T)\times\omega} |q(t,x)|^2 \, dt \, dx + Ce^{-(C/h)^{(2-\rho)}} \|q_T\|^2_{L^2(\Omega)},$$

for a solution to the adjoint problem

$$(S^*) \begin{cases} -\partial_t q - \overline{D}(\gamma D q) = 0 & \text{in } Q \\ q_0 = q_{N+1} = 0 & \\ q(T) = q_T & \text{in } \Omega \end{cases}$$

$\rightarrow$ constructive algorithm.

We address quasi-uniform meshes

Let $0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1$:   primal mesh

Set $h_{i+\frac{1}{2}} = x_{i+1} - x_i$ and $x_{i+\frac{1}{2}} = (x_{i+1} + x_i)/2$

$\overline{\mathfrak{M}} := \{x_{i+\frac{1}{2}};\ i = 0, \dots, N\}$ dual mesh

$h = \max_{0 \le i \le N} h_{i+\frac{1}{2}}$

Assumption:

$$\operatorname{reg}(\mathfrak{M}) = \max\left(\sup_{1 \le i \le N} \left(\frac{h}{h_{i+\frac{1}{2}}} \right), \sup_{1 \le i \le N} \left(\frac{|h_{i+\frac{1}{2}} - h_{i-\frac{1}{2}}|}{h_{i-\frac{1}{2}} h_{i+\frac{1}{2}}} \right) \right),$$

remains bounded when the mesh size h tends to 0.

Change of variable in space: the operator $\overline{D}(\gamma D)$ can be put in the form $\frac{1}{\xi_1}\overline{D}(\xi_2 D)$ on a uniform mesh

$$0 < \xi_{\min}(\operatorname{reg}(\mathfrak{M})) \le \xi_1, \xi_2 \le \xi_{\max}(\operatorname{reg}(\mathfrak{M})) < \infty,$$
$$|D\xi_1|, |\overline{D}\xi_2| \le L(\operatorname{reg}(\mathfrak{M})) < \infty.$$