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Well posedness of finite difference schemes for a singular diffusion problem with a free boundary

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1 Introduction

Nonlinear diffusion equation with a transport term

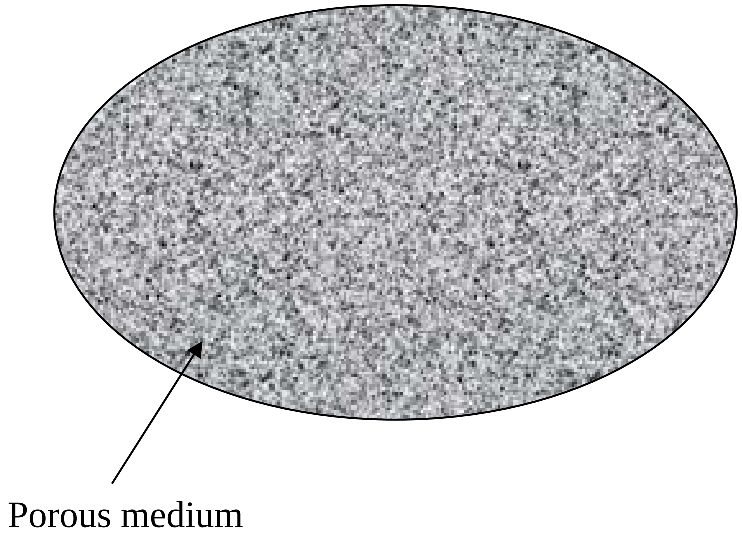
$$\frac{\partial \theta}{\partial t} - \Delta \beta^*(\theta) + \nabla \cdot K(\theta) \ni f \quad \text{in } Q = (0,T) \times \Omega,$$

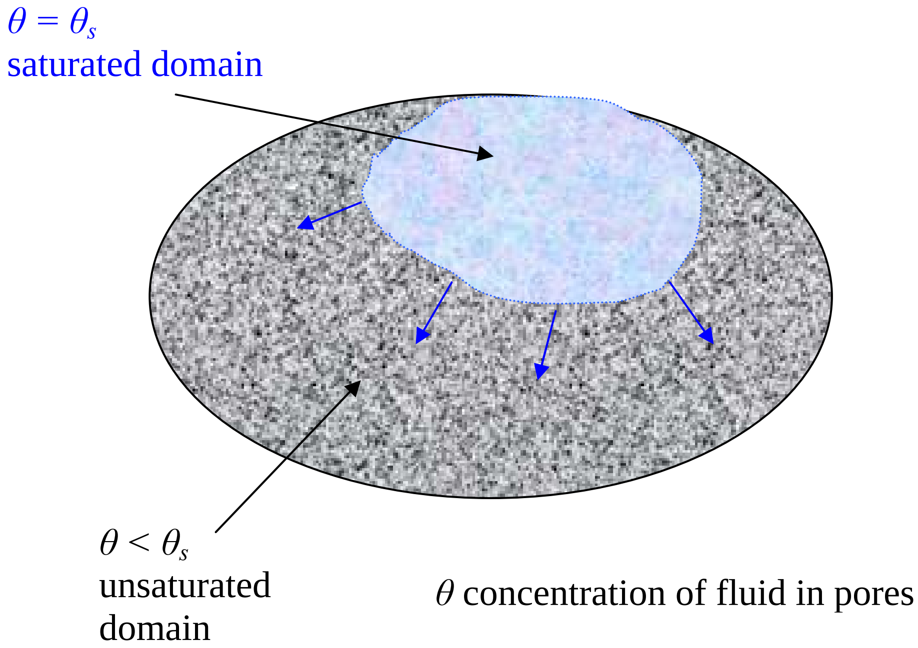
$$\theta(t,x) = \theta_0 \quad \text{in } \Omega, \tag{bvp}$$

$$(K(\theta) - \nabla \beta^*(\theta)) \cdot \nu - \alpha \beta^*(\theta) \ni f_\alpha \quad \text{on } \Sigma = (0,T) \times \Gamma$$

$\Omega \subset \mathbf{R}^N$ open, bounded, $\Gamma = \partial\Omega$ smooth, ν unit normal vector to Γ, T finite

$$K = (K_i)_{i=1,\dots,N}.$$

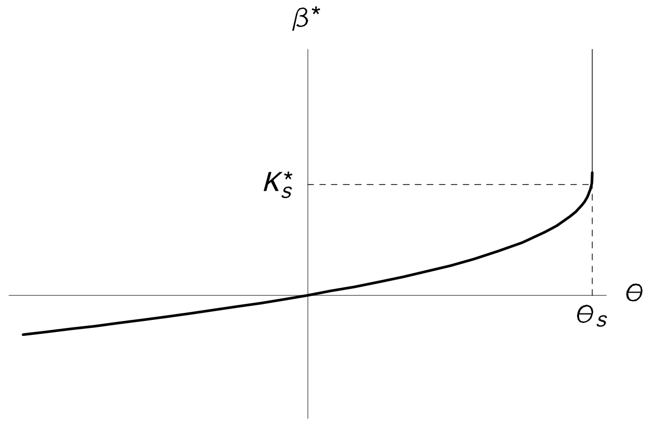

Porous medium


$\theta = \theta_s$
saturated domain
$\theta < \theta_s$
unsaturated domain
θ concentration of fluid in pores

In this model

$$\beta^* : \mathbf{R} \to \mathbf{R} \text{ is multivalued}$$

$$\beta^*(r) = \begin{cases} \int_0^r \beta(\xi) d\xi, & \text{if } r < \theta_s, \\ [K_s^*, +\infty), & \text{if } r = \theta_s, \\ \emptyset, & \text{if } r > \theta_s, \end{cases}$$



- β is the diffusion coefficient

$$\beta : (-\infty, \theta_s) \to \mathbf{R}_{+} \text{continuous}$$

Nondegenerate case

$\beta(r) > 0$

Degenerate case

$\beta(r) \geq 0$

▶ β is the diffusion coefficient

$\beta : (-\infty, \theta_s) \to \mathbf{R}_+$ continuous

Nondegenerate case

$\beta(r) > 0$

Degenerate case

$\beta(r) \geq 0$

$$\lim_{r \nearrow \theta_s} \beta(r) = +\infty$$

$$\lim_{r \nearrow \theta_s} \int_0^r \beta(\xi) d\xi = K_s^*.$$

$$(K(\theta) - \nabla\beta^*(\theta)) \cdot \nu - \alpha\beta^*(\theta) \ni f_\alpha \text{ on } \Sigma = (0,T) \times \Gamma,$$

$$\alpha : \Gamma \longrightarrow [\alpha_m, \alpha_M]$$

is continuous and it is positive at least on a nonzero measure subset of Γ.

The nondegenerate case with K a general nonlinear Lipschitz function having a nonzero component only along Ox_3 was treated (G.M., 2005, 2006) in relation with the 3D-model of water infiltration in soils.

$$\frac{d\theta}{dt}(t) + A\theta(t) \ni f(t), \text{ a.e. } t \in (0,T),$$

$$\theta(0) = \theta_0.$$

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$$\frac{d\theta_\varepsilon}{dt}(t) + A_\varepsilon\theta(t) \ni f(t), \text{ a.e. } t \in (0,T),$$

$$\theta_\varepsilon(0) = \theta_0.$$

For numerical purposes the method can have no great efficiency due to the fact that

β_ε blows up as $\varepsilon \to 0$.

Purpose of this work:

- Propose time discretization schemes for both
  - the nondegenerate case ($\beta(r) > 0$)
  - the degenerate case ($\beta(r) \geq 0$)
- Study existence and stability of the discretized schemes
- Convergence of the discretized solutions.

Functional framework

$V = H^1(\Omega)$ $\qquad$ $\|\psi\|_V = \left(\int_\Omega |\nabla \psi(x)|^2 \, dx + \int_\Gamma \alpha(x) \, |\psi(x)|^2 \, d\sigma \right)^{1/2}, \ \forall \psi \in V$

V' the dual of V $\qquad$ $(\theta, \overline{\theta})_{V'} = \left\langle \theta, A_\Delta^{-1} \overline{\theta} \right\rangle_{V',V}, \ \ \forall \, \theta, \overline{\theta} \in V'$

$$A_\Delta : V \to V'$$

$$\langle A_\Delta \psi, \phi \rangle_{V',V} = \int_\Omega \nabla \psi \cdot \nabla \phi dx + \int_\Gamma \alpha(x) \psi \phi d\sigma, \ \ \forall \psi, \phi \in V.$$

The abstract Cauchy problem

$$\frac{d\theta}{dt}(t) + A\theta(t) \ni g(t), \text{ a.e. } t \in (0,T), \qquad (\mathrm{O}P)$$

$$\theta(0) = \theta_0.$$

where

$$g = f + f_{\Gamma_\alpha}.$$

$$f_{\Gamma_\alpha} \in L^2(0,T;V')$$

$$f_{\Gamma_\alpha}(t)(\psi) = -\int_\Gamma f_\alpha \psi d\sigma, \text{ for any } \psi \in V.$$

$$A : D(A) \subset V' \to V' \quad \text{multivalued}$$

$$D(A) = \{\theta \in L^2(\Omega);\ \text{there exists } \eta \in V,\ \eta(x) \in \beta^*(\theta(x)) \text{ a.e. } x \in \Omega\},$$

$$\langle A\theta, \psi \rangle_{V',V} = \int_\Omega (\nabla\eta - K(\theta)) \cdot \nabla\psi dx + \int_\Gamma \alpha\eta\psi d\sigma,\ \psi \in V.$$

2 Time discretized schemes for the nondegenerate and degenerate cases

2.1 Hypotheses

Nondegenerate case

$\beta(r) \geq \beta(0) = \rho > 0$ for $r < \theta_s$

Degenerate case

$\beta(r) \geq \beta(0) = 0$ for $r < \theta_s$

Nondegenerate case

$\beta(r) \geq \beta(0) = \rho > 0$ for $r < \theta_s$

$\beta(r) = \rho$ for $r \leq 0$

Degenerate case

$\beta(r) \geq \beta(0) = 0$ for $r < \theta_s$

$\beta(r) \geq c_\beta \, |r|^\gamma$ for $r < 0, \gamma > 0$

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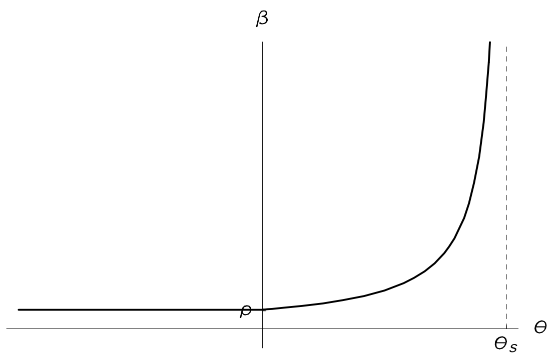
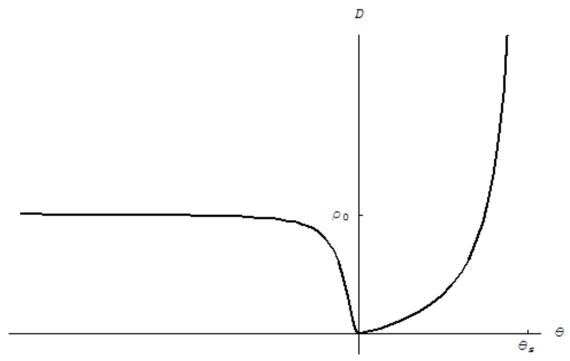


Degenerate case

$\beta(r) \geq \beta(0) = 0$ for $r < \theta_s$

$\beta(r) \geq c_\beta \, |r|^\gamma$ for $r < 0, \gamma > 0$



Nondegenerate case

K_i nonlinear, Lipschitz with the constant M_i

Degenerate case

$K_i(x, r) = a_i(x)r, \quad i = 1, ..., N$

$a_i \in W^{1,\infty}(\Omega)$

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$a \cdot \nu \leq 0$ on Γ

Nondegenerate case

K_i nonlinear, Lipschitz with the constant M_i

$f \in L^2(0,T;V')$

$f_\alpha \in L^2(0,T;L^2(\Gamma))$

Degenerate case

$K_i(x,r) = a_i(x)r, \quad i = 1, ..., N$

$a_i \in W^{1,\infty}(\Omega)$

$a \cdot \nu \leq 0$ on Γ

$f \in L^2(0,T;L^2(\Omega))$

$f_\alpha = 0$

Definition (solution in the nondegenerate case). Let

$$\theta_0 \in L^2(\Omega),\ \theta_0 \leq \theta_s \text{ a.e. } x \in \Omega,\ f \in L^2(0,T;V').$$

A *solution* to (bvp) is a pair (θ, η), satisfying

$$\theta \in C([0,T]; L^2(\Omega)) \cap W^{1,2}(0,T;V') \cap L^2(0,T;V),$$

$$\eta \in L^2(0,T;V), \quad \eta(t,x) \in \beta^*(\theta(t,x)) \text{ a.e. } (t,x) \in Q,$$

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$$\left\langle \frac{d\theta}{dt}(t), \psi \right\rangle_{V',V} + \int_\Omega (\nabla\eta(t) - K(\theta(t))) \cdot \nabla\psi dx$$

$$= \langle g(t), \psi \rangle_{V',V} - \int_\Gamma \alpha\eta(t)\psi d\sigma,\ \text{a.e. } t \in (0,T),\ \forall\psi \in V,$$

$$\theta(0,x) = \theta_0 \text{ in } \Omega,$$

$$\theta(t,x) \le \theta_s \text{ a.e. } (t,x) \in Q.$$

Definition (solution in the degenerate case). Let

$$\theta_0 \in L^2(\Omega),\ \theta_0 \le \theta_s \text{ a.e. } x \in \Omega,\ f \in L^2(0,T;L^2(\Omega)).$$

A *solution* to (bvp) is a pair (θ, η), satisfying

$$\theta \in C([0,T];V')) \cap W^{1,2}(0,T;V'),$$

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$$= \langle f(t), \psi \rangle_{V',V} - \int_\Gamma \alpha\eta(t)\psi d\sigma,\ \text{a.e. } t \in (0,T),\ \forall \psi \in V,$$

$$\theta(0,x) = \theta_0 \text{ in } \Omega,$$

$$\theta(t,x) \le \theta_s \text{ a.e. } (t,x) \in Q.$$

2.2 Stability of the discretization schemes

Let

$$D_A^h(0 = t_0 \le t_1 \le t_2 \le \ldots \le t_n; g_1^h, \ldots, g_n^h), \qquad h = \frac{T}{n},$$

$$g_i^h = \frac{1}{h}\int_{(i-1)h}^{ih} g(s)ds.$$

In the degenerate case

$$g = f.$$

Nondegenerate case

$$\left(\frac{1}{h}I + A\right)\theta_i^h \ni g_i^h + \frac{1}{h}\theta_{i-1}^h$$

$$A : D(A) \subset V' \to V'$$

Degenerate case

$$\left(\frac{1}{h}I + A^h\right)\theta_i^h \ni f_i^h + \frac{1}{h}\theta_{i-1}^h$$

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$$A^h : D(A^h) \subset V' \to V'$$

$$\langle A^h\theta, \psi\rangle_{V',V} = \int_\Omega \left(\nabla\eta + \sqrt{h}\nabla\theta - a(x)\theta\right)\cdot\nabla\psi dx + \int_\Gamma \alpha(\eta + \sqrt{h}\theta)\psi d\sigma,\ \psi \in V$$

$$D(A^h) = \{\theta \in V;\ \text{there exists } \eta \in V,\ \eta(x) \in \beta^*(\theta(x)) \text{ a.e. } x \in \Omega\}.$$

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M.G. Crandall and T.M. Liggett (1971). Generation of semigroups of nonlinear transformations in general Banach spaces. Amer. J. Math. 93: 265-298.

M.G. Crandall and L.C. Evans (1975). On the relation of the operator $\partial/\partial s + \partial/\partial t$ to evolution governed by accretive operators. Israel J. Math. 21: 261-278.

Y. Kobayashi (1975). Difference approximation of Cauchy problem for quasi-dissipative operators and generation of nonlinear semigroups. J. Math. Soc. Japan 27: 641-663.

This work

- $\gg$ specifies the precise nature of the convergence
- $\gg$ computes the error
- $\gg$ indicates a numerical algorithm without approximating the multivalued function β^*.

Quasi $m-$accretiveness

Under the appropriate hypotheses made for each case

Nondegenerate case

A *is quasi* m*-accretive*

Degenerate case

A^h *is quasi* m*-accretive for each* $h > 0$

Quasi $m-$accretiveness

Under the appropriate hypotheses made for each case

Nondegenerate case		Degenerate case
A *is quasi* m*-accretive*		A^h *is quasi* m*-accretive for each* $h > 0$
	If	
$h < \frac{\rho}{kM^2}$		$h < \frac{1}{k^2 M^4}$
	then	
$\frac{1}{h}I + A$ *is invertible*		$\frac{1}{h}I + A^h$ *is invertible*
$M = \sum_{i=1}^{N} M_i, \quad k > 1$		$M = \\|a\\|_\infty, \quad k > 1.$

Stability of the scheme

Each discretization scheme has a unique solution θ_i^h and it is stable, i.e.,

$$\left\|\theta_p^h\right\|^2 \leq C,$$

$$h \sum_{i=1}^{p} \left\|\eta_i^h\right\|_V^2 \leq C,$$

$$h \sum_{i=1}^{p} \left\|\frac{\theta_i^h - \theta_{i-1}^h}{h}\right\|_{V'}^2 \leq C,$$

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$$h\sum_{i=1}^{p}\left\|\theta_i^h\right\|_V^2 \leq C \qquad \sqrt{h}h\sum_{i=1}^{p}\left\|\theta_i^h\right\|_V^2 \leq C$$

$$for\ any\ p = 1, ..., n$$

where C denotes several constants independent on p and h.

2.3 Convergence of the discretization schemes

We define

$$\begin{aligned} \theta^h(t,x) &= \theta_i^h(x), \quad \text{for } t \in ((i-1)h, ih], \\ \eta^h(t,x) &= \eta_i^h(x), \quad \text{for } t \in ((i-1)h, ih], \\ g^h(t,x) &= g_i^h(x), \quad \text{for } t \in ((i-1)h, ih], \end{aligned}$$

for $i = 1, ..., n$.

Theorem (convergence of the discretized schemes)

Under the appropriate hypotheses, the original problem (OP) *has at least a solution*

$\theta \in C([0,T];L^2(\Omega)) \cap W^{1,2}(0,T;V') \cap L^2(0,T;V)$ $\qquad$ $\theta \in C([0,T];V') \cap W^{1,2}(0,T;V')$

Theorem (convergence of the discretized schemes)

Under the appropriate hypotheses, the original problem (OP) *has at least a solution*

$\theta \in C([0,T]; L^2(\Omega)) \cap W^{1,2}(0,T;V') \cap L^2(0,T;V)$

$\theta = \lim_{h\to 0} \theta^h$ *strongly in* $L^2(0,T;L^2(\Omega))$

$\theta \in C([0,T]; V') \cap W^{1,2}(0,T;V')$

$\theta = \lim_{h\to 0} \theta^h$ *weakly in* $L^2(0,T;L^2(\Omega))$

$$as\ h \to 0.$$

Sketch of the proof.

$$\left\|\theta^h(t)\right\| \leq C \text{ for any } t \in (0,T),$$

$$\int_0^T \left\|\eta^h(t)\right\|_V^2 dt \leq C,\ \eta^h(t,x) \in \beta^*(\theta^h(t,x)) \text{ a.e. } (t,x) \in Q,$$

$$\int_0^T \left\|\frac{\theta^h(t) - \theta^h(t-h)}{h}\right\|_{V'}^2 dt \leq C,$$

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$$\int_0^T \left\|\theta^h(t)\right\|_V^2 dt \le C$$

$$h^{\frac{1}{2}} \int_0^T \left\|\theta^h(t)\right\|_V^2 dt \le C$$

$\Longrightarrow \theta^h$ is a h-approximate solution to the Cauchy problem (OP).

We can select a subsequence of $\{\theta^h\}_{h>0}$ such that

$\theta^h \to \theta$ weak-star in $L^\infty(0,T;L^2(\Omega))$ as $h \to 0$,

$\eta^h \to \eta$ weakly in $L^2(0,T;V)$ as $h \to 0$, $\eta \in \beta^*(\theta^h)$ a.e. in Q,

$$\frac{\theta^h(t)-\theta^h(t-h)}{h} \to \frac{d\theta}{dt} \text{ weakly in } L^2(0,T;V') \text{ as } h \to 0.$$

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Helly's theorem

$$\theta^h(t) \to \theta(t) \text{ strongly in } V' \text{ for } t \in [0,T] \text{ as } h \to 0.$$

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Lions' lemma

$\theta^h \to \theta$ strongly in $L^2(0,T;L^2(\Omega))$

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$\theta^h \to \theta$ weakly in $L^2(0,T;V)$

$h^{\frac{1}{4}}\theta^h \to \kappa$ weakly in $L^2(0,T;V)$

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$\eta \in \beta^*(\theta)$ a.e. on Q

(G.M., 2006)

$K_i(\theta^h) \to K_i(\theta)$ strongly in $L^2(0,T;L^2(\Omega))$

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$\eta \in \beta^*(\theta)$ a.e. on Q

(G.M., 2006)

$$\limsup_{h\to 0} \int_0^T \left(\eta^h(t), \theta^h(t)\right)_{L^2(\Omega)} dt \le \int_0^T (\eta(t), \theta(t))_{L^2(\Omega)} dt$$

$K_i(\theta^h) \to K_i(\theta)$ strongly in $L^2(0,T;L^2(\Omega))$

$a_i\theta^h \to a_i\theta$ weakly in $L^2(0,T;L^2(\Omega))$

$$\int_0^T \left\langle \frac{\theta^h(t)-\theta^h(t-h)}{h}, \phi(t) \right\rangle_{V',V} dt \quad + \int_Q (\nabla \eta^h - K(\theta^h)) \cdot \nabla \phi dx dt$$
$$+ \int_\Sigma \alpha(x) \eta^h \phi d\sigma dt \; = \; \int_0^T \left\langle g^h(t), \phi(t) \right\rangle_{V',V} dt, \; \forall \phi \in L^2(0,T;V)$$

$$\int_0^T \left\langle \frac{\theta^h(t)-\theta^h(t-h)}{h}, \phi(t) \right\rangle_{V',V} dt \quad + \int_Q (\nabla \eta^h - K(\theta^h)) \cdot \nabla \phi dx dt$$
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We pass to the limit as $h \to 0$, and deduce that

$$\int_0^T \left\langle \frac{d\theta}{dt}(t), \phi(t) \right\rangle_{V',V} dt \quad + \int_Q (\nabla \eta - K(\theta)) \cdot \nabla \phi dx dt$$
$$+ \int_\Sigma \alpha(x) \eta \phi d\sigma dt \; = \; \int_0^T \langle g(t), \phi(t) \rangle_{V',V} dt, \; \forall \phi \in L^2(0,T;V).$$

$$\int_0^T \left\langle \frac{\theta^h(t) - \theta^h(t-h)}{h}, \phi(t) \right\rangle_{V',V} dt + \int_Q (\nabla \eta^h - a(x)\theta^h) \cdot \nabla \phi dx dt + \int_\Sigma \alpha \eta^h \phi d\sigma dt$$

$$h^{1/4} \int_Q h^{1/4} \nabla \theta^h \cdot \nabla \phi dx dt + h^{1/4} \int_\Sigma \alpha(x) h^{1/4} \theta^h \phi d\sigma dt$$

$$= \int_0^T \left\langle g^h(t), \phi(t) \right\rangle_{V',V} dt, \ \forall \phi \in L^2(0,T;V)$$

$$\int_0^T \left\langle \frac{\theta^h(t) - \theta^h(t-h)}{h}, \phi(t) \right\rangle_{V',V} dt + \int_Q (\nabla \eta^h - a(x)\theta^h) \cdot \nabla \phi dx dt + \int_\Sigma \alpha \eta^h \phi d\sigma dt$$
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$$\int_0^T \left\langle \frac{d\theta}{dt}(t), \phi(t) \right\rangle_{V',V} dt + \int_Q (\nabla \eta - a(x)\theta) \cdot \nabla \phi dx dt + \int_\Sigma \alpha(x) \eta \phi d\sigma dt$$
$$= \int_0^T \langle g(t), \phi(t) \rangle_{V',V} dt, \ \forall \phi \in L^2(0,T;V).$$

2.4 Uniqueness

Under the assumptions of the nondegenerate case the solution to the N-D problem (OP) *is unique*.

Under the assumptions of the nondegenerate case the solution to the N*-D problem* (OP) *is unique*.

Under the assumptions of the degenerate case the solution to the N*-D problem* (OP) *is unique if* $K = 0$.

Under the assumptions of the nondegenerate case the solution to the N*-D problem* (OP) *is unique*.

Under the assumptions of the degenerate case the solution to the N*-D problem* (OP) *is unique if* $K = 0$.

Under the assumptions of the degenerate case the solution to problem (OP) *is unique if* $N = 1$ *and*

$$a(x) = 0 \text{ on } \Gamma.$$

2.5 Error estimate

Nondegenerate case

$$\left\|\theta(t_i) - \theta_i^h\right\|_{V'} = O(h^{1/4}) \; as \; h \to 0,$$

$$\int_0^T \left\|\theta(t) - \theta^h(t)\right\|^2 dt \le O(h^{1/2}) \; as \; h \to 0,$$

Degenerate case

$N = 1$ and $a(x) = 0$ on Γ

$$\left\|\theta(t_i) - \theta_i^h\right\|_{V'} = O(h^{1/4}) \; as \; h \to 0,$$

Time step estimate

Nondegenerate case

$$h < \frac{1}{2k}\frac{\rho}{M^2}$$

Degenerate case

$$h < \min\left\{\frac{1}{k^2M^4}, \frac{1}{1+\|a\|_{1,\infty}}\right\}$$

Remark: another scheme in the degenerate case

If

$$\|a\|_\infty < 1$$

then, instead of

$$\begin{aligned}\frac{\theta_i^h - \theta_{i-1}^h}{h} - \sqrt{h}\Delta\theta_i^h - \Delta\eta_i^h + \nabla\cdot K(\theta_i^h) &= f_i^h \quad \text{in } \Omega,\ i = 1, ..., n\\ \left(a(x)\theta_i^h - \sqrt{h}\nabla\theta_i^h - \nabla\eta_i^h\right)\cdot\nu &= \alpha\eta_i^h + \sqrt{h}\alpha\theta_i^h \quad \text{on } \Gamma,\end{aligned}$$

Remark: another scheme in the degenerate case

If

$$\|a\|_\infty < 1$$

we can consider

$$\begin{aligned} \frac{\theta_i^h - \theta_{i-1}^h}{h} - h\Delta\theta_i^h - \Delta\eta_i^h + \nabla \cdot K(\theta_i^h) &= f_i^h \quad \text{in } \Omega,\ i = 1, ..., n \\ \big(a(x)\theta_i^h - h\nabla\theta_i^h - \nabla\eta_i^h\big) \cdot \nu &= \alpha\eta_i^h + h\alpha\theta_i^h \quad \text{on } \Gamma, \end{aligned}$$

and all result remain valid.

Time discretized systems: `algorithms`

Nondegenerate case

$$\frac{\theta_i^h-\theta_{i-1}^h}{h} - \Delta\eta_i^h + \nabla\cdot K(\theta_i^h) = f_i^h \quad \text{in } \Omega$$

$$\big(K(\theta_i^h) - \nabla\eta_i^h\big)\cdot\nu = \alpha\eta_i^h + f_{\alpha,i}^h \quad \text{on } \Gamma$$

Degenerate case

$$\frac{\theta_i^h-\theta_{i-1}^h}{h} - \Delta\eta_i^h - \sqrt{h}\Delta\theta_i^h + \nabla\cdot\big(a(x)\theta_i^h\big) = f_i^h$$

$$\Big(a(x)\theta_i^h - \nabla\eta_i^h - \sqrt{h}\nabla\theta_i^h\Big)\cdot\nu = \alpha\eta_i^h + \sqrt{h}\alpha\theta_i^h$$

$$\eta_i^h(x) \in \beta^*(\theta_i^h(x)) \text{ a.e. } x\in\Omega,$$

$$\theta_0^h = \theta_0 \text{ in } \Omega.$$

2.6 Algorithm in the nondegenerate case

$$\zeta_i^h \in \beta^*(\theta_i^h) \qquad G(\zeta_i^h) := (\beta^*)^{-1}(\zeta_i^h) \qquad K_G(\zeta_i^h) := K(G(\zeta_i^h))$$

where

$$G(r) := \begin{cases} (\beta^*)^{-1}(r) & \text{if } r < K_s^* \\ \theta_s & \text{if } r \geq K_s^*. \end{cases}$$

We are led to solve the following elliptic boundary value problem

$$G(\zeta_i^h) - h\Delta\zeta_i^h + h\nabla \cdot K_G(\zeta_i^h) \;=\; \int_{t_{i-1}}^{t_i} g(s)ds + \theta_{i-1}^h \quad \text{in } \Omega,\ i = 1, ..., n,$$

$$h(K_G(\zeta_i^h) - \nabla\zeta_i^h) \cdot \nu \;=\; h\alpha\zeta_i^h + \int_{t_{i-1}}^{t_i} g(s)ds \quad \text{on } \Gamma,$$

and set

$$\theta_i^h := \begin{cases} (\beta^*)^{-1}(\zeta_i^h) & \text{if } \zeta_i^h < K_s^* \\ \theta_s & \text{if } \zeta_i^h \geq K_s^*. \end{cases}$$

2.7 Algorithm in the degenerate case

$$\zeta_i^h \in \widetilde{\beta}^*(\theta_i^h) = \beta^*(\theta_i^h) + \sqrt{h}\theta_i^h \qquad G(\zeta_i^h) := (\widetilde{\beta}^*)^{-1}(\zeta_i^h) \qquad K_G(\zeta_i^h) := a(x)G(\zeta_i^h)$$

$$G(r) := \begin{cases} (\widetilde{\beta}^*)^{-1}(r) & \text{if } r < K_s^* + \sqrt{h}\theta_s \\ \theta_s & \text{if } r \geq K_s^* + \sqrt{h}\theta_s. \end{cases}$$

We are led to solve the following elliptic boundary value problem

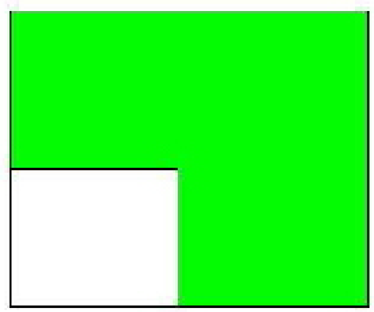
$$G(\zeta_i^h) - h\Delta\zeta_i^h + h\nabla \cdot K_G(\zeta_i^h) = \int_{t_{i-1}}^{t_i} f(s)ds + \theta_{i-1}^h \quad \text{in } \Omega, \ i = 1, ..., n,$$

$$h(K_G(\zeta_i^h) - \nabla\zeta_i^h) \cdot \nu = h\alpha\zeta_i^h \quad \text{on } \Gamma,$$

and set

$$\theta_i^h := \begin{cases} (\widetilde{\beta}^*)^{-1}(\zeta_i^h) & \text{if } \zeta_i^h < K_s^* + \sqrt{h}\theta_s \\ \theta_s & \text{if } \zeta_i^h \geq K_s^* + \sqrt{h}\theta_s. \end{cases}$$

3 Numerical results



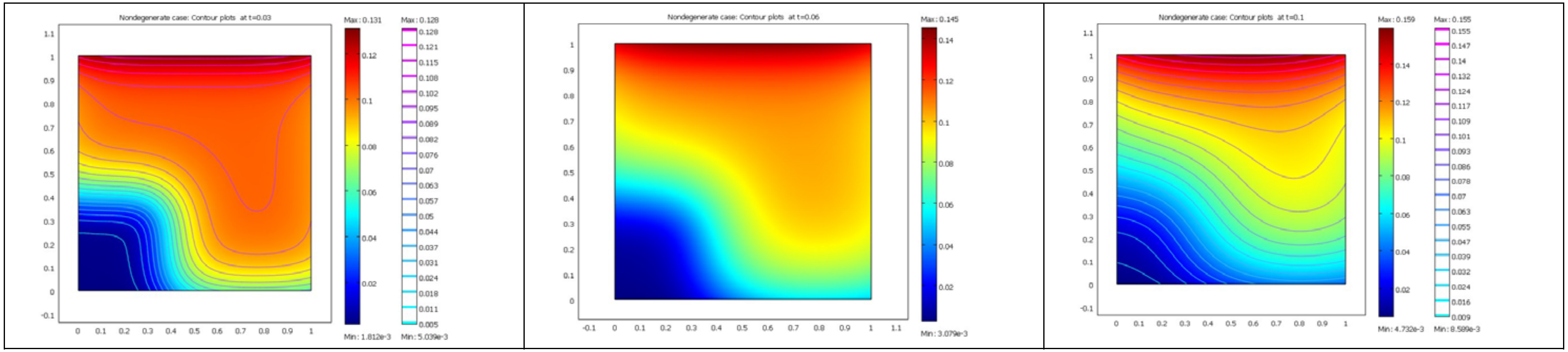
$$\theta_0(x,y) = \begin{cases} 0, \text{ on } \{(x,y); 0 \le x \le 0.4,\ 0 \le y \le 0.4\} \\ 0.1 \text{ otherwise.} \end{cases}$$

$$f = 0.1,\ \alpha = 1,\ f_\alpha = 0, \quad \text{on } \Sigma, \quad h = 10^{-4}$$

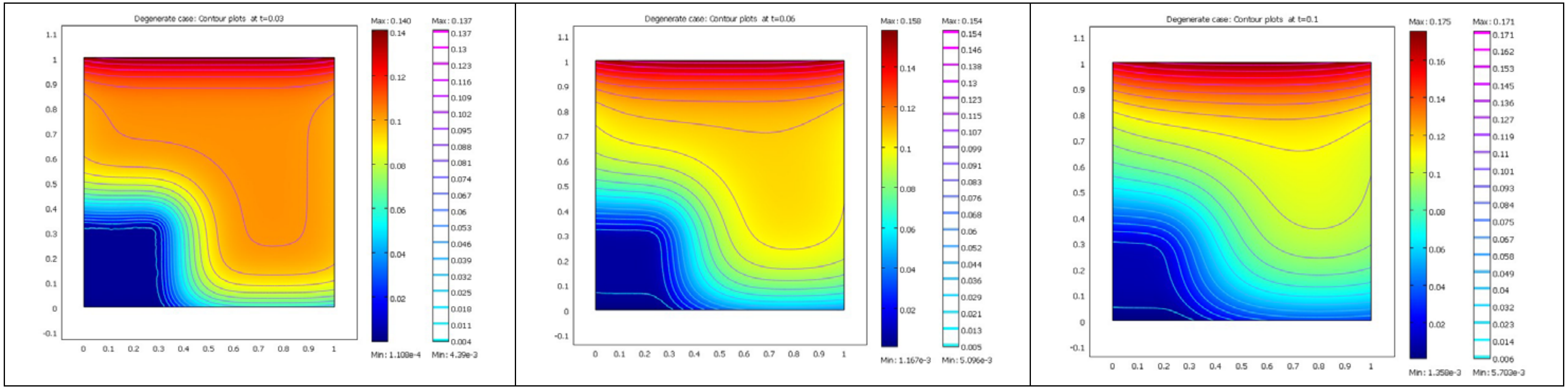
$$\beta_{\text{non deg}}(r) = 2r + 0.1 \qquad \text{for } r \in [0,1) \qquad \beta_{\deg}(r) = 2r$$

$$K(r) = \Big\{ a(x,y)r,\ r \in [0,1]\ a(x,y) = \Big\{ 1, \text{ in } \mathring{\Omega};\ 0, \text{ otherwise}$$

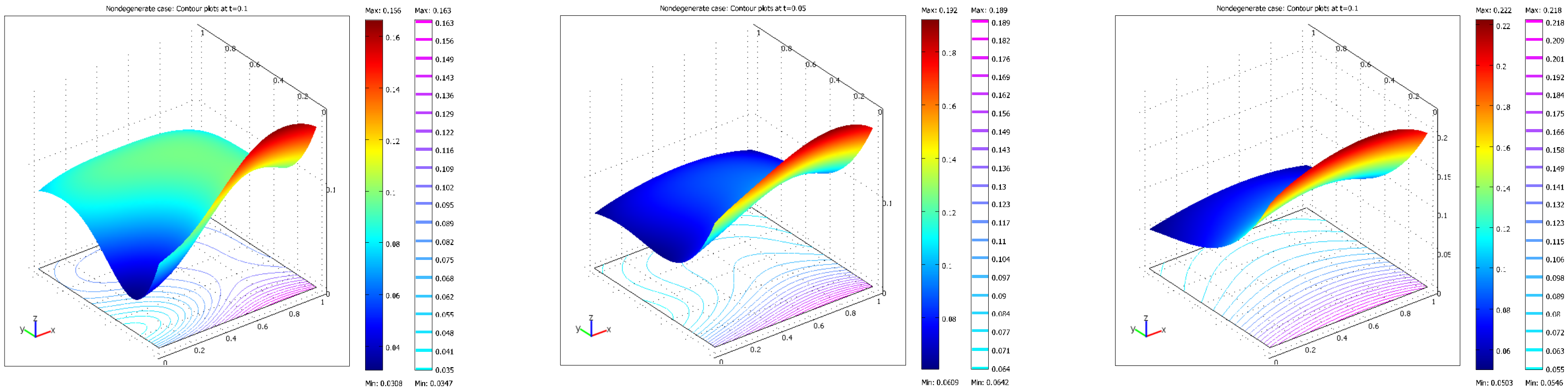
Nondegenerate case



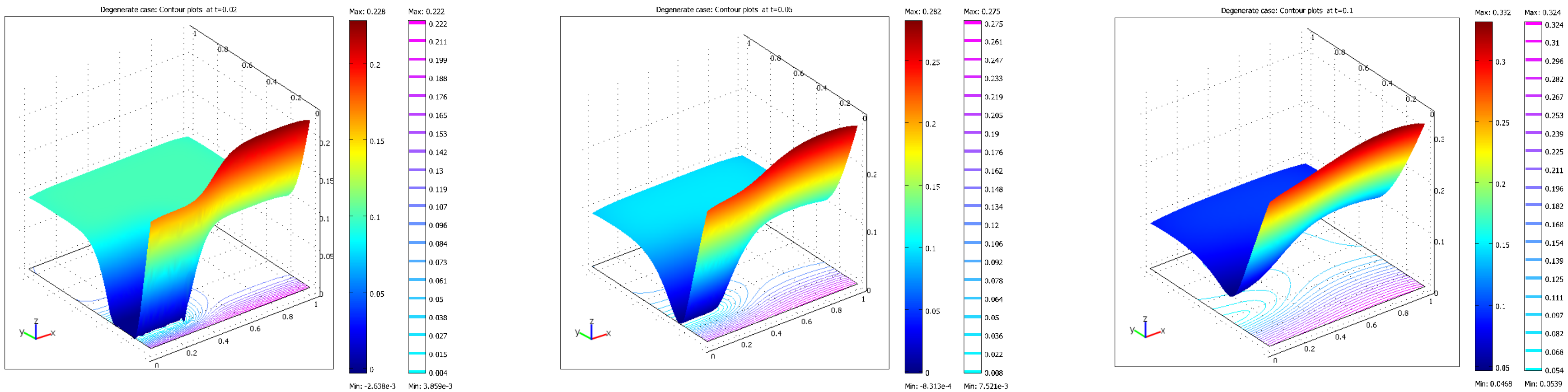
Degenerate case



Nondegenerate case



Degenerate case



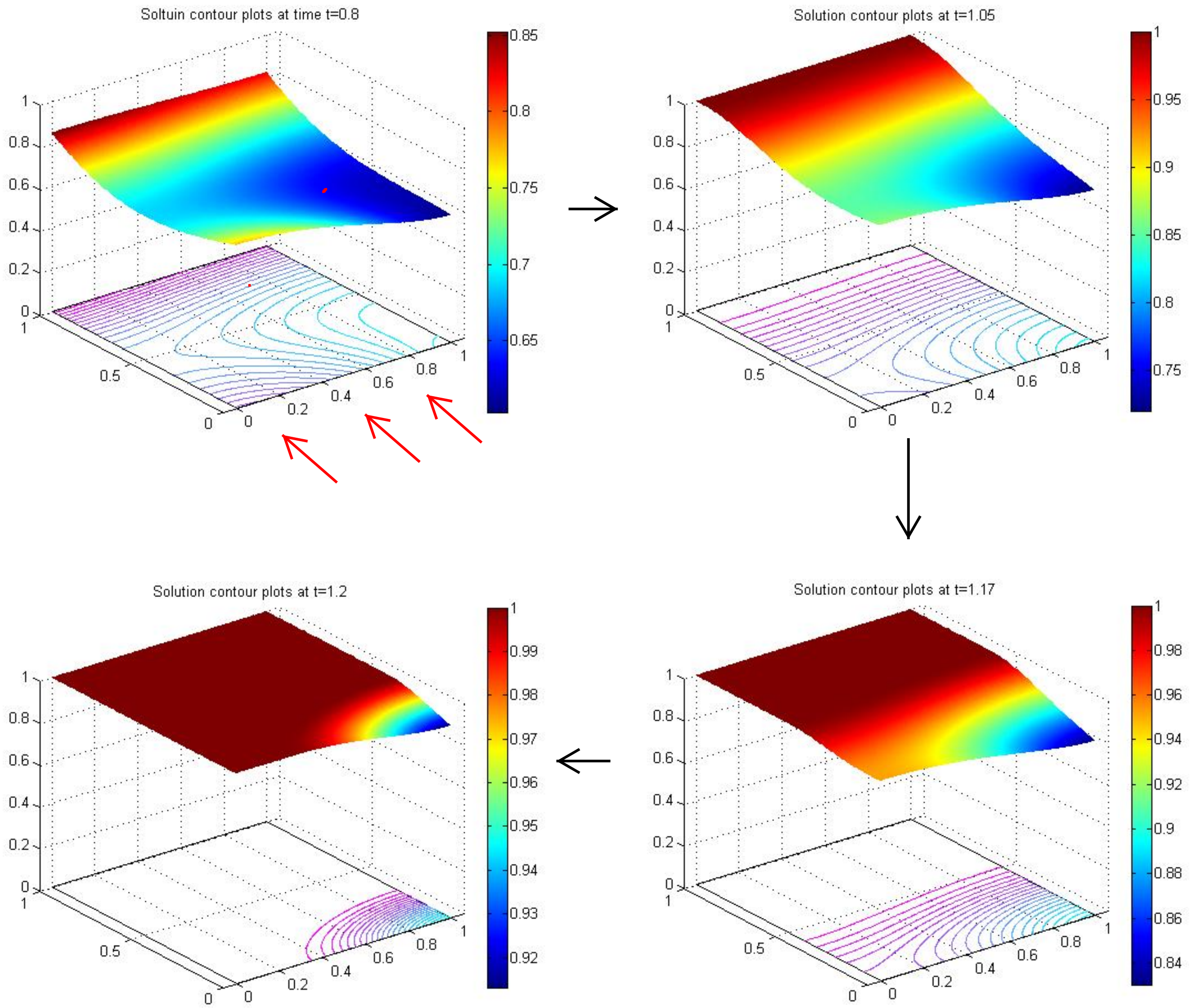
$$\beta(r) = \frac{1}{2\sqrt{1-r}} \text{ for } r \in [0,1), \quad \beta^*(r) = \begin{cases} 1-\sqrt{1-r}, & r \in [0,1) \\ [1,\infty), & r = 1. \end{cases}$$

$$\theta_0 = 0.1,$$

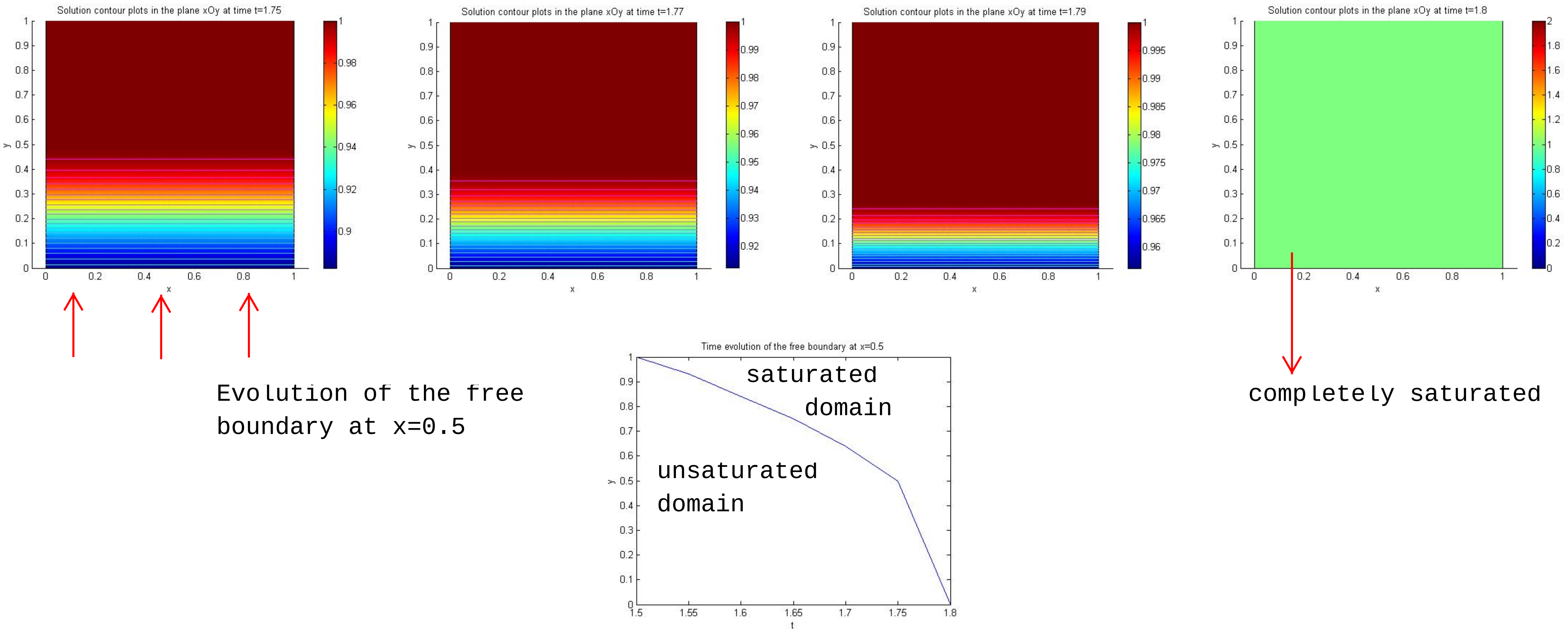
$$f = 0,$$

$$f_\alpha = 0, \ \alpha = 10^{-8}.$$

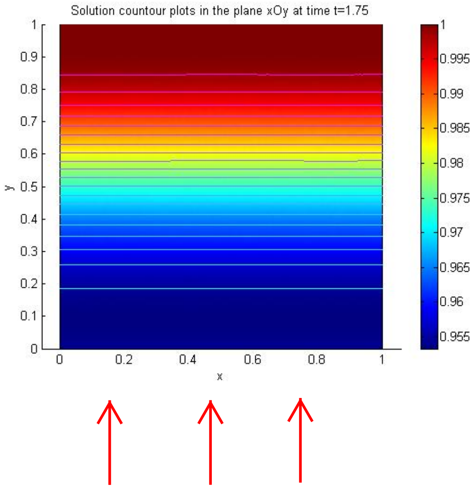
Evolution of the saturated domain

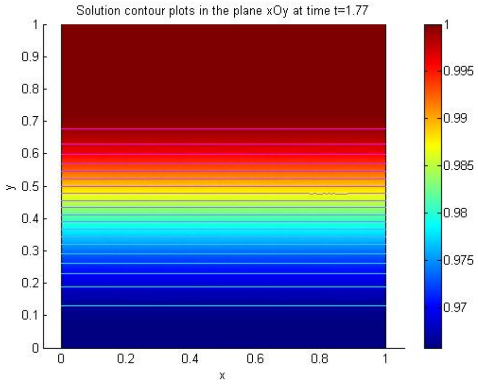


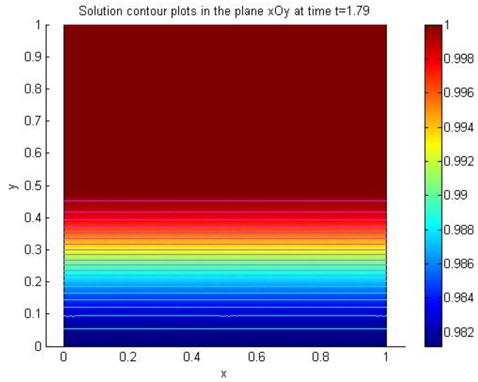
$$L=1,\ K(\theta)=\begin{cases}\dfrac{e^{r}-1}{e-1}, & r\in[0,1)\\ 1, & r=1.\end{cases}\ \text{and}\ u=0.5$$

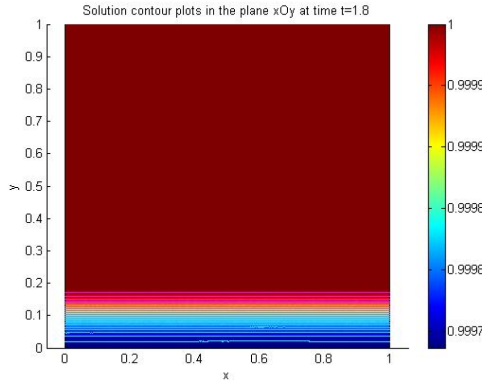


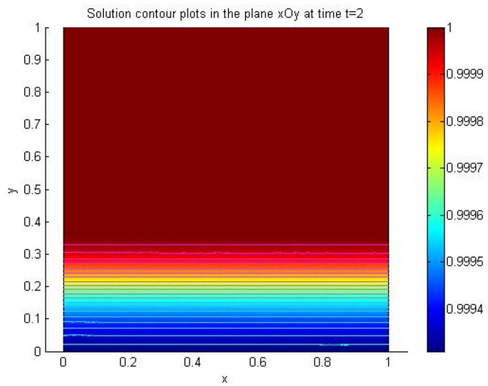
$$L = 1,\ K(\theta) = \frac{\theta^2}{2} \text{ and } u = 0.5$$

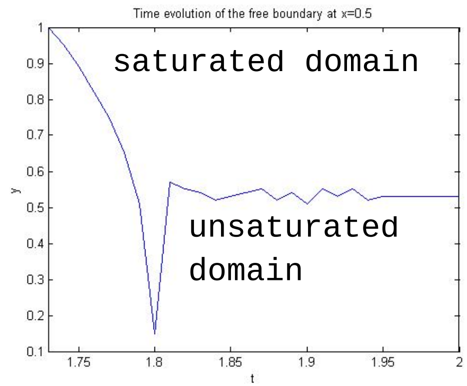












Evolution of the free boundary at x=0.5

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Thank you for your attention !