Null controllability properties of some degenerate parabolic equations in space dimension 2

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Presentation of the problem

Ω bounded, smooth domain of $\mathbb{R}^2$ $(\mathbb{R}^n)$; ω nonempty open subset of Ω.

$$A : \overline{\Omega} \to S_2(\mathbb{R}), \quad A(x) \geq 0,$$

but non uniformly positive :

$$\forall x \in \partial\Omega, \quad \det(A(x)) = 0.$$

Null controllability properties of

$$\begin{cases} u_t - \text{div } (A(x)\nabla u) = h(x,t)\chi_\omega, \\ \text{boundary conditions,} \\ \text{initial condition} \end{cases} \qquad ?$$

(First step to exact controllability to trajectories...)
Uniformly positive case : heat equation Lebeau-Robbiano (95), general case : Fursikov-Imanuvilov (95,96)

Some examples in 1D

- aeronautics : the Crocco type equation (boundary layer model) :

  $$u_t + a(y)u_x - (b(y)u_y)_y = \text{localized control}, x \in (0, L), y(0, 1)$$

  with $a(1) = 0 = b(1)$;
- climatology : the Budyko-Sellers model :

  $$RT_t - ((1 - x^2)T_x)_x - QS(1 - \alpha) = -I(T), x \in (-1, 1);$$
- economics : the Black-Scholes equation of the type :

  $$u_t - x^2 u_{xx} + \cdots = \cdots, x \in (0, L).$$

An example in N-D

- biology : the Fleming Viot model :

$$u_t - \text{Tr}\,(C(x)D^2 u) \cdots = f,$$

where

$$C(x) = (c_{ij}(x))_{i,j}, \quad c_{ij}(x) = x_i(\delta_{ij} - x_j),$$

and

$$x \in \{x_i \in [0,1], \sum_i x_i \leq 1\};$$

example : N=2 : degenerate along the sides of the triangle.

Main results on the 1D degenerate problems : the simplest problem

The simplest problem in divergence form (Cannarsa, Martinez, Vancostenoble (2008)) :

$$u_t - (x^\alpha u_x)_x = h(x,t)\chi_{(a,b)}, x \in (0,1), t > 0 :$$

- $\alpha \in [0,1[$: well-posed with the Dirichlet boundary condition $(u(0,t) = 0 = u(t,1)$, and null controllable ;
- $\alpha \in [1,2[$: well-posed with the Neumann boundary condition $(x^\alpha u_x)(0,t) = 0 = u(1,t)$, and null controllable ;

Main tools : Carleman estimates associated to the degenerate problem, and Hardy type inequalities ;

- $\alpha \geq 2$: well-posed with the Neumann boundary condition $(x^\alpha u_x)(0,t) = 0 = u(1,t)$, and not null controllable ;
  Main tools : application of a result of Escauriaza, Seregin, Sverak (2004) related to the backward uniqueness properties of the heat equation in half space.
  Remark : Strong connection between degenerate problems in bounded domains and nondegenerate problems in unbounded domains

Main results on the 1D degenerate problems : other problems

- in divergence form (Martinez, Vancostenoble (2006)) :

$$u_t - (a(x)u_x)_x = \text{localized control},$$

  with $a(0) = 0 = a(1)$;
- with semilinear terms (Alabau Boussouira, Cannarsa, Fragnelli (2006)) :

$$u_t - (a(x)u_x)_x + f(u) = \text{localized control},$$

- in nondivergence form, with drift (Cannarsa, Fragnelli, Rocchetti (2007))

$$u_t - a(x)u_{xx} + b(x)u_x = \text{localized control}.$$

The problem in 2D : simplest assumptions on the degeneracy

$$A(x) \sim \begin{pmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(x) \end{pmatrix} \quad \text{with } 0 \leq \lambda_1(x) \leq \lambda_2(x).$$

$\varepsilon_1(x)$ denotes the eigenvector associated to $\lambda_1(x)$, and $\varepsilon_2(x)$ the eigenvector associated to $\lambda_2(x)$.

Simplest assumptions $(H_s(A))$: Ω is of class C^4, and there exists some $\alpha \geq 0$, and some neighborhood of the boundary Γ such that :

- $\lambda_1(x) = d(x, \Gamma)^\alpha$ for all $x \in \mathcal{V}(\Gamma)$,
- $\varepsilon_1(x) = \nu(p_\Gamma(x))$ for all $x \in \mathcal{V}(\Gamma)$, where $p_\Gamma(x)$ is the projection of x on the boundary Γ,
- $0 < m \leq \lambda_2(x) \leq M$ for all $x \in \overline{\Omega}$.

More general assumptions on the degeneracy

$(H_g(A))$: there exists some $\alpha \geq 0$ such that :

- $\lambda_1(x) \sim d(x,\Gamma)^\alpha$ as $x \to \Gamma$,
- $\varepsilon_1(x) - \nu(p_\Gamma(x)) \to 0$ as $x \to \Gamma$,
- $0 < m \leq \lambda_2(x) \leq M$ for all $x \in \overline{\Omega}$.

(work in progress)

The weakly degenerate control problem

We consider

$$\begin{cases} u_t - \text{div } (A(x)\nabla u) = h(x,t)\chi_\omega, \\ \text{boundary conditions,} \\ \text{initial condition,} \end{cases}$$

under $(H_s(A))$ (resp. $(H_g(A))$), with $\alpha \in [0,1)$.

Question : given $T > 0$, $u_0 \in ??$, does there exist $h \in ??$ such that $u(T) = 0$?

The functional setting for the well-posedness

$$H^1_A(\Omega) := \{u \in L^2(\Omega) \cap H^1_{\text{loc}}(\Omega), A\nabla u \cdot \nabla u \in L^1(\Omega)\},$$
$$= \{u \in L^2(\Omega) \cap H^1_{\text{loc}}(\Omega), \int_{\mathcal{V}(\Gamma)} d(x,\Gamma)^\alpha (\nabla u, \varepsilon_1)^2 + (\nabla u, \varepsilon_2)^2 < \infty\},$$

$$H^2_A(\Omega) := \{u \in H^1_A(\Omega) \cap H^2_{\text{loc}}(\Omega),\ \text{div}\ (A\nabla u) \in L^2(\Omega)\},$$

endowed with their natural norms.

Proposition

- *$H^1_A(\Omega)$ and $H^2_A(\Omega)$ are Hilbert spaces ; $\mathcal{C}^\infty(\overline{\Omega})$ is dense in both ;*
- *the trace operator $\gamma : H^1(\Omega) \to L^2(\Gamma)$ can be extended into $\gamma_A : H^1_A(\Omega) \to L^2(\Gamma)$;*
- *$H^1_{A,0}(\Omega) := \overline{\mathcal{D}(\Omega)}^{H^1_A} = \{u \in H^1_A(\Omega), \gamma_A(u) = 0\}$.*

The functional setting : regularity up to the boundary

Difficulty : $H^2(\Omega) \neq H^2_A(\Omega)$.
Directional derivative : $\partial_{\varepsilon_i} u := (\nabla u, \varepsilon_i)$; then

Proposition

For all $u \in H^2_A(\Omega)$, we have :

$$\partial_{\varepsilon_1}\Big(d(x,\Gamma)^\alpha \partial_{\varepsilon_1} u\Big), \quad d(x,\Gamma)^{\alpha/2}\partial^2_{\varepsilon_1,\varepsilon_2} u, \quad \partial^2_{\varepsilon_2} u \in L^2(\Omega).$$

- a more general study (weakened conditions on λ_1, λ_2) can be found in the thesis of D. Rocchetti (2008) ;
- this regularity is essential in the study of the null controllability problem

Well-posedness

Under assumptions $(H_s(A))$:

Proposition

The unbounded operator $(\mathcal{A}_1, D(\mathcal{A}_1))$ defined by

$$D(\mathcal{A}_1) := H_A^2(\Omega) \cap H_{A,0}^1(\Omega),$$

and

$$\forall u \in D(\mathcal{A}_1), \quad \mathcal{A}_1 u = \mathrm{div}(A \nabla u),$$

is m-dissipative and self-adjoint, with dense domain in $L^2(\Omega)$.

Therefore $(\mathcal{A}_1, D(\mathcal{A}_1))$ generates a strongly C_0-semi-group in $L^2(\Omega)$ that can be proved to be analytic.

Well-posedness

Consequently :

Proposition

Let $h \in L^2(\Omega_T)$ *be given. Then for all* $u_0 \in L^2(\Omega)$, *the problem*

$$\begin{cases} u_t - div\,(A(x)\nabla u) = h(x,t), x \in \Omega, t > 0 \\ u(t,x) = 0 \text{ on } \Gamma, \\ u(0,x) = u_0(x) \end{cases}$$

has a unique mild solution satisfying

$$u \in \mathcal{C}^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1_{A,0}(\Omega)).$$

Moreover, if $u_0 \in H^2_A(\Omega) \cap H^1_{A,0}(\Omega)$, *then*

$$u \in \mathcal{C}^0([0,T]; H^2_A(\Omega) \cap H^1_{A,0}(\Omega)) \cap H^1(0,T; H^1_{A,0}(\Omega)).$$

Null controllability result

Theorem

Assume that A satisfies Hypothesis $(H_s(A))$. Let $T > 0$ be given and consider ω any non-empty open subset of Ω. Then for all $u_0 \in L^2(\Omega)$, there exists $h \in L^2(\Omega_T)$ such that the solution u of

$$\begin{cases} u_t - div\,(A(x)\nabla u) = h(x,t)\chi_\omega, \quad x \in \Omega, t > 0 \\ u(t,x) = 0 \text{ on } \Gamma, \\ u(0,x) = u_0(x) \end{cases}$$

satisfies $u(T,\cdot) = 0$ in $L^2(\Omega)$.

Associated observability estimate

The null controllability result derives from the following observability estimate :

Theorem

Under $(H_s(A))$, there is some $C(\alpha, T)$ such that, given $v_T \in L^2(\Omega)$, the solution v of the adjoint problem

$$\begin{cases} v_t + div\,(A(x)\nabla v) = 0, & x \in \Omega, t > 0 \\ v(t,x) = 0 \text{ on } \Gamma, \\ v(T,x) = v_T(x) \end{cases}$$

satisfies :

$$\int_\Omega v(0,x)^2\,dx \leq C(\alpha, T) \int_0^T \int_\omega v(t,x)^2\,dx\,dt.$$

Hence : usual observability estimate, *but in the degenerate context*.

Extensions and other results

As in the 1D case :

- extension to the weakened assumptions $(H_g(A))$;
- null controllability result in the strongly degenerate case $\alpha \in [1, 2[$ (with the generalized Neumann boundary condition $A\nabla u \cdot \nu = 0$ on Γ) ;
- negative result in the case $\alpha \geq 2$.

Main steps in the proof of the positive controllability result

The observability results derives from a Carleman estimate related to the degenerate problem :

- awful computations,
- some useful tools : a special geometrical lemma, a special Hardy type inequality,
- awful computations

What does not help :

- the operator is degenerate,
- the solution is not regular enough up to the boundary,

What helps :

- the degeneracy occurs only on the boundary, in a very specified way ($(H_s(A))$ or $(H_g(A))$),
- the solution has some regularity up to the boundary.

The useful geometrical lemma

Lemma

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $\mathcal{C}^4$, let $\omega_0 \subset \Omega$ be an open set away from Γ, and let $\alpha \in [0,2)$.
Then there exists a positive number $\eta > 0$ and a function $\phi \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^4(\Omega)$ such that

$$\begin{cases} (i) \quad \forall x \in C(\Gamma,\eta) \quad \phi(x) = \dfrac{1}{2-\alpha} d(x,\Gamma)^{2-\alpha}, \\ (ii) \quad \{x \in \Omega \mid \nabla\phi(x) = 0\} \subset \omega_0. \end{cases}$$

In particular, by (i), ϕ also satisfies

$$\forall x \in C(\Gamma,\eta) \quad \nabla\phi(x) = -d(x,\Gamma)^{1-\alpha}\nu(p_\Gamma(x)) = -d(x,\Gamma)^{1-\alpha}\varepsilon_1(x).$$

Remark : In the nondegenerate case, Fursikov-Imanuvilov : (i) was :

$$\phi(x) = 0 \ \text{ and } \ \nabla\phi(x)\cdot\nu(x) < 0 \ \text{ for all } x \in \Gamma.$$

Ideas of the proof of the geometrical lemma

- sufficient to prove it in the case $\alpha = 1$;
- $x \mapsto d(x, \Gamma)$ is C^4 in some $\{x \in \overline{\Omega}, d(x,\Gamma) < \eta_1\}$, and can be extended in a C^4 function on $\overline{\Omega}$;
- density of the Morse functions and suitable convex combination : there is some Morse function $\tilde{\theta} : \overline{\Omega} \to \mathbb{R}$, C^4, such that

$$\tilde{\theta}(x) = d(x, \Gamma) \quad \text{on a neighborhood of } \Gamma \text{ ;}$$

- construction of some diffeomorphism that moves all the critical points of $\tilde{\theta}$ on ω_0.

Remarks : - the construction is much more simple when Ω is convex ;

- this function is essential to "balance" the effect of the degeneracy.

The useful Hardy type inequality

Theorem

There is some $\eta_0 > 0$ *such that, given* $0 < \eta < \eta_0$ *and* $\alpha \in [0, 1)$, *there is a positive constant* $C_H(\alpha)$ *(independent of* η*) such that, for all functions* $z \in H^1_{A,0}(\Omega)$,

$$\int_{C(\Gamma,\eta)} d(x,\Gamma)^{\alpha-2} z(x)^2 \, dx \leq C_H(\alpha) \int_{C(\Gamma,\eta)} d(x,\Gamma)^{\alpha} (\nabla z(x) \cdot \varepsilon_1(x))^2 \, dx.$$

Remarks : - $C_H(\alpha) = \frac{C}{(\alpha-1)^2}$;

- the usual Hardy type inequality (Opic-Kufner) involves *all the derivatives* of z :

$$\int_\Omega w_1(x) z(x)^2 \, dx \leq C \int_\Omega w_2(x) |\nabla z|^2 \, dx;$$

- it is essential for us to use only the "normal" derivative $\partial_{\varepsilon_1} z$.

Ideas of the proof of the Hardy type inequality

- normal parametrization of the boundary by arclength : $\Gamma = \gamma([0, \ell(\Gamma)])$, $\gamma(0) = \gamma(\ell(\Gamma))$, $|\gamma'(t)| = 1$;
- this gives a parametrization of the neighborhood $C(\Gamma, \eta)$ of Γ :

$$\psi[0, \ell(\Gamma)] \times (0, \eta) \to C(\Gamma, \eta), \quad \psi(s, t) = \gamma(s) - t\nu(\gamma(s)) :$$

  ψ is a C^1-diffeomorphism between $(0, \ell(\Gamma)) \times (0, \eta)$ and $C(\Gamma, \eta) \setminus \psi(\{0\} \times (0, \eta))$.

Then

$$\iint_{C(\Gamma,\eta)} d(x, \Gamma)^{\alpha-2} z(x)^2 \, dx$$

$$= \int_0^{\ell(\Gamma)} \int_0^{\eta} t^{\alpha-2} z(\psi(s, t))^2 |J_\psi(s, t)| \, dt \, ds$$

$$\leq C_0 \int_0^{\ell(\Gamma)} \Big(\int_0^{\eta} t^{\alpha-2} z(\psi(s, t))^2 \, dt \Big) \, ds.$$

Since $z \in H^1_{A,0}(\Omega)$: a.e. $s \in (0, \ell(\Gamma))$, the function $Z_s : Z_s(t) = z(\psi(s,t))$ is absolutely continuous on $(0,\eta)$, and $Z_s(0) = 0$. Hence the 1D Hardy type inequality says :

$$\begin{aligned}\int_0^\eta t^{\alpha-2} Z_s(t)^2 \, dt \le C_H(\alpha) \int_0^\eta t^\alpha \frac{dZ_s}{dt}(t)^2 \, dt \\ = C_H(\alpha) \int_0^\eta t^\alpha (\nabla z(\psi(s,t)), \nu(\gamma(s)))^2 \, dt.\end{aligned}$$

We integrate with respect to $s \in (0, \ell(\Gamma))$:

$$\begin{aligned}&\int_0^{\ell(\Gamma)} \Big(\int_0^\eta t^{\alpha-2} z(\psi(s,t))^2 \, dt \Big) \, ds \\ &\quad \le C_H(\alpha) \int_0^{\ell(\Gamma)} \int_0^\eta t^\alpha (\nabla z(\psi(s,t)), \nu(\gamma(s)))^2 \, dt \, ds \\ &\quad \le C_H(\alpha) C_0 \iint_{C(\Gamma,\eta)} d(x,\Gamma)^\alpha (\nabla z(x) \cdot \varepsilon_1(x))^2 \, dx.\end{aligned}$$

The standard form of the Carleman estimate

We consider the solution of the adjoint problem

$$\begin{cases} w_t + \text{div}\,(A(x)\nabla w) = f, \quad x \in \Omega, t > 0, \\ w(t,x) = 0 \text{ on } \Gamma, \\ w(T,x) = w_T(x). \end{cases}$$

A Carleman estimate will be of the type

$$\iint_{\Omega_T} \text{weight}_0 w^2 + \text{weight}_1 |\nabla w|^2 + \text{weight}_2 |D^2 w|^2 + \text{weight}_3 w_t^2 \\ \leq C \iint_{\Omega_T} \text{weight}_4 f^2 + C \iint_{\omega_T} \text{weight}_5 w^2,$$

hence a parabolic regularity type result, but without involving $\|w_T\|$.

How to obtain the Carleman estimate : the starting point

Some useful weight functions :

$$\theta(t) = \left(\frac{1}{t(T-t)}\right)^4, \quad \sigma(t,x) = \theta(t)\Big(e^{2S\|\phi\|_\infty} - e^{S\phi(x)}\Big),$$

and

$$z(t,x) := e^{-R\sigma} w.$$

Then z satisfies the differential problem

$$P_R^+ z + P_R^- z = f e^{-R\sigma},$$

and so

$$\|P_R^+ z\|^2 + \|P_R^- z\|^2 + 2\langle P_R^+ z, P_R^- z\rangle = \|f e^{-R\sigma}\|^2,$$

and the Carleman estimates comes from a suitable lower bound of the scalar product $\langle P_R^+ z, P_R^- z\rangle$.

How to obtain the Carleman estimate : description of the method

- the solution is not enough regular to compute everything directly : we first compute

$$\iint_{\Omega_T^\delta} P_R^+ z, P_R^- z,$$

where $\Omega_T^\delta = \{x \in \Omega, d(x, \Gamma) > \delta\}$;

- this brings

$$\iint_{\Omega_T^\delta} P_R^+ z, P_R^- z = DT_0^\delta(z) + DT_1^\delta(\nabla z) + BT^\delta(z, \nabla z);$$

- we let $\delta \to 0$:
  - $DT_0^\delta(z) \to DT_0(z)$, to be bound from below (Hardy, ϕ,...) ;
  - $DT_1^\delta(\nabla z) \to DT_1(\nabla z)$, to be bound from below (...) ;
  - $BT^\delta(z, \nabla z) \to 0$, thanks to the choice of ϕ and the regularity results up to the boundary.

Example of the convergence of the boundary terms

In particular :

$$\delta \int_0^T \int_{\Gamma^\delta} (\nabla z, \varepsilon_2(\gamma))^2 d\gamma\, dt \to 0 \quad \text{as } \delta \to 0 :$$

indeed, if $z \in \mathcal{C}^\infty(\overline{\Omega})$, then

$$\begin{aligned}
\delta \int_{\Gamma^\delta} (\nabla z, \varepsilon_2(\gamma))^2 d\gamma\, dt &= \text{value at } \delta = 0 + \int_0^\delta \text{derivative } /\delta \\
&= 0 + \iint_{C(\Gamma,\delta)} \cdots \\
&\leq C \iint_{C(\Gamma,\delta)} (\nabla z, \varepsilon_2)^2 + C \iint_{C(\Gamma,\delta)} (d(x,\Gamma)^{\alpha/2} \partial^2_{\varepsilon_1,\varepsilon_2} z)^2;
\end{aligned}$$

hence

$$\delta \int_{\Gamma^\delta} (\nabla z, \varepsilon_2(\gamma))^2 d\gamma\, dt \to 0 \quad \text{as } \delta \to 0$$

for all $z \in \mathcal{C}^\infty(\overline{\Omega})$, but also by density for all $z \in H^2_A(\Omega)$, thanks to the results of regularity up to the boundary of elements of $H^2_A(\Omega)$.

The Carleman estimate

Consider

$$\rho = RS\theta e^{S\phi}.$$

then, for sufficiently large parameters R, S, we obtain :

$$\iint_{\Omega_T} \frac{\rho}{\theta} A\nabla w \cdot \nabla w e^{-2R\sigma} \leq C \iint_{\Omega_T} f^2 e^{-2R\sigma} + C \iint_{\omega_T} \rho^3 w^2 e^{-2R\sigma},$$

which implies the desired observability estimate (using the Hardy type inequality).

Same bound for

$$S \iint_{\Omega_T} (A\nabla\phi, \nabla\phi)^2 \rho^3 w^2 e^{-2R\sigma},$$

$$\iint_{C(\Gamma,\eta)_T} d(x,\Gamma)^{2-\alpha} \rho^3 w^2 e^{-2R\sigma}, \quad \iint_{C(\Gamma,\eta)_T} d(x,\Gamma)^{\alpha} \rho (\nabla w, \varepsilon_1)^2 e^{-2R\sigma},$$

and similar estilates for w_t and $D^2 w$ (useful to obtain the first estimate).

No null controllability when $\alpha \geq 2$

Construction of an explicit example :

- $\Omega =$ disc of radius 1 ; explicit matrix $A(x)$ whose eigenvalues are $\lambda_1(x) = (1 - r)^\alpha$, $\lambda_2(x) = 1$;
- transformation to write the problem in polar coordinates : the associated function $v(r, \theta)$ is solution of a degenerate parabolic equation in 2D ;
- the means $w(r) = \int_0^{2\pi} v(r, \theta)\, d\theta$ is solution of a degenerate parabolic equation in 1D, for which we already know that there is no null controllability ;
- return to the initial problem.