An identification problem with evolution on the boundary of hyperbolic type

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`The identification problem`

Determine two functions $u : [0,T] \times \Omega \to \mathbf{R}$ *and* $k : [0,T] \to \mathbf{R}$ *solutions to*

$$(P) \begin{cases} A(u + k * u) = f, \text{ in } (0,T) \times \Omega, \\ D_t^2 u + b(x) D_{\nu_A} u = g, \text{ on } (0,T) \times \partial\Omega, \\ u(0,\cdot) = u_0, \text{on } \partial\Omega, \\ D_t u(0,\cdot) = u_1, \text{on } \partial\Omega, \\ \Phi[u(t,\cdot)] = l(t),\ t \in [0,T], \end{cases}$$

where Ω is an open bounded domain in $\mathbf{R}^n$ with a boundary $\partial\Omega$ of class C^2, A is a linear uniformly elliptic operator of second order.
Moreover ν_A denotes the conormal unit vector related to A and $b \in C^2(\overline{\Omega})$, $b(x) > 0$ for all $x \in \overline{\Omega}$ and $D_{\nu_A} b \geq 0$ on $\partial\Omega$.
Additional information: $\Phi \in H^{1/2+\varepsilon}(\partial\Omega)^*$, $\varepsilon \in (0, 1/2)$, e.g.,

$$\Phi[z] = \int_\Omega \phi_0(x) z(x)\, dx + \int_{\Omega\times\Omega} \phi_1(x,y) |x-y|^{-(n+2\varepsilon)} [z(x) - z(y)]\, dxdy,$$

ϕ_0 and ϕ_1 being two given functions in $L^2(\Omega)$ and $L^2(\Omega \times \Omega)$, respectively.

A. Lorenzi, F. M.: *A new identification problem with evolution on the boundary*, to appear in Advances in Differential Equations.

Assumptions on the operator A

Let $A : H^2(\Omega) \to L^2(\Omega)$ be a linear (uniformly) elliptic operator of the following form:

$$A = \sum_{i,j=1}^{n} D_{x_i}[a_{i,j}(x) D_{x_j}] + \sum_{j=1}^{n} a_j(x) D_{x_j} + a_0(x),$$

where $a_{i,j} \in C^2(\overline{\Omega})$, $a_j \in C^1(\overline{\Omega})$ $i, j = 1, \ldots, n$, $a_0 \in C(\overline{\Omega})$ and

$$a_0(x) - \frac{1}{2} \sum_{j=1}^{n} D_{x_j} a_j(x) \le 0, \quad x \in \overline{\Omega}, \qquad \sum_{i,j=1}^{n} a_{i,j}(x) \xi_i \xi_j \ge \kappa |\xi|^2, \quad (x, \xi) \in \overline{\Omega} \times \mathbf{R}^n,$$

for some positive constant κ. According to Theorem 9.11 in

D.Gilbarg,N.S.Trudinger: Elliptic partial differential equations of second order, Classics in Mathematics, Springer-Verlag, Berlin, 2001, xiv+517,

from our assumptions on the coefficients $a_{i,j}$ and a_0 we can conclude that, for any pair $(f, w) \in L^2(\Omega) \times H^{3/2}(\partial\Omega)$, the Dirichlet problem

$$Av = f, \quad \text{in } \Omega, \quad v = w \quad \text{on } \partial\Omega, \tag{1}$$

admits a unique solution $v \in H^2(\Omega)$ satisfying the estimate
$\|v\|_{H^2(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|w\|_{H^{3/2}(\partial\Omega)})$.
Moreover, according to the fundamental result in

J. L. Lions, E. Magenes: *Problemi ai limiti non omogenei (V)*, (in Italian) Ann. Sc. Norm. Sup. Pisa, 16 (1962), 1-44,

the solution to the elliptic boundary value problem (1) admits the representation

$$v = L_0 w + L_1 f,$$

where

$$L_0 \in \mathcal{L}(H^{s-1/2}(\partial\Omega); H^s(\Omega)), \quad 1/2 < s \le 2, \ s \ne 3/2,$$

$$L_1 \in \mathcal{L}(H^s(\Omega); H^{2+s}(\Omega)), \quad 1/2 < s \le 2, \ s \ne 3/2.$$

We emphasize that $L_0 w$ stands for the solution to problem (1) with $f = 0$, while $L_1 f$ stands for the solution to problem (1) with $w = 0$, i.e.

$$\begin{cases} AL_0 w = 0 & \text{in } \Omega, \\ L_0 w = w & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} AL_1 f = f & \text{in } \Omega, \\ L_1 f = 0 & \text{on } \partial\Omega. \end{cases}$$

`Main result`

To be able to state our fundamental result we need some notation.

$$\|g\|_{t,m,\beta,Y} = \|g\|_{C^m([0,t];H^\beta(Y))},\ \beta \geq 0, \qquad \|l\|_{t,m} = \|l\|_{C^m([0,t])}, \qquad t \in (0,T].$$

For $Y \in \{\partial\Omega, \Omega\}$ we set

$$\|u_0\|_{\beta,Y} = \|u_0\|_{H^\beta(Y)}, \qquad \beta \geq 0,$$

To define in a precise way the admissible space for our data we need to introduce the following non-standard vector space, where $\alpha \in (0,1)$, $\beta \in (0,+\infty)$, $s \in \mathbf{N}$:

$$C^s_\alpha([0,T];H^\beta(Y)) := \{v \in C^s([0,T];H^\beta(Y)) :$$

$$|D^s_t v|_{\alpha,\beta,Y} = \sup_{t\in(0,T]} t^{-\alpha}\|D^s_t v(t) - D^s_t v(0)\|_{\varepsilon,Y} < +\infty\}.$$

When endowed with the norm

$$\|v\|_{C^s_\alpha([0,T];H^\beta(Y))} := \|v\|_{T,s,\beta,Y} + |D^s_t v|_{\alpha,\beta,Y}$$

$C^s_\alpha([0,T];H^\beta(Y))$ turns out to be a Banach space.
We can now introduce the space of *admissible data:*

$$\mathcal{D}(\varepsilon,\alpha,R_0) := \{\mathbf{d} = (u_0,u_1,f,g,l) : u_0 \in H^{2+\varepsilon}(\partial\Omega),\ u_1 \in H^{1+\varepsilon}(\partial\Omega),$$

$$f \in C^2_\alpha([0,T];H^\varepsilon(\Omega)), g \in C_\alpha([0,T];H^{1/2+\varepsilon}(\partial\Omega)), l \in C^2_\alpha([0,T]),$$

$$\|\mathbf{d}\|_{\beta,\varepsilon} \leq R_0, |\Phi[L_1(bf(0,\cdot))]| \geq m_1 > 0,$$

$$|\Phi[L_0u_0 + L_1 f(0,\cdot)]| \geq m_2 > 0\},$$

where $\varepsilon \in [0,1/2)$ and

$$\|\mathbf{d}\|_{\alpha,\varepsilon} := \|u_0\|_{2+\varepsilon,\partial\Omega} + \|u_1\|_{1+\varepsilon,\partial\Omega} + \|f\|_{C^2_\alpha([0,T];H^\varepsilon(\Omega))} + \|g\|_{C_\alpha([0,T];H^{1/2+\varepsilon}(\partial\Omega))} + \|l\|_{C^2_\alpha([0,T])}.$$

Theorem 2.1. *Let* $\mathbf{d} = (u_0,u_1,f,g,l) \in \mathcal{D}(\varepsilon,\alpha,R_0)$. *Then problem* $P)$ *admits a unique solution* $(u,k) \in C([0,T];H^{1/2+\varepsilon}(\Omega)) \times C^1([0,T])$, *such that* $D_{\nu_A}u \in C([0,T];H^\varepsilon(\partial\Omega))$ *continuously depending on the data* $\mathbf{d}$ *with respect to the natural metrics of the spaces pointed out.*

`An equivalent differential problem`

Assume that $k \in L^p((0,T))$. Then there exists a unique solution $h \in L^p((0,T))$ solving the convolution equation

$$h + k + h * k = 0 \quad \text{in } (0,T).$$

Introduce the new unknown

$$v = u + k * u \quad \Longleftrightarrow \quad u = v + h * v.$$

Determine two functions $v : [0,T] \times \Omega \to \mathbf{R}$ *and* $h : [0,T] \to \mathbf{R}$ *solutions to*

$$(P') \quad \begin{cases} Av = f \quad \text{in } (0,T) \times \Omega, \\ D_t^2 v + D_t h * D_t v + h(0) D_t v + u_0 D_t h + b D_{\nu_A} v + b h * D_{\nu_A} v = g \\ \text{on } (0,T) \times \partial\Omega, \\ v(0,\cdot) = u_0 \quad \text{on } (0,T) \times \partial\Omega, \\ D_t v(0,\cdot) = u_1 - h(0) u_0 \quad \text{on } (0,T) \times \partial\Omega, \\ \Phi[v(t,\cdot)] + h * \Phi[v(t,\cdot)] = l(t) \quad t \in [0,T]. \end{cases}$$

Indeed, $v(0) = u(0) = u_0$ on $\partial\Omega$ and

$$u = v + h * v,$$

$$D_t u = D_t v + h * D_t v + h v(0), \; D_t^2 u = D_t^2 v + D_t h * D_t v + h(0) D_t v + D_t h v(0).$$

We prove that problems (P) in (u,k) and (P') in (v,h) are equivalent.

`First equivalent boundary integrodifferential problem`

$$v = L_0 w + L_1 f$$

Introduce the new unknown

$$w = v \quad \text{on } \partial\Omega.$$

Determine two functions $w : [0,T] \times \partial\Omega \to \mathbf{R}$ *and* $h : [0,T] \to \mathbf{R}$ *solutions to*

$$(P'') \quad \begin{cases} D_t^2 w - Bw = [g - bD_{\nu_A} L_1 f] + [-h(0)D_t w - bh * D_{\nu_A} L_1 f - D_t h * D_t w \\ +h * Bw] - u_0 D_t h =: F_1 + F_2(w, D_t w, Bw, h, D_t h) - u_0 D_t h \qquad \text{on } \partial\Omega, \\ w(0,\cdot) = u_0, \quad \text{on } \partial\Omega, \\ D_t w(0,\cdot) = u_1 - h(0)u_0, \quad \text{on } \partial\Omega, \\ \Phi[L_0 w(t,\cdot) + L_1 f(t,\cdot)] + h * \Phi[L_0 w(t,\cdot) + L_1 f(t,\cdot)] = l(t) \quad t \in [0,T]. \end{cases}$$

We denote

$$B := -bD_{\nu_A} L_0.$$

We prove that problems (P') in (v,h) and (P'') in (w,h) are equivalent.

`Second equivalent boundary integrodifferential problem`

Then, by differentiation, we obtain the following new problem:
Determine two functions $w : [0,T] \times \partial\Omega \to \mathbf{R}$ *and* $h : [0,T] \to \mathbf{R}$ *solutions to*

$$
(P''') \quad \begin{cases}
D_t^2 w - Bw = [g - bD_{\nu_A} L_1 f] + [-h(0)D_t w - bh * D_{\nu_A} L_1 f - D_t h * D_t w \\
+h * Bw] - u_0 D_t h =: F_1 + F_2(w, D_t w, Bw, h, D_t h) - u_0 D_t h \qquad \text{on } \partial\Omega, \\
w(0,\cdot) = u_0, \quad \text{on } \partial\Omega, \\
D_t w(0,\cdot) = u_1 - h_0 u_0, \quad \text{on } \partial\Omega, \\
\Phi[L_0 D_t^2 w(t,\cdot)] + h_0 \Phi[L_0 D_t w(t,\cdot)] + D_t h * \Phi[L_0 D_t w(t,\cdot)] + \Phi[L_1 D_t^2 f(t,\cdot)] \\
+D_t h * \Phi[L_1 D_t f(t,\cdot)] + h_0 \Phi[L_1 D_t f(t,\cdot)] + D_t h(t)\{\Phi[L_0 u_0] + \Phi[L_1 f(0,\cdot)]\} \\
= l''(t), \quad t \in [0,T].
\end{cases}
$$

We denote

$$
h(0) = \{\Phi[L_0 u_0 + L_1 f(0,\cdot)]\}^{-1}\{l'(0) - \Phi[L_0 u_1 + L_1 D_t f(0,\cdot)]\} =: h_0.
$$

We prove that problems (P'') and (P''') are equivalent.

Then we observe that the first equation in (P''') is equivalent to the following system for $(w, D_t w)$

$$D_t \begin{pmatrix} w \\ D_t w \end{pmatrix} = \mathcal{B}_\beta \begin{pmatrix} w \\ D_t w \end{pmatrix} + \begin{pmatrix} 0 \\ F_1 + F_2(w, D_t w, Bw, h, D_t h) - u_0 D_t h \end{pmatrix}$$

where

$$\mathcal{D}(\mathcal{B}_\beta) = H^{\beta+1/2}(\partial\Omega) \times H^\beta(\partial\Omega), \quad \mathcal{B}_\beta = \begin{pmatrix} 0 & 1 \\ B & 0 \end{pmatrix}, \quad \beta \geq 0.$$

We note that $\mathcal{B}_\beta \in \mathcal{L}(H^{\beta+1/2}(\partial\Omega) \times H^\beta(\partial\Omega); H^\beta(\partial\Omega) \times H^{\beta-1/2}(\partial\Omega))$, for all $\beta \geq 0$. Indeed, $B \in \mathcal{L}(H^{\beta+1/2}(\partial\Omega); H^{\beta-1/2}(\partial\Omega))$, $\beta \geq 0$, cf.

T. Hintermann: *Evolution equations with dynamic boundary conditions,* Proc. Roy. Soc. Edinburgh Sect. A 113 (1989), no. 1-2, 43–60.

Therefore $\mathcal{B}_\beta$ generates a continuous semigroup

$$\mathcal{S}(t) = \begin{pmatrix} S_{11}(t) & S_{12}(t) \\ S_{21}(t) & S_{22}(t) \end{pmatrix}$$

of linear bounded operators from $H^{\beta+1/2}(\partial\Omega) \times H^\beta(\partial\Omega)$ into itself, for all $\beta \geq 0$.

For $\beta \geq 0$

$$S_{21}(t)x = S_{12}(t)Bx, \quad x \in H^{\beta+1}(\partial\Omega),$$

$$S_{22}(t)y = S_{11}(t)y, \quad y \in H^{\beta+1/2}(\partial\Omega),$$

$$S_{22}(t)Bx = BS_{11}(t)x, \quad x \in H^{\beta+1}(\partial\Omega),$$

$$S_{21}(t)y = BS_{12}(t)y, \quad y \in H^{\beta+1/2}(\partial\Omega).$$

$$BS_{11}(t)x = S_{11}(t)Bx \quad x \in H^{\beta+1}(\partial\Omega),$$

$$BS_{12}(t)y = S_{12}(t)By, \quad y \in H^{\beta+1/2}(\partial\Omega).$$

$$D_t[S_{11}(t)x] = S_{12}(t)Bx, \quad D_t[S_{21}(t)x] = BS_{11}(t)x, \quad x \in H^{\beta+1}(\partial\Omega),$$

$$D_t[S_{12}(t)y] = S_{11}(t)y, \quad y \in H^{\beta+1/2}(\partial\Omega),$$

$$D_t[S_{22}(t)y] = BS_{12}(t)y, \quad y \in H^{\beta+1/2}(\partial\Omega).$$

$$\|S_{11}(t)x\|_{H^{\beta+1/2}(\partial\Omega)} \leq Me^{wt}\|x\|_{H^{\beta+1/2}(\partial\Omega)}, \quad x \in H^{\beta+1/2}(\partial\Omega),$$

$$\|S_{12}(t)y\|_{H^{\beta+1/2}(\partial\Omega)} \leq Me^{wt}\|y\|_{H^{\beta}(\partial\Omega)}, \quad y \in H^{\beta}(\partial\Omega),$$

$$\|S_{21}(t)x\|_{H^{\beta}(\partial\Omega)} \leq Me^{wt}\|x\|_{H^{\beta+1/2}(\partial\Omega)}, \quad x \in H^{\beta+1/2}(\partial\Omega),$$

$$\|S_{22}(t)y\|_{H^{\beta}(\partial\Omega)} \leq Me^{wt}\|y\|_{H^{\beta}(\partial\Omega)}, \quad y \in H^{\beta}(\partial\Omega).$$

According well-known results in Semigroup Theory, the first two equations in problem (P''') are equivalent to the following fixed-point equation:

$$
\begin{aligned}
w(t) = u_0 + \Big\{[S_{11}(t) - I]u_0 + S_{12}(t)(u_1 - h_0 u_0) + \int_0^t S_{12}(t-s)F_1(s)\,ds\Big\} \\
+ \int_0^t S_{12}(t-s)F_2(w, D_t w, Bw, h, D_t h)(s)\,ds - \int_0^t D_t h(s) S_{12}(t-s) u_0\,ds \\
=: u_0 + w_1(t) + \widetilde{N}_1(w, D_t w, Bw, h, D_t h)(t) - \int_0^t D_t h(s) S_{12}(t-s) u_0\,ds \\
=: u_0 + w_1(t) + N_1(w, D_t w, Bw, h, D_t h)(t).
\end{aligned}
$$

Differentiating the previous equation and using the relations on $S_{ij}(t)$ and their derivatives we get

$$
\begin{aligned}
D_t w(t) = \Big\{S_{12}(t)Bu_0 + S_{11}(t)(u_1 - h_0 u_0) + \int_0^t S_{11}(t-s)F_1(s)\,ds\Big\} \\
+ \int_0^t S_{11}(t-s)F_2(w, D_t w, Bw, h, D_t h)(s)\,ds - \int_0^t D_t h(s) S_{11}(t-s) u_0\,ds \\
= u_1 - h_0 u_0 + \Big\{S_{12}(t)Bu_0 + [S_{11}(t) - I](u_1 - h_0 u_0) + \int_0^t S_{11}(t-s)F_1(s)\,ds\Big\} \\
+ \int_0^t S_{11}(t-s)F_2(w, D_t w, Bw, h, D_t h)(s)\,ds - \int_0^t D_t h(s) S_{11}(t-s) u_0\,ds \\
=: u_1 - h_0 u_0 + w_2(t) + \widetilde{N}_2(w, D_t w, Bw, h, D_t h)(t) - \int_0^t D_t h(s) S_{11}(t-s) u_0\,ds \\
=: u_1 - h_0 u_0 + w_2(t) + N_2(w, D_t w, Bw, h, D_t h)(t).
\end{aligned}
$$

We need now to compute $D_t^2 w$. It seems more convenient to differentiate from equation for $D_t w$ with respect to t. We get

$$\begin{aligned}
D_t^2 w(t) &= Bu_0 + F_1(0) - h_0(u_1 - h_0 u_0) + \Big\{[S_{11}(t) - I]Bu_0 + S_{12}(t)[Bu_1 - h_0 Bu_0] \\
&\quad + \int_0^t S_{11}(s) D_t F_1(t-s)\,ds + [S_{11}(t) - I]F_1(0) \\
&\quad - h_0 \int_0^t S_{11}(s) b D_{\nu_A} L_1 f(t-s)\,ds - h_0[S_{11}(t) - I](u_1 - h_0 u_0)\Big\} \\
&\quad + \Big\{ \int_0^t S_{11}(t-s) F_3(w, D_t w, D_t^2 w, Bw, h, D_t h)(s)\,ds \\
&\quad - \int_0^t D_t h(s) S_{12}(t-s) Bu_0\,ds \Big\} - D_t h(t) u_0 \\
&=: Bu_0 + F_1(0) - h_0(u_1 - h_0 u_0) + \widetilde{w}_3(t) + \widetilde{N}_3(w, D_t w, D_t^2 w, Bw, D_t h)(t) \\
&\quad - D_t h(t) u_0,
\end{aligned}$$

where

$$\begin{aligned}
F_3(w, D_t w, D_t^2 w, Bw, h, D_t h)(t) &=: D_t F_2(w, D_t w, Bw, h, D_t h)(t) + h_0 b D_{\nu_A} L_1 f(t) \\
&= -h_0 D_t^2 w(t) - b D_t h * D_{\nu_A} L_1 f(t) - D_t h * D_t^2 w(t) \\
&\quad - D_t h(t)(u_1 - h_0 u_0) + D_t h * Bw(t) + h_0 Bw(t).
\end{aligned}$$

Then we replace the expression of $D_t^2 w$ into the last equation of (P^{III}) and we obtain the following fixed-point equation for $D_t h$, $t \in [0, T]$:

$$\begin{aligned}
D_t h(t) &= \chi^{-1}\{ - \Phi[L_0 B u_0] - \Phi[L_0 F_1(0)] + \Phi[L_0 h_0(u_1 - h_0 u_0)] - \Phi[L_1 D_t^2 f(0,\cdot)] \\
&\quad - h_0 \Phi[L_1 D_t f(0,\cdot)] + l''(0)\} + \chi^{-1}\{ - \Phi\{L_1 D_t^2 [f(t,\cdot) - f(0,\cdot)]\} \\
&\quad - h_0 \Phi\{L_1[D_t f(t,\cdot) - D_t f(0,\cdot)]\} + l''(t) - l''(0) - \Phi[L_0 \widetilde{w}_3(t)]\} \\
&+ \chi^{-1}\{h_0 \Phi[L_0 S_{11} * D_t^2 w(t)] + \Phi[L_0 D_t h * S_{11} * b D_{\nu_A} L_1 f(t)] + \Phi[L_0 D_t h * S_{11} * D_t^2 w(t)] \\
&+ \Phi[L_0 D_t h * S_{11}(u_1 - h_0 u_0)] - \Phi[L_0 D_t h * S_{11} * Bw(t)] - h_0 \Phi[L_0 S_{11} * Bw(t)]] \\
&- D_t h * \Phi[L_0 D_t w(t,\cdot)] - D_t h * \Phi[L_1 D_t f(t,\cdot)]\} - h_0 \Phi[L_0 D_t w(t,\cdot)] \\
&=: \chi^{-1}\{ - \Phi[L_0 B u_0] - \Phi[L_0 F_1(0)] + \Phi[L_0 h_0(u_1 - h_0 u_0)] - \Phi[L_1 D_t^2 f(0,\cdot)] \\
&- h_0 \Phi[L_1 D_t f(0,\cdot)] + l''(0)\} + \widetilde{h}(t) + \widetilde{N}_4(w, D_t w, D_t^2 w, Bw, D_t h)(t) - h_0 \Phi[L_0 D_t w(t,\cdot)].
\end{aligned}$$

Up to now we have proved that the sixtuple $(w, D_t w, D_t^2 w, Bw, h, D_t h)$ satisfies the equivalent new system:

$$(P^{IV})\begin{cases} w(t) = u_0 + w_1(t) + N_1(w, D_t w, Bw, h, D_t h)(t), \\ D_t w(t) = u_1 - h_0 u_0 + w_2(t) + N_2(w, D_t w, Bw, h, D_t h)(t), \\ D_t^2 w(t) + D_t h(t) u_0 = Bu_0 + F_1(0) - h_0(u_1 - h_0 u_0) + \widetilde{w}_3(t) \\ \qquad\qquad + \widetilde{N}_3(w, D_t w, D2_t w, Bw, D_t h)(t), \\ Bw(t) - h_0 D_t w(t) = Bu_0 - h_0(u_1 - h_0 u_0) + Bw_1(t) + \widetilde{w}_5(t) \\ \qquad\qquad + \widetilde{N}_5(D_t w, D_t^2 w, Bw, h, D_t h)(t), \\ h(t) = h_0 + 1 * D_t h(t), \\ D_t h(t) + h_0 \Phi[L_0 D_t w(t, \cdot)] = \chi^{-1}\{ - \Phi[L_0 B u_0] - \Phi[L_0 F_1(0)] \\ +\Phi[L_0 h_0(u_1 - h_0 u_0)] - \Phi[L_1 D_t^2 f(0, \cdot)] - h_0 \Phi[L_1 D_t f(0, \cdot)] + l''(0)\} + \widetilde{h}(t) \\ +\widetilde{N}_4(w, D_t w, D_t^2 w, Bw, D_t h)(t). \end{cases}$$

Let us now consider the auxiliary system

$$\begin{aligned} D_t w(t) &= f_1(t), \\ D_t^2 w(t) + D_t h(t) u_0 &= f_2(t), \\ Bw(t) - h_0 D_t w(t) &= f_3(t), \\ D_t h(t) + h_0 \chi^{-1} \Phi[L_0 D_t w(t, \cdot)] &= f_4(t). \end{aligned}$$

The solution of such a system is given by

$$D_t w(t) = f_1(t), \quad D_t^2 w(t) = f_2(t) + h_0 \chi^{-1} \Phi[L_0 f_1(t)] u_0 - f_4(t) u_0,$$

$$Bw(t) = f_3(t) + h_0 f_1(t), \quad D_t h(t) = -h_0 \chi^{-1} \Phi[L_0 f_1(t)] + f_4(t).$$

Consequently, system (P^{IV}) is equivalent to the following one, where we have separated the values at $t=0$ of the right-hand sides - which are independent of the unknowns - from the terms vanishing at $t=0$:

$$(P^V)\begin{cases} w(t) = z_{0,1} + w_1(t) + N_1(w, D_t w, Bw, h, D_t h)(t), \\ D_t w(t) = z_{0,2} + w_2(t) + N_2(w, D_t w, Bw, h, D_t h)(t), \\ D_t^2 w(t) = z_{0,3} + w_3(t) + N_3(w, D_t w, D_t^2 w, Bw, h, D_t h)(t), \\ Bw(t) = z_{0,4} + w_4(t) + N_4(w, D_t w, D_t^2 w, Bw, h, D_t h)(t), \\ h(t) = z_{0,5} + N_5(D_t h)(t), \\ D_t h(t) = z_{0,6} + w_6(t) + N_6(w, D_t w, D_t^2 w, Bw, h, D_t h)(t), \end{cases}$$

Now we the following abstract system for $\mathbf{z}$

$$(P^{VI})\begin{cases} z_1(t) = z_{0,1}(\mathbf{d}) + w_1(\mathbf{d})(t) + N_1(\mathbf{z}, \mathbf{d})(t), \\ z_2(t) = z_{0,2}(\mathbf{d}) + w_2(\mathbf{d})(t) + N_2(\mathbf{z}, \mathbf{d})(t), \\ z_3(t) = z_{0,3}(\mathbf{d}) + w_3(\mathbf{d})(t) + N_3(\mathbf{z}, \mathbf{d})(t), \\ z_4(t) = z_{0,4}(\mathbf{d}) + w_4(\mathbf{d})(t) + N_4(\mathbf{z}, \mathbf{d})(t), \\ z_5(t) = z_{0,5}(\mathbf{d}) + N_5(\mathbf{z}, \mathbf{d})(t), \\ z_6(t) = z_{0,6}(\mathbf{d}) + w_6(\mathbf{d})(t) + N_6(\mathbf{z}, \mathbf{d})(t), \end{cases}$$

where we have replaced $(w, D_t w, Bw, D_t^2 w, h, D_t h)$ with $\mathbf{z}$.

Theorem 4.1. *Let* $\mathbf{d} = (u_0, u_1, f, g, l) \in \mathcal{D}(\varepsilon, \alpha, R_0)$. *Then the fixed-point system* (P^{VI}) *admits a unique solution* $\mathbf{z} \in [C([0,T]; H^{\varepsilon}(\partial\Omega)]^4 \times [C([0,T])]^2$ *such that* $z_1 \in C([0,T]; H^{1/2+\varepsilon}(\partial\Omega)]$. $\mathbf{z}$ *continuously depends on the data* $\mathbf{d}$ *with respect to the metrics of the spaces pointed out.*

Main result

Theorem 0.
Let (f, u_0, l, g) *satisfy assumptions H1–H5 and consistency conditions (**??**), (**??**) and (**??**) for some* $\alpha \in (0, 1/2)$*. Then there exists a unique pair*

$$(u, k) \in C^{1+\alpha}([0,T]; W^{2,p}(\Omega)) \times C^{1+\alpha}([0,T]),$$

solving problem (P)*.*

Auxiliary theorems

Theorem 1. *Let (f, u_0, l, g) satisfy assumptions H1–H5 and consistency conditions (??), (??) and (??). Then problems (P) and (P^{iv}) are equivalent and their solutions*

$$(u, k) \in C^{1+\alpha}([0,T]; W^{2,p}(\Omega)) \times C^{1+\alpha}([0,T]) \tag{2}$$

and

$$(z, q) \in C^{\alpha}([0,T]; W^{2-1/p,p}(\partial\Omega)) \times C^{\alpha}([0,T]) \tag{3}$$

are related by formulae (??)–(??).

Theorem 2. *Let (f, u_0, l, g) satisfy assumptions H1–H5 and set*

$$\begin{aligned} Z^{\alpha}(\partial\Omega) \times Q^{\alpha} \;:=\; & [C^{1+\alpha}([0,T]; W^{1-1/p,p}(\partial\Omega)) \cap C^{\alpha}([0,T]; W^{2-1/p,p}(\partial\Omega))] \\ & \times C^{\alpha}([0,T]). \end{aligned}$$

Then there exists a unique pair

$$(z, q) \in Z^{\alpha}(\partial\Omega) \times Q^{\alpha} \tag{4}$$

solving problem (P^{iv}).

`Sketch of the proof of Theorem 1.`

We make use of

i) consistency conditions (**??**), (**??**) and (**??**);

ii) well-known properties concerning convolutions;

iii) well-known properties of the Dirichlet problem for elliptic operators;

iv) well-known results for abstract semigroups.

Sketch of the proof of Theorem 2.

We consider the following fixed point problem

$$(P^{iv}) \quad \begin{cases} z(t) =: \zeta(t) - N_1(z,q)(t), \\ q(t) =: \kappa(t) - N_2(z,q)(t). \end{cases}$$

Recalling that

$$Z^\alpha(\partial\Omega) := [C^{1+\alpha}([0,T]; W^{1-1/p,p}(\partial\Omega)) \cap C^\alpha([0,T]; W^{2-1/p,p}(\partial\Omega))],$$

$$Q^\alpha := C^\alpha([0,T]),$$

we endow the vector spaces $Z^\alpha(\partial\Omega)$ and Q^α with the following weighted norms:

$$\|z\|_{Z^\alpha_\rho} = \|e^{-\rho\cdot} z\|_{C^\alpha([0,T];X)} + \|e^{-\rho\cdot} D_t z\|_{C^\alpha([0,T];X)} + \|e^{-\rho\cdot} Bz\|_{C^\alpha([0,T];D(B))},$$

$$\|q\|_{Q^\alpha_\rho} = \|e^{-\rho\cdot} q\|_{C^\alpha([0,T])},$$

where we denote them by $Z^\alpha_\rho(\partial\Omega)$ and Q^α_ρ.

The following formulae hold true

$$D_t \int_0^t S(t-s)h(s)\,ds = \int_0^t S'(t-s)[h(s)-h(t)]\,ds + S(t)h(t), \tag{5}$$

$$B \int_0^t S(t-s)h(s)\,ds = \int_0^t S'(t-s)[h(s)-h(t)]\,ds + S(t)h(t) - h(t), \tag{6}$$

see, for example,
E. Sinetrari: *On the abtract Cauchy problem of parabolic type in spaces of continuous funtions*, J. Math. Anal. Appl., 107 1985, 16-66.

Making use of (5) and (6) we estimate

$\|N_1(z,q)\|_{Z^\alpha_\rho}$, $\|N_1(z_1,q_1) - N_1(z_2,q_2)\|_{Z^\alpha_\rho}$, $\|N_2(z,q)\|_{Q^\alpha_\rho}$, $\|N_2(z_1,q_1) - N_2(z_2,q_2)\|_{Q^\alpha_\rho}$.

Consequently we prove that N_1 maps $Z^\alpha_\rho \times Q^\alpha_\rho$ into Z^α_ρ and N_2 maps $Z^\alpha_\rho \times Q^\alpha_\rho$ into Q^α_ρ.
Moreover the vector operator (N_1, N_2) is a contraction mapping from $Z^\alpha_\rho \times Q^\alpha_\rho$ into itself.