Stabilization of second order evolution equations with unbounded feedback with delay

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Outline of the talk

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- Exponential stability
- Polynomial stability
- Some examples

The abstract problem

Let H be a real Hilbert space with norm and i. p. $\|.\|$ and $(.,.)$; $A : D(A) \to H$ a self-adjoint positive op. with a compact inverse in H; $V := D(A^{\frac{1}{2}})$. For $i = 1, 2$, let U_i be a real Hilbert space (identified to its dual space) with norm and i. p. $\|.\|_{U_i}$ and $(.,.)_{U_i}$ and let $B_i \in \mathcal{L}(U_i, V')$.

We consider the closed loop system

$$\begin{cases} \ddot{\omega}(t) + A\omega(t) + B_1 B_1^* \dot{\omega}(t) + B_2 B_2^* \dot{\omega}(t-\tau) = 0, \; t > 0 \\ \omega(0) = \omega_0, \; \dot{\omega}(0) = \omega_1, \; B_2^* \dot{\omega}(t-\tau) = f^0(t-\tau), \; 0 < t < \tau. \end{cases} \quad (1)$$

where τ is a positive constant which represents the delay, $\omega : [0, \infty) \to H$ is the state of the system.

Example

The wave equation with internal feedbacks

$$\begin{cases} \frac{\partial^2\omega}{\partial t^2} - \frac{\partial^2\omega}{\partial x^2} + \alpha_1\frac{\partial\omega}{\partial t}(\xi,\, t)\delta_\xi + \alpha_2\frac{\partial\omega}{\partial t}(\xi,\, t-\tau)\delta_\xi = 0,\ 0 < x < 1, \\ \omega(0,\, t) = \omega(1,\, t) = 0,\ t > 0 \\ I.C. \end{cases}$$

where $\xi \in (0,\, 1)$, $\alpha_1,\, \alpha_2 > 0$ and $\tau > 0$.

$$H = L^2(0,\, 1), A : H^2(0,\, 1) \cap H_0^1(0,1) \to H\ :\ \varphi \mapsto -\frac{d^2}{dx^2}\varphi$$

$$V = H_0^1(0,1); U_1 = U_2 = \mathbb{R},$$

$$B_i : \mathbb{R} \to V'\ :\ k \mapsto \sqrt{\alpha_i}k\delta_\xi, i = 1, 2.$$

Some instabilities

If $\alpha_1 = 0$, the previous system is unstable, cfr. [Datko-Lagnese-Polis 1986].

If $\alpha_2 > \alpha_1$, the previous system may be unstable, cfr. [N.-Pignotti 2006, N.-Valein 2007].

Hence some conditions between B_1 and B_2 have to be imposed to get stability.

Well-posedness (1)

Let us set $z(\rho, t) = B_2^*\dot{\omega}(t-\tau\rho)$ for $\rho \in (0, 1)$ and $t > 0$.
Then (1) is equivalent to

$$\left\{\begin{array}{c} \ddot{\omega}(t) + A\omega(t) + B_1B_1^*\dot{\omega}(t) + B_2z(1, t) = 0,\ t > 0 \\ \tau\frac{\partial z}{\partial t} + \frac{\partial z}{\partial \rho} = 0,\ t > 0,\ 0 < \rho < 1 \\ \omega(0) = \omega_0,\ \dot{\omega}(0) = \omega_1,\ z(\rho, 0) = f^0(-\tau\rho),\ 0 < \rho < 1 \\ z(0, t) = B_2^*\dot{\omega}(t),\ t > 0. \end{array}\right. \tag{2}$$

Well-posedness (2)

Therefore the problem can be rewritten as

$$\begin{cases} U' = \mathcal{A}U \\ U(0) = (\omega_0, \, \omega_1, f^0(-\tau.)), \end{cases} \tag{3}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} \omega \\ u \\ z \end{pmatrix} = \begin{pmatrix} u \\ -A\omega - B_1 B_1^* u - B_2 z(1) \\ -\frac{1}{\tau}\frac{\partial z}{\partial \rho} \end{pmatrix},$$

$$D(\mathcal{A}) := \{(\omega, \, u, \, z) \in V \times V \times H^1((0,\,1),\, U_2); \, z(0) = B_2^* u, \\ A\omega + B_1 B_1^* u + B_2 z(1) \in H\}.$$

Well-posedness (3)

Denote by $\mathcal{H}$ the Hilbert space $\mathcal{H} = V \times H \times L^2((0,\,1),\,U_2)$.
Let us now suppose that

$$\exists 0 < \alpha \leq 1,\ \forall u \in V,\ \|B_2^* u\|_{U_2}^2 \leq \alpha \, \|B_1^* u\|_{U_1}^2 \,. \tag{4}$$

We fix a positive real number ξ such that

$$1 \leq \xi \leq \frac{2}{\alpha} - 1. \tag{5}$$

We now introduce the following inner product on $\mathcal{H}$:

$$\langle \begin{pmatrix} \omega \\ u \\ z \end{pmatrix}, \begin{pmatrix} \tilde{\omega} \\ \tilde{u} \\ \tilde{z} \end{pmatrix} \rangle = (A^{\frac{1}{2}}\omega,\, A^{\frac{1}{2}}\tilde{\omega}) + (u,\, \tilde{u}) + \tau\xi \int_0^1 (z(\rho),\, \tilde{z}(\rho))_{U_2}\, d\rho$$

Well-posedness (4)

We show that $\mathcal{A}$ generates a C_0 semigroup on $\mathcal{H}$, by showing that $\mathcal{A}$ is dissipative in $\mathcal{H}$ for the above inner product and that $\lambda I - \mathcal{A}$ is surjective for any $\lambda > 0$. By Lumer-Phillips Theorem $\Rightarrow$

Thm 1. *Under the assumption (4), for an initial datum* $U_0 \in \mathcal{H}$, *there exists a unique solution* $U \in C([0, +\infty), \mathcal{H})$ *to system (3). Moreover, if* $U_0 \in D(\mathcal{A})$, *then*

$$U \in C([0, +\infty), D(\mathcal{A})) \cap C^1([0, +\infty), \mathcal{H}).$$

Ex. (4) $\Leftrightarrow \alpha_2 \leq \alpha_1$.

Energy decay

We restrict the ass. (4) to obtain the decay of the energy:

$$\exists 0 < \alpha < 1, \; \forall u \in V, \; \|B_2^* u\|_{U_2}^2 \leq \alpha \, \|B_1^* u\|_{U_1}^2 \qquad (6)$$

We define the energy as

$$E(t) := \frac{1}{2}\left(\left\|A^{\frac{1}{2}}\omega\right\|_H^2 + \|\dot\omega\|_H^2 + \tau\xi \int_0^1 \|B_2^*\dot\omega(t-\tau\rho)\|_{U_2}^2 \, d\rho\right),$$

where ξ is a positive constant satisfying $1 < \xi < \frac{2}{\alpha} - 1$.

Prop 1. *For any regular sol. of (1), the energy is non increasing and*

$$E'(t) \sim -\left(\|B_1^*\dot\omega(t)\|_{U_1}^2 + \|B_2^*\dot\omega(t-\tau)\|_{U_2}^2\right). \qquad (7)$$

Ex. (6) $\Leftrightarrow \alpha_2 < \alpha_1$.

Energy decay to 0

Prop 2. *Assume that (6) holds. Then, for all initial data in* $\mathcal{H}$*,* $\lim_{t\to\infty} E(t) = 0$ *iff*

$$\forall \textit{(non zero) eigenvector } \varphi \in D(A) : B_1^* \varphi \neq 0. \tag{8}$$

Pf. $\boxed{\Leftarrow}$ We closely follow [Tucsnak-Weiss 03].

$\boxed{\Rightarrow}$ We use a contradiction argument. ■

Rk This NSC is the same than without delay; therefore, (1) with delay is st. stable (i.e. the energy tends to zero) iff the system without delay (i.e. for $B_2 = 0$) is st. stable. ■

Ex. $\varphi_k = \sin(k\pi\cdot)\forall k \in \mathbb{N}^*$:

(8) $\Leftrightarrow \sin(k\pi\xi) \neq 0 \forall k \in \mathbb{N}^* \Leftrightarrow \xi \notin \mathbb{Q}$.

Cons. syst. (1)

The stability of (1) is based on some observability estimates for the associated conservative system: We split up ω sol of (1) in the form

$$\omega = \phi + \psi,$$

where ϕ is solution of the problem without damping

$$\begin{cases} \ddot{\phi}(t) + A\phi(t) = 0 \\ \phi(0) = \omega_0,\ \dot{\phi}(0) = \omega_1. \end{cases} \tag{9}$$

and ψ satisfies

$$\begin{cases} \ddot{\psi}(t) + A\psi(t) = -B_1B_1^*\dot{\omega}(t) - B_2B_2^*\dot{\omega}(t-\tau) \\ \psi(0) = 0,\ \dot{\psi}(0) = 0. \end{cases} \tag{10}$$

Cons. syst. (2)

By setting $B = (B_1\, B_2) \in \mathcal{L}(U, V')$ where $U = U_1 \times U_2$, ψ is solution of

$$\begin{cases} \ddot{\psi}(t) + A\psi(t) = Bv(t) \\ \psi(0) = 0,\, \dot{\psi}(0) = 0, \end{cases} \qquad v(t) = (-B_1^*\dot{\omega}(t), -B_2^*\dot{\omega}(t-\tau))^\top. \tag{11}$$

[Ammari-Tucsnak 01] $\Rightarrow$ if B satisfies: $\exists \beta > 0$:

$$\lambda \in \{\mu \in \mathbb{C}\, |\Re\mu = \beta\} \to H(\lambda) = \lambda B^*(\lambda^2 I + A)^{-1} B \in \mathcal{L}(U) \text{ is bd}, \tag{12}$$

then ψ satisfies

$$\int_0^T \sum_{i=1,2} \|(B_i^*\psi)'\|_{U_i}^2\, dt \le Ce^{2\beta T} \int_0^T (\|B_1^*\dot{\omega}(t)\|_{U_1}^2 + \|B_2^*\dot{\omega}(t-\tau)\|_{U_2}^2) dt.$$

Le 1. *Suppose that the assumption (12) is satisfied. Then the solutions ω of (1) and ϕ of (9) satisfy*

$$\int_0^T \sum_{i=1,2} \|(B_i^*\phi)'\|_{U_i}^2\, dt \le Ce^{2\beta T} \int_0^T (\|B_1^*\dot{\omega}(t)\|_{U_1}^2 + \|B_2^*\dot{\omega}(t-\tau)\|_{U_2}^2) dt,$$

with $C > 0$ independent of T.

Exp. stab. (1)

Thm 2. *Assume that (6) and (12) are satisfied. If* $\exists T > \tau > 0$ *and a constant* $C > 0$ *ind. of* τ *s. t. the observability estimate*

$$\left\|A^{\frac{1}{2}}\omega_0\right\|_H^2 + \|\omega_1\|_H^2 \leq C\int_0^T \left\|B_1^*\dot{\phi}(t)\right\|_{U_1}^2 dt \tag{13}$$

holds, where ϕ *is solution of (9), then the system (1) is exp. stable. in energy space.*

Pf. Le 1 $\Rightarrow E(0) - E(T) \geq CE(T) \geq CE(0) +$ inv. by translation $\Rightarrow$ exp. stab.

Rk. Notice that the SC (13) is the same than the case without delay [Ammari-Tucsnak 01]. Therefore, if (12) holds, then (1) is exp. stable if the dissipative system without delay (i.e. with $B_2 = 0$) is exp. stable. ■

Exp. stab. (2)

Ingham's $\leq \Rightarrow$

Prop 3. *Assume that the eigenvalues* $\lambda_k, k \in \mathbb{N}^*$ *are simple and that the standard gap condition*

$$\exists \gamma_0 > 0, \, \forall k \geq 1, \, \lambda_{k+1} - \lambda_k \geq \gamma_0$$

holds. Then (13) holds iff

$$\exists \alpha > 0, \, \forall k \geq 1, \, \|B_1^* \varphi_k\|_{U_1} \geq \alpha.$$

Ex. Not exp. stable for any $\xi \in (0,1)$ because
$\not\exists \alpha > 0 : |\sin(k\pi\xi)| \geq \alpha \quad \forall k$.

Pol. stab. (1)

$(\omega_0, \omega_1, f^0(-\tau.)) \in D(\mathcal{A}) \not\Rightarrow \omega_0 \in D(A)$, we can not use standard interpolation inequalities. Therefore we need to make the following hypo.: $\exists m, C > 0 \forall (\omega_0, \omega_1, z) \in D(\mathcal{A})$:

$$\|\omega_0\|_V^{m+1} \leq C \left\|(\omega_0, \omega_1, z)\right\|_{D(\mathcal{A})}^m \|\omega_0\|_{D(A^{\frac{1-m}{2}})} . \quad (14)$$

Pol. stab. (2)

Thm 3. *Let* ω *sol. of (1) with* $(\omega_0,\, \omega_1,\, f^0(-\tau\cdot)) \in D(\mathcal{A})$. *Assume that (6), (12) and (14) are verified. If* $\exists m > 0$, *a time* $T > 0$ *and* $C > 0$ *ind. of* τ *s. t.*

$$\int_0^T \|(B_1^*\phi)'(t)\|_{U_1}^2 dt \geq C(\|\omega_0\|^2_{D(A^{\frac{1-m}{2}})} + \|\omega_1\|^2_{D(A^{-\frac{m}{2}})}) \tag{15}$$

holds where ϕ *is sol. of (9), then the energy decays polynomially, i.e.,* $\exists C > 0$ *depending on* m *and* τ *s. t.*

$$E(t) \leq \frac{C}{(1+t)^{\frac{1}{m}}} \left\|(\omega_0,\, \omega_1,\, f^0(-\tau\cdot))\right\|^2_{D(\mathcal{A})}, \forall t > 0.$$

Pol. stab. (3)

Ingham's $\leq \Rightarrow$

Prop 4. *Assume that the eigenvalues* $\lambda_k, k \in \mathbb{N}^*$ *are simple and that the standard gap condition*

$$\exists \gamma_0 > 0, \, \forall k \geq 1, \, \lambda_{k+1} - \lambda_k \geq \gamma_0$$

holds. Then (15) holds iff

$$\exists \alpha > 0, \, \forall k \geq 1, \, \|B_1^* \varphi_k\|_{U_1} \geq \frac{\alpha}{\lambda_k^m}.$$

Ex. If $\xi \in S$ (containing the quadratic irrational numbers), then energy decays as t^{-1}, because $\exists \alpha > 0 : |\sin(k\pi\xi)| \geq \frac{\alpha}{k} \quad \forall k$.

Ex 1: Distributed dampings

$$\left\{\begin{array}{l} \frac{\partial^2\omega}{\partial t^2} - \frac{\partial^2\omega}{\partial x^2} + \alpha_1 \frac{\partial\omega}{\partial t}(x,\,t)\chi_{|I_1} \\ +\alpha_2 \frac{\partial\omega}{\partial t}(x,\,t-\tau)\chi_{|I_2} = 0 \quad \text{in } (0,\,1)\times(0,\,\infty) \\ \omega(0,\,t) = \omega(1,\,t) = 0 \quad t>0 \\ \omega(x,\,0) = \omega_0(x),\ \frac{\partial\omega}{\partial t}(x,\,0) = \omega_1(x) \quad \text{in } (0,\,1) \\ \frac{\partial\omega}{\partial t}(x,\,t-\tau) = f^0(x,\,t-\tau) \quad \text{in } I_2\times(0,\,\tau), \end{array}\right.$$

where $\chi_{|I}$ = characteristic fct of I.
We assume that $0 < \alpha_2 < \alpha_1$, $\tau > 0$ and

$$I_2 \subset I_1 \subset [0,\,1], \quad \exists\delta \in [0,\,1], \epsilon > 0 :\ [\delta,\,\delta+\epsilon] \subset I_1.$$

Ex 1 continued

$H = L^2(0,\,1), V = H_0^1(0,\,1), D(A) =$
$H_0^1(0,\,1) \cap H^2(0,\,1)$, $A : D(A) \to H\ : \varphi \mapsto -\frac{d^2}{dx^2}\varphi$
$U_i = L^2(I_i), B_i : U_i \to H \subset V'\ :\ k \mapsto \sqrt{\alpha_i}\tilde{k}\chi_{|I_i}$.
$B_i^*(\varphi) = \sqrt{\alpha_i}\varphi_{|I_i} \Rightarrow B_iB_i^*(\varphi) = \alpha_i\varphi\chi_{|I_i}\forall\varphi \in V$.

The problem is exponentially stable since $\|B_1^* \sin(k\pi\cdot)\|_{U_1} \geq \alpha_1\frac{\epsilon}{2}$, for $k \ggg$.

Ex 2: Distributed dampings

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, with a C^2 bdy Γ. We assume that $\Gamma = \Gamma_D \cup \Gamma_N$, with $\bar{\Gamma}_D \cap \bar{\Gamma}_N = \emptyset$ and $\Gamma_D \neq \emptyset$. $\exists x_0 \in \mathbb{R}^n$ is s. t. $(x - x_0) \cdot \nu(x) \leq 0$, $\forall x \in \Gamma_D$. Let $O_2 \subset O_1 \subset \Omega$ s. t. $\Gamma_N \subset \partial O_1$.

$$\begin{cases} \frac{\partial^2 \omega}{\partial t^2} - \Delta\omega + \alpha_1 \frac{\partial \omega}{\partial t}(x, t)\chi_{|O_1} \\ +\alpha_2 \frac{\partial \omega}{\partial t}(x, t-\tau)\chi_{|O_2} = 0 \quad \text{in } \Omega \times (0, \infty), \\ \omega(x, t) = 0 \quad \text{on } \Gamma_D \times (0, \infty), \\ \frac{\partial \omega}{\partial \nu}(x, t) = 0 \quad \text{on } \Gamma_N \times (0, \infty), \\ I.C., \end{cases}$$

$0 < \alpha_2 < \alpha_1$, $\tau > 0$.

Ex 2 continued

$$H = L^2(\Omega), V = H^1_{\Gamma_D}(\Omega),$$
$$A : D(A) \to H \ : \ \varphi \mapsto -\Delta\varphi$$
$$U_i = L^2(O_i), B_i : U_i \to H \subset V' \ : \ k \mapsto \sqrt{\alpha_i}\tilde{k}\chi_{|0_i}.$$
$$B_i^*(\varphi) = \sqrt{\alpha_i}\varphi_{|O_i}.$$

The obs. est. (13) was proved in [Lasiecka-Triggiani-Yao 99] $\Rightarrow$ the pb is exponentially stable.

Rk. Generalization of [N.-Pignotti 2006] where $O_2 = O_1$.