Linear evolution equations of hyperbolic type with application to Schrödinger equations

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This is a joint work with my student Kentaro Yoshii.

We consider the existence and uniqueness of solutions to Cauchy problem for Schrödinger Evolution Equations:

$$(\mathrm{SE}) \qquad \begin{cases} i\dfrac{\partial u}{\partial t} = -\Delta u + V(x,t)u, & (x,t) \in \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x). \end{cases}$$

Here $V(x,t)$ is a time-dependent real-valued potential.

Plan of the talk

1. Abstract results — a modified Kato's theory

2. Proof of main theorems (outline)

3. Applications to Schrödinger evolution equations

1. Abstract result

X— a separable complex Hilbert space.
$\{A(t); 0 < t < T\}$ — a family of closed linear operators in X.
We are concerned with linear evolution equations of the form

(E) $$\frac{d}{dt}u(t) + A(t)u(t) = f(t) \quad \text{on} \quad (0, T).$$

S — selfadjoint in X, with $(u, Su) \geq \|u\|^2 \ \forall\, u \in D(S)$.
Then we introduce the second Hilbert space

$$Y := D(S^{1/2}) \text{ —— the space of initial values}$$

with $(u, v)_Y := (S^{1/2}u, S^{1/2}v) \ \forall\, u, v \in Y$ and $\|v\|_Y := (v, v)_Y^{1/2}$;
note that $Y \hookrightarrow X$: dense and continuous.

Assumption of Theorem 1 (Main Theorem).

Let X, Y and $\{A(t), S\}$ be as above. Then we assume

(I) $0 \le \exists\, \alpha \in L^1(0,T)$;

$$|\,\mathrm{Re}(A(t)v, v)| \le \alpha(t)\|v\|^2 \quad \forall\, v \in D(A(t)),\ \text{a.a. } t \in (0,T).$$

(II) $Y = D(S^{1/2}) \subset D(A(t))$, a.a. $t \in (0,T)$.

(III) $\alpha \le \exists\, \beta \in L^1(0,T)$;

$$|\,\mathrm{Re}(A(t)u, Su)| \le \beta(t)\big\|S^{1/2}u\big\|^2 \quad \forall\, u \in D(S),\ \text{a.a. } t \in (0,T).$$

(IV) $A(\cdot) \in L^2_*\big(0,T; L(Y,X)\big)$

$\Big($more precisely, $A(t)$ is strongly measurable in t

with values in $L(Y,X)$ and $\big\|A(t)\big\|_{L(Y,X)} \in L^2(0,T)\Big)$.

Remark 1.

Conditions (I), (II) and (III) with $t = t_0$ fixed

$\Longrightarrow$ m-accretivity of $\alpha(t_0) \pm A(t_0)$,

i.e. $A(t_0)$ cannot be an analytic generator.

This is the meaning of "hyperbolic" in the title.

Therefore, the term "hyperbolic" may be replaced

with "non-parabolic".

Remark 2.

Condition (III) is a consequence of conditions (I), (II) and "**commutator type relation**":

(K) $\exists$ $B(\cdot) \in L^{\infty}_{*}(0, T; L(X))$

$\Big($i.e. $B(\cdot)$ is strongly measurable in t with values in $L(X)$ and $\|B(t)\|_{L(X)}$ is essentially bounded in $t\Big)$, and

$$S^{1/2}A(t)S^{-1/2} = A(t) + B(t), \quad \text{a.a. } t \in (0, T),$$

in which the domain relation is exact.

Theorem 1. Under conditions (I)–(IV)

$$\exists\,!\ \{U(t,s);\ (t,s)\in\Delta\} \text{ ——— } \textbf{evolution operators},$$

where $\Delta := \{(t,s); 0 \le s \le t \le T\}$, having the properties:

(i) $U(\cdot,\cdot)$: strongly continuous on Δ to $L(X)$,

$$\|U(t,s)\|_{L(X)} \le \exp\Big(\int_s^t \alpha(r)\,dr\Big), \qquad (t,s)\in\Delta.$$

(ii) $U(t,r)U(r,s) = U(t,s)$ on Δ and $U(s,s) = 1$ (the identity).

(iii) $U(t,s)Y \subset Y$ and

$U(t,s)$: strongly continuous on Δ to $L(Y)$,

$$\|U(t,s)\|_{L(Y)} \le \exp\Big(\int_s^t \beta(r)\,dr\Big), \qquad (t,s)\in\Delta.$$

Furthermore, $v \in Y \Rightarrow U(\cdot,\cdot)v \in H^1(\Delta; X)$, with

(iv) $(\partial/\partial t)U(t,s)v = -A(t)U(t,s)v$, a.a. $t \in (s,T)$.

(v) $(\partial/\partial s)U(t,s)v = U(t,s)A(s)v$, a.a. $s \in (0,t)$.

Theorem 2 (Inhomogeneous Equation).

Let $\{U(t,s);\ (t,s) \in \Delta\}$ be as in Theorem 1.

Let $u_0 \in Y$ and $f(\cdot) \in L^2(0,T;X) \cap L^1(0,T;Y)$. Put

$$u(t) := U(t,0)u_0 + \int_0^t U(t,s)f(s)\,ds$$

$\Longrightarrow u(\cdot) \in H^1(0,T;X) \cap C([0,T];Y)$

is a unique strong solution to

(IVP)
$$\begin{cases} \dfrac{d}{dt}u(t) + A(t)u(t) = f(t) \quad \text{a.e. on } (0,T), \\ u(0) = u_0. \end{cases}$$

2. Proof of main theorems

2.1. Proof of Theorem 1

2.2. Proof of Theorem 2

2.1. Proof of Theorem 1.

Approximation of $A(\cdot) \in L^2_*(0,T;B(Y,X))$

Definition 1 **[Modified Yosida approximation (S. Ishii, 1982)].**

$$A_n(t) := A(t)\Big(1 + \frac{1}{\nu_n(t)}A(t)\Big)^{-1} = \nu_n(t)\Big[1 - \Big(1 + \frac{1}{\nu_n(t)}A(t)\Big)^{-1}\Big],$$

where $0 < \nu_n(t) := 2\beta(t) + n \in L^1(0,T) \;\; \forall\, n \in \mathbb{N}$.

Then **(a)** $A_n(\cdot) \in L^2_*(0,T;L(Y,X))$, **(b)** $A_n(\cdot) \in L^1_*(0,T;L(X))$:

(a) $\forall\, y \in Y \;\; A_n(t)y$ is strongly measurable in t and

$$\|A_n(t)\|_{L(Y,X)} \le 2\,\|A(t)\|_{L(Y,X)} \in L^2(0,T);$$

(b) $\forall\, x \in X \;\; A_n(t)x$ is strongly measurable in t and

$$\|A_n(t)\|_{L(X)} \le \nu_n(t) \in L^1(0,T).$$

Lemma 1. Let $\{A_n(t)\}$ be the modified Yosida approximation. Let $s \in [0, T)$. Then the approximate problem:

$$\text{(AP)} \qquad \begin{cases} \dfrac{d}{dt}u_n(t) + A_n(t)u_n(t) = 0 \quad \text{a.e. on } (s, T), \\ u_n(s) = w \end{cases}$$

has a unique strong solution $u_n \in H^1(s, T; X) \subset C([s, T]; X)$.

- $(d/dt)u_n(t)$: strong derivative.

Proof. Define the mapping $\Phi : C([s, T]; X) \to C([s, T]; X)$ as

$$(\Phi\, u)(t) := w - \int_s^t A_n(r)u(r)\, dr, \qquad \text{a.a. } t \in (s, T).$$

Then $\|\Phi u - \Phi v\|_E \le \left[1 - e^{-\|\nu_n\|_{L^1(s,T)}}\right] \|u - v\|_E$, where

$$E := \left\{u \in C([s, T]; X); \|u\|_E := \sup_{s\le t\le T} \|u(t)\|\, e^{-\int_s^t \nu_n(r)dr} < \infty\right\}. \quad \blacksquare$$

We define the "solution operator" of (AP) by

$$U_n(t,s)w := u_n(t) \quad \text{for } (t,s) \in \Delta.$$

The main properties of $U_n(t,s)$:

$\{U_n(t,s)\}$ — bounded linear operators on X, with

- $\|U_n(t,s)\|_{L(X)} \le \exp\Big[\int_s^t \alpha(r)\big(1-\nu_n(r)^{-1}\alpha(r)\big)^{-1}\,dr\Big]$ on Δ.
- $U_n(t,r)U_n(r,s) = U_n(t,s)$ on Δ and $U_n(s,s) = 1$.
- $U_n(t,s)Y \subset Y$ and
  $\|U_n(t,s)\|_{L(Y)} \le \exp\Big[\int_s^t \beta(r)\big(1-\nu_n(r)^{-1}\beta(r)\big)^{-2}\,dr\Big]$ on Δ.

Furthermore, $\forall\ w \in Y$

- $\dfrac{\partial}{\partial t}U_n(t,s)w = -A_n(t)U_n(t,s)w$, a.e. on Δ.
- $\dfrac{\partial}{\partial s}U_n(t,s)w = U_n(t,s)A_n(s)w$, a.e. on Δ.

Lemma 2. Let $\{U_n(t,s)\}$ be as above. Then $\exists\ \{U(t,s);\ (t,s)\in\Delta\}$ in $B(X)$;

$$U(t,s) := \operatorname*{s\text{-}lim}_{n\to\infty} U_n(t,s)$$

(convergence is uniform on Δ).

Proof. It suffices to show that

$$(\#)\qquad \|U_n(t,s)v - U_m(t,s)v\|^2$$
$$\le a_{nm}\|A(\cdot)\|^2_{L^2(0,T;L(Y,X))}\exp\Big[4\int_s^t\{\alpha(r)+2\beta(r)\}\,dr\Big]\|v\|_Y^2.$$

where (we have used $\big\|A(t)\big\|_{L(Y,X)}\in L^2(0,T)$ and)

$$a_{nm} := 4\Big|\frac{1}{n}-\frac{1}{m}\Big| + 2\Big(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{m}}\Big)^2.$$

Put $u_{nm}(r,s) := U_n(r,s)v - U_m(r,s)v$. Then

$$
\begin{aligned}
&\frac{\partial}{\partial r}\|u_{nm}(r,s)\|^2 \\
&= -2\,\mathrm{Re}\big(A_n(r)U_n(r,s)v - A_m(r)U_m(r,s)v, u_{nm}\big) \\
&\le 4\alpha(r)\|u_{nm}(r,s)\|^2 + \left|\frac{1}{n} - \frac{1}{m}\right| \max_{l=n,m} \|A_l(r)U_l(r,s)v\|^2 \\
&+ \frac{1}{2}\Big(\frac{1}{\sqrt{n}}\|A_n(r)U_n(r,s)v\| + \frac{1}{\sqrt{m}}\|A_m(r)U_m(r,s)v\|\Big)^2 \\
&\le 4\alpha(r)\|u_{nm}(r,s)\|^2 + \frac{1}{4}a_{nm} \max_{l=n,m} \|A_l(r)U_l(r,s)v\|^2.
\end{aligned}
$$

Integrating this inequality, we obtain (#). ■

2.2. Proof of Theorem 2.

(1) Homogeneous case: $f = 0$.

In Theorem 1 (iv) it is proved that $\forall\, u_0 \in Y$,

$$\frac{d}{dt} U(t,0)u_0 + A(t)U(t,0)u_0 = 0, \qquad \text{a.a. } t \in (0,T).$$

(2) Inhomogeneous term: $f(\cdot) \in L^2(0,T;X) \cap L^1(0,T;Y)$

$$v(t) := \int_0^t U(t,s) f(s)\, ds$$

$$\Longrightarrow \quad \frac{d}{dt} v(t) + A(t)v(t) = f(t), \qquad \text{a.a. } t \in (0,T).$$

Proof is based on the approximation:

- First we replace $v(\cdot)$ with

$$v_n(t) := \int_0^t U_n(t,s) f(s)\, ds,$$

where $\{U_n(t,s)\}$ is the approximate evolution operators. Then

$$\frac{d}{dt} v_n(t) + A_n(t) v_n(t) = f(t).$$

- Second we show the convergence of $\{v_n(\cdot)\}$ to $v(\cdot)$.

3. Applications to Schrödinger evolution equations

Put $\Sigma(n) := H^n(\mathbb{R}^N) \cap D(|x|^n)$ for $n \in \mathbb{N}$, where

$$D(|x|^n) := \{u \in L^2(\mathbb{R}^N);\ |x|^n u \in L^2(\mathbb{R}^N)\}.$$

The Cauchy problem for Schrödinger evolution equations:

(SE) $$\begin{cases} i\dfrac{\partial u}{\partial t} - (-\Delta_x + V(x,t))u(x,t) = 0, & (x,t) \in \mathbb{R}^N \times \mathbb{R}_+, \\ u(x,0) = u_0(x) \in \Sigma(n) \end{cases}$$

in $X := L^2(\mathbb{R}^N)$. Put

$$A(t) := i(-\Delta_x + V(x,t)),$$

$$S := 1 + \sum_{k=1}^{N} \Big((-1)^n \frac{\partial^{2n}}{\partial x_k^{2n}} + x_k^{2n} \Big), \qquad n \geq 2.$$

Then S is selfadjoint on $D(S) := \Sigma(2n)$,

and $Y = D(S^{1/2}) = \Sigma(n)$ is equipped with norm

$$\|u\|_Y^2 = \left\|S^{1/2}u\right\|^2 = \|u\|^2 + \sum_{k=1}^{N} \left(\left\| \frac{\partial^n u}{\partial x_k^n} \right\|^2 + \left\| x_k^n u \right\|^2 \right).$$

Theorem 3.

(SE) with $u_0 \in \Sigma(n)$ has a unique strong solution

$$u(\cdot) \in H^1_{\mathrm{loc}}([0,\infty); L^2(\mathbb{R}^N)) \cap C([0,\infty); \Sigma(n))$$

under the following conditions on $V(x,t)$:

$V(x,t) : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$ — measurable

(V0) $V(\cdot,t) \in C^n(\mathbb{R}^N)$, a.a. $t \in \mathbb{R}_+$.

(V1) $V \in \big(L^{p_n}(\mathbb{R}^N) + \langle x\rangle^n L^\infty(\mathbb{R}^N)\big) \times L^2(0,T) \quad \forall\ T > 0.$

(V2) $\forall\ j \in \mathbb{N}\ (j \le n),\ \forall\ T > 0$

$$\sum_{k=1}^{N}\left|\frac{\partial^j V}{\partial x_k^j}\right| \in \big(L^{p_j}(\mathbb{R}^N) + \langle x\rangle^j L^\infty(\mathbb{R}^N)\big) \times L^1(0,T).$$

Here $\langle x\rangle := (1+|x|^2)^{1/2}$,

$$\langle x\rangle^j L^\infty(\mathbb{R}^N) := \{u \in L^1_{\mathrm{loc}}(\mathbb{R}^N); \langle x\rangle^{-j} u \in L^\infty(\mathbb{R}^N)\},$$

and p_j $(1 \le j \le n)$ is a constant such that

$$p_j \in \begin{cases} [2,\infty) & (j < N/2), \\ (2,\infty) & (j = N/2), \\ [N/j,\infty) & (j > N/2). \end{cases}$$

Before verifying conditions (I)–(IV) in Theorem 1,
we need to show that S is selfadjoint.

Lemma 3 [Ok (1975)]

A, B — linear m-accretive operators in H.
D — core of B invariant under $\left(1+\frac{1}{n}A\right)^{-1}$ $(n \in \mathbb{N})$.

$\exists\ a \geq 0,\ \exists\ b \in [0,1];\quad \forall\ u \in D_0 := (1+A)^{-1}D,$

$$0 \leq \mathrm{Re}(Au, Bu) + a\|u\|^2 + b\|Au\|^2.$$

Then

- $b < 1 \Longrightarrow A + B$ is m-accretive.
- $b = 1 \Longrightarrow \overline{A + B}$ is m-accretive.

Lemma 4. For $n \in \mathbb{N}$ let

$$Au := (-1)^n \sum_{k=1}^{N} \frac{\partial^{2n} u}{\partial x_k^{2n}}, \qquad Bu := \sum_{k=1}^{N} x_k^{2n} u.$$

Then $S := 1 + A + B$ is selfadjoint in X.

Proof. Since $A + B \geq 0$ is symmetric,
it remains to prove the m-accretivity of $A + B$.
It suffices by Lemma 3 to show that $\exists\ C \geq 0$;

$$(*) \qquad \operatorname{Re}(Au, Bu) = (-1)^n \operatorname{Re} \sum_{j,k=1}^{N} \Big(\frac{\partial^{2n} u}{\partial x_j^{2n}}, x_k^{2n} u \Big) \geq -C\|u\|^2$$

for all $u \in D_0 = D := \mathcal{S}(\mathbb{R}^N)$ = Schwartz's space.

Integration by parts gives that $\mathrm{Re}\,(Au, Bu)$ is equal to

$$\sum_{\substack{j,k=1\\ j\neq k}}^{N} \left\| x_k^n \frac{\partial^n u}{\partial x_j^n} \right\|^2 + \sum_{j=1}^{N} \sum_{k=0}^{n} (-1)^{n-k} c_{n,k} \left\| x_j^k \frac{\partial^k u}{\partial x_j^k} \right\|^2$$

$$\geq (-1)^n N c_{n,0} \|u\|^2 + c_{n,n} \sum_{j=1}^{N} \left\| x_j^n \frac{\partial^n u}{\partial x_j^n} \right\|^2$$

$$+ \sum_{j=1}^{N} \sum_{k=1}^{n-1} (-1)^{n-k} c_{n,k} \left\| x_j^k \frac{\partial^k u}{\partial x_j^k} \right\|^2,$$

where $c_{n,k} > 0$ are constants ($0 \leq k \leq n$).

If $1 \leq k \leq n-1$ and $n-k$ is odd, then we have that

$$\left\| x_j^k \frac{\partial^k u}{\partial x_j^k} \right\|^2 \leq \|u\|^{2(1-k/n)} \left\| x_j^n \frac{\partial^n u}{\partial x_j^n} \right\|^{2k/n} \leq C_\varepsilon \|u\|^2 + \varepsilon \left\| x_j^n \frac{\partial^n u}{\partial x_j^n} \right\|^2.$$

Choosing $c_{n,k}\,\varepsilon \leq \frac{2}{n} c_{n,n}$, we obtain ($*$). ■

Verification of conditions (I)–(IV) in Theorem 1.

Given

$$A(t) = i(-\Delta_x + V(x,t)), \quad S = 1 + \sum_{k=1}^{N} \Big((-1)^n \frac{\partial^{2n}}{\partial x_k^{2n}} + x_k^{2n} \Big),$$

assume that (V0)–(V2) are satisfied. Then $\forall\ T > 0$

(I) $\mathrm{Re}(A(t)v, v) = 0, \ v \in D(A(t)), \ \text{a.a. } t \in (0,T).$

(II) $Y = D(S^{1/2}) \subset D(A(t)), \ \text{a.a. } t \in (0,T).$

(III) $0 \le \exists\ \beta \in L^1(0,T);\ \forall\ u \in D(S)$

$$|\,\mathrm{Re}(A(t)u, Su)| \le \beta(t) \big\| S^{1/2} u \big\|^2, \quad \text{a.a. } t \in (0,T).$$

(IV) $A(\cdot) \in L^2_*(0,T; B(Y, L^2(\mathbb{R}^N)))$.

- (I) is clear by symmetry of $A(t) = -\Delta + V(x,t)$.

• (II) and (IV) follows from the measurability of $V(x,t)$ and (V1).

To see this put $(V(t)v)(x) := V(x,t)v(x)\ \forall\ v \in Y$.

Then $A(t) = i(-\Delta + V(t))$ and we have

Lemma 5. $Y \subset D(A(t))$, a.a. $t \in (0,T)$, $A(t)$ is strongly measurable with values in $L(Y,X)$, and $\exists\ \gamma \in L^2(0,T);\ \forall\ v \in Y = \Sigma(n)$

$$(**) \qquad \|A(t)v\| \le \gamma(t)\|v\|_Y, \ \text{ a.a. } t \in (0,T).$$

In fact, (V1) implies that $Y \subset D(A(t))$ (a.e.) with $(**)$.

Since $V(t)$ is w-measurable and Y is separable, $V(t)$ is s-measurable and $\|V(t)\|_{L(Y,X)}$ is measurable.

Proof of (III). Let $A(t)$ and S as above.

We shall prove that $\exists\ \beta_n \in L^1(0,T)$; $\forall\ u \in D(S)$

$$(\bigstar) \qquad \left|\mathrm{Im}\left(-\Delta u + V(t)u, Su\right)\right| \le \beta_n(t)\left\|S^{1/2}u\right\|^2.$$

To this end we start with

$$\begin{aligned}
&(-\Delta u + V(t)u, Su) - \left\|\nabla u\right\|^2 \\
&= \sum_{k=1}^{N}\left[\left\|\nabla \frac{\partial^n u}{\partial x_k^n}\right\|^2 + (-\Delta u, x_k^{2n} u)\right. \\
&+ \left(V(t)u, (-1)^n \frac{\partial^{2n} u}{\partial x_k^{2n}}\right) + \left.\int_{\mathbb{R}^N} V(x,t)(1 + x_k^{2n})|u(x)|^2\,dx\right].
\end{aligned}$$

Here 1st and 4th terms on RHS are <u>real</u>.

We estimate 2nd and 3rd terms using Lin's lemma.

Lemma 6 **[C.S. Lin (1986)]** **(Interpolation inequality)**

$\psi \in \Sigma(m)$

$\Longrightarrow 0 \leq \forall\, j \leq m$

$$\sum_{|\alpha|=j} \left\| |x|^{m-j} D^{\alpha}\psi \right\| \leq C(j,m) \sum_{|\beta|=m} \left\| D^{\beta}\psi \right\|^{j/m} \left\| |x|^{m}\psi \right\|^{1-j/m}.$$

$$\text{2nd term} = (-\Delta u, x_k^{2n} u)$$
$$= \sum_{j=1}^{N} \Big(\frac{\partial u}{\partial x_j}, 2n x_k^{2n-1} \delta_{jk} u \Big) + (\nabla u, x_k^{2n} \nabla u)$$
$$= 2n \Big(x_k^{n-1} \frac{\partial u}{\partial x_k}, x_k^n u \Big) + \big\| x_k^n \nabla u \big\|^2.$$

The second term on RHS is real. Now Lemma 6 yields

$$|\operatorname{Im}(-\Delta u, x_k^{2n} u)| \le 2n \Big\| x_k^{n-1} \frac{\partial u}{\partial x_k} \Big\| \big\| x_k^n u \big\|$$
$$\le 2n C(1,n) \Big\| \frac{\partial^n u}{\partial x_k^n} \Big\|^{1/n} \big\| x_k^n u \big\|^{1-1/n} \big\| x_k^n u \big\|$$
$$= 2n C(1,n) \Big\| \frac{\partial^n u}{\partial x_k^n} \Big\|^{1/n} \big\| x_k^n u \big\|^{2-1/n}.$$

By Young's inequality we obtain

$$\sum_{k=1}^{N} |\operatorname{Im}(-\Delta u, x_k^{2n} u)| \le (2n-1) C(1,2n) \big\| S^{1/2} u \big\|^2.$$

$$\text{3rd term} = \Big(V(t)u, (-1)^n \frac{\partial^{2n} u}{\partial x_k^{2n}}\Big) = \Big(\frac{\partial^n}{\partial x_k^n}(V(t)u), \frac{\partial^n u}{\partial x_k^n}\Big)$$

$$= \int_{\mathbb{R}^N} V(t)\Big|\frac{\partial^n u}{\partial x_k^n}\Big|\, dx + \sum_{j=1}^{n} \binom{n}{j} \Big(\frac{\partial^j V(t)}{\partial x_k^j}\frac{\partial^{n-j} u}{\partial x_k^{n-j}}, \frac{\partial^n u}{\partial x_k^n}\Big).$$

The first term is <u>real</u>. For every j $(1 \le j \le n)$ by (V2):

$$\Big|\Big(\frac{\partial^j V(t)}{\partial x_k^j}\frac{\partial^{n-j} u}{\partial x_k^{n-j}}, \frac{\partial^n u}{\partial x_k^n}\Big)\Big| \le \Big\|\Big|\frac{\partial^j V(t)}{\partial x_k^j}\Big|\frac{\partial^{n-j} u}{\partial x_k^{n-j}}\Big\|\Big\|\frac{\partial^n u}{\partial x_k^n}\Big\|$$

$$\le \Big(\Big\|V_{j1}(t)\frac{\partial^{n-j} u}{\partial x_k^{n-j}}\Big\| + \Big\|V_{j2}(t)\frac{\partial^{n-j} u}{\partial x_k^{n-j}}\Big\|\Big)\|S^{1/2}u\|,$$

where $V_{j1}(\cdot,\cdot) \in L^{p_j}(\mathbb{R}^N) \times L^1_{\mathrm{loc}}(\mathbb{R}^+)$,

$V_{j2}(\cdot,\cdot) \in \langle x\rangle^j L^\infty(\mathbb{R}^N) \times L^1_{\mathrm{loc}}(\mathbb{R}^+)$.

- For the function with $V_{j1}(t)$ we have

$$\begin{aligned}\left\|V_{j1}(t)\frac{\partial^{n-j}u}{\partial x_k^{n-j}}\right\| &\le \left\|V_{j1}(t)\right\|_{p_j}\left\|\frac{\partial^{n-j}u}{\partial x_k^{n-j}}\right\|_{q_j}\\ &\le c_1\left\|V_{j1}(t)\right\|_{p_j}\left\|\frac{\partial^{n-j}u}{\partial x_k^{n-j}}\right\|_{H^j}\\ &\le c_1\left\|V_{j1}(t)\right\|_{p_j}\|u\|_{H^n}\\ &\le c_1'\left\|V_{j1}(t)\right\|_{p_j}\left\|S^{1/2}u\right\|,\end{aligned}$$

where q_j satisfies $1/p_j + 1/q_j = 1/2$
and constants c_1, c_1' depend on j.

- For the function with $V_{j2}(t)$ we have

$$\begin{aligned}
\left\|V_{j2}(t)\frac{\partial^{n-j}u}{\partial x_k^{n-j}}\right\| &\le \|\langle x\rangle^{-j}V_{j2}(t)\|_\infty \left\|\langle x\rangle^{j}\frac{\partial^{n-j}u}{\partial x_k^{n-j}}\right\| \\
&\le c_2\|\langle x\rangle^{-j}V_{j2}(t)\|_\infty \left(\left\|\frac{\partial^{n-j}u}{\partial x_k^{n-j}}\right\| + \sum_{l=1}^{N}\left\|x_l^j\frac{\partial^{n-j}u}{\partial x_k^{n-j}}\right\|\right) \\
&\le c_2\|\langle x\rangle^{-j}V_{j2}(t)\|_\infty \left(\|u\|^{\frac{j}{n}} + \sum_{l=1}^{N} C(n-j,n)\|x_l^n u\|^{\frac{j}{n}}\right)\left\|\frac{\partial^{n}u}{\partial x_k^{n}}\right\|^{1-\frac{j}{n}} \\
&\le c_2'\|\langle x\rangle^{-j}V_{j2}(t)\|_\infty \|S^{1/2}u\|,
\end{aligned}$$

where constants c_2, c_2' depend on j.

Combining those mentioned above, we obtain (★).

4. Resumé

The Cauchy problem for Schrödinger evolution equations:

$$\text{(SE)} \quad \begin{cases} i\,\partial u/\partial t - \big(-\Delta + V(x,t)\big)u = 0, \quad \text{a.a. } t \in \mathbb{R}_+, \\ u(x,0) = u_0(x) \in H^n(\mathbb{R}^N) \cap D(|x|^n) \end{cases}$$

in $X = L^2(\mathbb{R}^N)$. (SE) has a **unique strong solution**

$$u(\cdot) \in H^1_{\mathrm{loc}}([0,\infty); L^2(\mathbb{R}^N)) \cap C([0,\infty); \Sigma(n))$$

under the following conditions on $V(x,t)$:

- $V(x,t) : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$ — measurable.

(V0) $V(\cdot,t) \in C^n(\mathbb{R}^N)$, a.a. $t \in \mathbb{R}_+$.

(V1) $V \in \big(L^{p_n}(\mathbb{R}^N) + \langle x\rangle^n L^\infty(\mathbb{R}^N)\big) \times L^2(0,T)$ $\forall\, T > 0$.

(V2) For $j \in \mathbb{N}$ $(1 \le j \le n)$, $\forall\, T > 0$

$$\sum_{k=1}^N \left|\frac{\partial^j V}{\partial x_k^j}\right| \in \big(L^{p_j}(\mathbb{R}^N) + \langle x\rangle^j L^\infty(\mathbb{R}^N)\big) \times L^1(0,T).$$