Riesz basis for heat equation with memory

Application to an inverse problem

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The system

$$\theta_t = \int_0^t N(t-s)\Delta\theta \, \mathrm{d}s \,. \qquad (1)$$

Here, $\theta = \theta(t,x)$ with $x \in (0,\pi)$ and $t > 0$. Initial and boundary conditions are

$$\theta(0) = \theta(0,x) = 0 \quad x \in (0,\pi)\,, \quad \theta(t,\pi) = 0\,, \quad t > 0\,.$$

and

$$\theta(t,0) = u(t)\,.$$

Here u is locally square integrable.

The control problem

- control problem: find u which drives the initial condition to a prescribed target $\eta \in L^2(0,\pi)$, in time T.
- Fact: This problem is solvable for any $\eta \in L^2(0,\pi)$ provided that $T > \pi$. Barbu, Yong and Zhang (also P. who does not identify $T = \pi$)
- It is easily seen that the reachable η are dense in $L^2(0,\pi)$ if $T = \pi$.

Moment problem-1

The controllability problem is equivalent to a *moment problem.* This is seen as follows:
Let $h(t)$ solve

$$h'(t) = -\lambda \int_t^T N(r-t)h(r)\,\mathrm{d}r\,, \qquad h(T) = 1 \tag{2}$$

Let $\phi(x)$ solve

$$\Delta\phi = -n^2\phi \text{ with } \phi(0) = \phi(\pi) = 0$$

i.e.

$$\phi(x) = \phi_n(x) = \sin nx\,.$$

Moment problem-2

- Multiply both the sides of

$$\theta_t = \int_0^t N(t-s)\Delta\theta \; \mathrm{d}s$$

with $h(t)\phi(x)$.

- Partial integration on $[0,T] \times [0,\pi]$ gives the equality

$$\int_0^\pi \phi(x)\theta(T,x)\;\mathrm{d}x = \phi'(0)\int_0^T h(r)\tilde{v}(r)\;\mathrm{d}r\,,$$
$$\tilde{v}(r) = \int_0^r N(r-s)u(s)\;\mathrm{d}s\,. \qquad (3)$$

Moment problem-3

We can go the other way around and we get the following

Theorem:

L et $\eta \in L^2(0,\pi)$ be the prescribed target. A square integrable control u which transfer $\theta(0) = 0$ to η in time T exists if and only if for every n we can solve the moment problem

$$\int_0^T h_n(r)\tilde{v}(r)\,\mathrm{d}r = \frac{1}{\phi_n'(0)} \int_0^\pi \phi_n(x)\eta(x)\,\mathrm{d}x \tag{4}$$

and then we can solve

$$\tilde{v}(r) = \int_0^r N(r-s)u(s)\,\mathrm{d}s\,. \tag{5}$$

We recapitulate

From the known results on the controllability problem we know:

- The moment problem is solvable in time $T > \pi$.
- If $T = \pi$ the moment problem is solvable provided that the right hand side belong to a dense subspace of l^2.

From Avdonin-Ivanov book:

- the set $\{h_n\}$ is a minimal basis in its span (as a subspace of $L^2(0, \pi)$);
- It is ω-minimal if $T = \pi$. This means that if

$$\sum \alpha_k^2 < +\infty \quad \text{and} \quad \sum \alpha_k h_k = 0$$

then $\alpha_k = 0$ for every k.

The goal

Fact:

- computation with minimal or ω-independent basis practically impossible.
- Feasible computations are possible with Riesz basis.

A basis is Riesz when it is the image of an orthogonal basis under a linear continuous invertible transformation.

So, the goal

To prove that $\{h_n\}$ is a Riesz basis.

A transformation

It is more natural if we transform $z_n(t) = h_n(T-t)$. The function $z_n(t)$ solves

$$z_n'(t) = -n^2 \int_0^t N(t-s) z_n(s)\, \mathrm{d}s\,, \qquad z_n(0) = 1 \tag{6}$$

- The moment problem is (Note the factor $1/n$)

$$\langle z_n, v\rangle = \frac{1}{n} c_n\,, \qquad c_n = \langle \eta, \phi_n\rangle$$

- The control steering to η is given by

$$\int_0^t N(t-s)u(s)\, \mathrm{d}s = v(t)\,.$$

A WARNING

Assumptions on $N(t)$: it is of class C^3 with $N(0) = 1$.

WARNING!

We shall present computations using the condition $N'(0) = 0$. This assumption is used SOLELY to present simpler formulas. It is not at all needed for the results.

A Volterra equation for $z_n(t)$

- We have

$$z_n'(t) = -n^2 \int_0^t N(t-s) z_n(s) \,\mathrm{d}s\,, \qquad z_n(0) = 1$$

so that $z_n(0) = 1$, $z_n'(0) = 0$ and

$$z_n''(t) = -n^2 z_n(t) - n^2 \int_0^t N'(t-s) z_n(s) \,\mathrm{d}s\,,$$

- and

$$z_n(t) = \cos nt - n \int_0^t \sin n(t-s) \int_0^s N'(s-r) z_n(r) \,\mathrm{d}r \;\mathrm{d}s\,.$$

This suggest that we compare $z_n(t)$ and $\cos nt$.

Comparison and Bari Theorem

Is comparison of any use?

Answer YES, thanks to Bari Theorem whose (loose) statement is

- If $\{\epsilon_n\}$ is orthogonal with constant norm;
- if $\{z_n\}$ is ω-independent (WE KNOW THAT IT IS!) and L^2-close to $\{\epsilon_n\}$, i.e.

$$\sum ||z_n - \epsilon_n||^2 < +\infty$$

then $\{z_n\}$ is a Riesz basis.

The comparison

- Write

$$e_n(t) = (z_n(t) - \cos nt) =$$
$$-n \int_0^t \sin n(t-s) \int_0^s N'(s-r)(e_n(r) + \cos nr)\ \mathrm{d}r\ \ \mathrm{d}s\,.$$

- use $n \sin n(t-s) = \frac{\mathrm{d}}{\mathrm{d}s} \cos n(t-s)$. Integrate by parts.

$$e_n(t) = -\int_0^t N'(t-r)e_n(r)\ \mathrm{d}r + \int_0^t \cos n(t-r) \int_0^r N''(r-s)e_n(s)$$
$$-\int_0^t N'(t-s)\cos ns\ \mathrm{d}s + \int_0^t \cos n(t-r) \int_0^r N''(r-s)\cos ns\ \mathrm{d}s\ \ \mathrm{d}r$$

Note that the last integrals are of the order $1/n$.

Conclusion

Gronwall inequality implies

- $$\sup_{t\in[0,T]} |e_n(t)| \leq \frac{M}{n}, \qquad M = M(T)\,.$$
- In particular:

the sequence $\{z_n\}$ is bounded on $L^2(0,T)$ for every T and it is a Riesz basis (in its span) if $T = \pi$, thanks to Bari theorem.

Further properties

Similar methods, bases on differentiation/partial integration, can be used to prove that

- $\{z_n'(t)\}$ is L^2-close to $\{-\sin t\}$.
- $\{n\int_0^t z_n(s)\,\mathrm{d}s\}$ is L^2-close to $\{\sin nt\}$.

the sequence of the functions

$$n\int_0^t z_n(s)\,\mathrm{d}s$$

is a Riesz basis.

Consequences on controllability

The control steering to η is obtained from

$$\int_0^t N(t-s)u(s)\,\mathrm{d}s = v(t)$$

where $v(t)$ solves the moment problem. If we can prove that v' exists, then it can be practically computed.

- Fact: $v(t)$ has a representation in terms of the biorthogonal basis of $\{z_n\}$ (this means$\langle z_n, \zeta_k\rangle = \delta_{n,k}$)

$$v(t) = \sum v_n \zeta_n \,.$$

- Due to the form of the moment problem, $v_n = \hat{v}_n/n$, $\{\hat{v}_n\} \in l^2$.

Differentiability of v

Using

$$n \int_0^t z_n(s)\ \mathrm{d}s$$

Riesz basis, let $\{w_k\}$ be its biorthogonal sequence.

$$\delta_{nj} = n \int_0^T w_j(t) \int_0^t z_n(\tau)\ \mathrm{d}\tau\ ,\ \mathrm{d}t = \int_0^T z_n(\tau) \left[n \int_\tau^T w_j(s)\ \mathrm{d}s \right]\ \mathrm{d}\tau$$

so that we can choose $\zeta_n(t) = n \int_t^T w_n(s)\ \mathrm{d}s$.

So: $\{z_n(t)\}$ has a differentiable biorthogonal sequence.

Determination of the control

Now,

$$v(t) = \sum_n \frac{c_n}{n} \zeta_n(t) = \sum_n c_n \int_t^T w_n(s)\, \mathrm{d}s$$

shows that $v'(t) \in L^2(0,\pi)$ so that the Volterra equation of the *first kind* for u is equivalent to the solvable Volterra equation of the *second kind*

$$u(t) + \int_0^t N'(t-s)u(s)\, \mathrm{d}s = v'(t)$$

and *the control problem can be solved in time* π *for every target* $\eta \in L^2(0,\pi)$. Note the crucial role of the factor $1/n$.

WE SUM UP

- The moment method gives a formula for the steering control.
- So, we can wonder whether this formula can be used to solve different problems: namely an inverse problem as in Yamamoto and Grasselli-Yamamoto papers.

Description of the inverse problem

$$\xi_t = \int_0^t N(t-s)\Delta\xi \, \mathrm{d}s + B(x)\sigma(t)\,.$$

Here, $\xi = \xi(t,x)$ with $x \in (0,\pi)$ and $t > 0$ and

$$\xi(0) = 0 \quad x \in (0,\pi)\,, \quad \xi(t,1) = 0\,, \quad \xi(t,0) = 0\,, \quad t > 0\,. \tag{7}$$

The function $\sigma(t)$ is known with suitable properties.
PROBLEM: To identify the function $B(x)$ from the observation

$$y(t) = \xi_x(t,0)\,, \qquad t \in [0,\pi]\,. \tag{8}$$

Properties of $\sigma(t)$: it is of class C^1 and $\sigma(0) = 1$.

Facts-1

It is convenient to recast the arguments of Yamamoto and Grasselli-Yamamoto in a more abstract form. So we note:

- $\xi(t)$ solves the heat equation with memory if and only if

$$\xi(t) = \int_0^t R_+(t-s)B\sigma(s)\,\mathrm{d}s + \int_0^t L(t-s)\xi(s)\,\mathrm{d}s$$

  where $R_+(t) = \frac{e^{\mathcal{A}t}+e^{-\mathcal{A}t}}{2}$, $\mathcal{A} = i(-A)^{1/2}$ and $L(t)$ is a bounded operator valued function, which leaves the domain of $\mathcal{A}$ invariant.

Facts-2

- So,

$$\xi(t) = \int_0^t M(t-s)B\sigma(s)\ \mathrm{d}s$$

where $M(t)$ is given by

$$M(t)B = R_+(t)B + \int_0^t H(t-\tau)R_+(\tau)B\ \mathrm{d}\tau\ ,$$

$H(t)$ being the resolvent kernel of $L(t)$.

- Regularity of $\sigma(t)$ implies that

$$\xi(t) \in C(0, T; \mathrm{dom}\, A)\ . \tag{9}$$

In particular: the output $y(t)$ makes sense.

Facts-3

A last piece of information we need is that

$$\xi \in \mathrm{domA} \implies \xi'(0) = -D^* A\xi \,, \tag{10}$$

where

$$(Du)(x) = (1-x)u \,.$$

Nothe that $g(x) = Du$ solves

$$\Delta g = 0 \,, \qquad g(0) = u \,, \quad g(\pi) = 0 \,.$$

Key idea for the reconstruction

- We collect the information contained in the output y in the following integral:

$$\int_0^T h(s)y(s)\,\mathrm{d}s\,.$$

- We prove that for $T \geq \pi$ it is possible to compute $h(t) = h_k(t)$ so to have ($\phi_k(x) = \sin kx$)

$$\int_0^T y(s)h_k(s)\,\mathrm{d}s = \int_0^\pi B(x)\phi_k(x)\,\mathrm{d}x\,.$$

so that

$$B(x) = \sum_k \left[\int_0^\pi y(s)h_k(s)\,\mathrm{d}s\right]\phi_k(x)\,.$$

Construction of h_k-1

- We note (the crochet denotes the inner product in $L^2(0,\pi)$)

$$\int_0^T h(s)y(s)\,\mathrm{d}s = \int_0^T \langle Dh(s), A\xi(s)\rangle\,\mathrm{d}s\,.$$

- Below $\langle\langle, \cdot, \cdot\rangle\rangle$ is the pairing of $[\mathrm{dom}(A)]'$ and $[\mathrm{dom}(A)]$.

$$\int_0^T \langle Dh(s), A\xi(s)\rangle\,\mathrm{d}s = \int_0^T \langle\langle ADh(s), \xi(s)\rangle\rangle\,\mathrm{d}s$$
$$= \int_0^T \langle\langle ADh(s), \int_0^s M(s-r)B\sigma(r)\,\mathrm{d}r\rangle\rangle\,\mathrm{d}s$$
$$= \int_0^T \langle\langle ADh(s), \int_0^s M(r)B\sigma(s-r)\,\mathrm{d}r\rangle\rangle\,\mathrm{d}s\,.$$

Construction of h_k-2

Let now B be “smooth” (at the end a limiting process can be used to remove this) and

$$u(T-r) = \int_0^{T-r} h(s)\sigma(T-r-s)\,\mathrm{d}s.$$

Note that u is smooth with $u(0) = 0$.

- We have

$$\int_0^T h(s)y(s)\,\mathrm{d}s = \left\langle\!\left\langle \int_0^T M(T-r)ADu(r)\,\mathrm{d}r, B \right\rangle\!\right\rangle .$$

- Introduce

$$\theta(t) = \int_0^t M(t-r)ADu(r)\,\mathrm{d}r\,.$$

Construction of h_k-3

- Recall:

$$\theta(t) = \int_0^t M(t-r) ADu(r)\, \mathrm{d}r\,.$$

- Using the definition of $M(t)$, $\theta(t)$ solves

$$\theta(t) = \int_0^t R_+(t-s) ADu(s)\, \mathrm{d}s + \int_0^t L(t-s)\theta(s)\, \mathrm{d}s$$

- Thanks to the differentiability of $u(t)$ we have

$$\theta(t) = -\mathcal{A} \int_0^t R_-(t-s) Du'(s)\, \mathrm{d}s$$

So, CONTROLLABILITY HERE!

- We repeate

$$\theta(t) = -\mathcal{A}\int_0^t R_-(t-s)Du'(s)\,\mathrm{d}s$$

i.e.

$$\theta'(t) = \int_0^t N(t-s)\theta_{xx}(s)\,\mathrm{d}s\,, \qquad \begin{cases} \theta(0,x) = 0 \\ \theta(t,\pi) = 0\,, \\ \theta(t,0) = u'(t)\,. \end{cases}$$

A CONTROLLABLE PROBLEM!

Conclusion-1

- Controllability of the heat equation with memory shows the existence of ${u_k}'(t)$ such that $\theta(T) = \phi_k$ so that

$$\int_0^T h(s)y(s)\,\mathrm{d}s = \langle\!\langle \int_0^T M(T-r)ADu(r)\,\mathrm{d}r, B \rangle\!\rangle = \langle \phi_k, B \rangle \,.$$

- Repalcement of $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ with $\langle \cdot, \cdot \rangle$ can be justified.

Conclusion-2

The previous argument identifies ${u_k}'(t)$

We need $h(t) = h_k(t)$ and we know that

$$u(T-r) = u_k(T-r) = \int_0^{T-r} h_k(s)\sigma(T-r-s)\,\mathrm{d}s.$$

Differentiability of $u_k(t)$ shows that $h_k(t)$ can be computed from

$$h_k(T-r) + \int_0^{T-r} \sigma'(T-r-s)h_k(s)\,\mathrm{d}s = u_k'(T-r)\,.$$

This we wanted to achieve.

Conclusion-2

- The previous arguments shows that the reconstruction ideas in Yamamoto and Grasselli-Yamamoto can be extended to the case of heat equations with memory;
- The previous arguments works equally well in the case of space variables in suitable regons $\Omega \subseteq \mathbb{R}^n$;
- The moment method however provides a practical formula for the computation of the functionsv $h_k(t)$, in the case $n = 1$.