Convergence to equilibria of solutions of nonlocal phase-separation models

$$\vartheta_t + \chi_t - \Delta\vartheta = f$$

$$\partial_t\chi + \chi - J * \chi + W'(\chi) = \vartheta,\ t > 0,\ x \in \Omega \subset R^3,$$

$$\varepsilon\partial_{tt}\chi + \partial_t\chi + \chi - J * \chi + W'(\chi) = \vartheta,$$

where

$$J * \chi(x,t) = \int_\Omega J(x-y)\chi(y,t)\ \mathrm{d}y \stackrel{def}{=} \mathcal{J}[\chi],$$

$$u(x,0) = u_0(x) \in L^\infty(\Omega)$$

==============================

$$u_t - \nabla\cdot(\mu\nabla v) = 0$$

$$v = f'(u) + \int_\Omega J(|x-y|)(1-2u(t,y))dy,$$

$$\mu\nu\cdot\nabla v = 0 \text{ in } (0,T)\times\partial\Omega,$$

$$f(u) = u\ln u + (1-u)\ln(1-u),\ \mu = \frac{a}{f''(u)},$$

$$u_0 \in L^\infty(\Omega),\ \ 0 \le u_0(x) \le 1,\ \ 0 < \int_\Omega u_0 = u_\alpha < 1.$$

$$\dot{u}(t) + M(u(t)) = g(t), \quad u(0) = u_0 \in H$$

$$\ddot{u}(t) + \dot{u}(t) + M(u(t)) = g(t), \;\; u(0) = u_0 \in V, \dot{u}(0) = u_1 \in H$$

$$V \subset H \subset V', \quad M = D\,E$$

$$|E(v) - E(\phi)|^{1-\theta} \leq C\|M(v)\|_{V'}, \quad \phi \in \omega(u)$$

$$\{u(t); t \geq 0\} \text{ is relatively compact in } V$$

Then

$$\lim_{t\to\infty} \|u(t) - \phi\|_V = 0$$

The energy functional takes a form

$$E[u] = \int_\Omega (\frac{1}{2}|\nabla u|^2 + F(u))\, \mathrm{d}x,$$

i.e., it is a quadratic form perturbed by a non-linear compact functional. The underlying function spaces are the Sobolev spaces $W^{1,p}(\Omega)$ on which E is analytic provided p is large enough and F is an analytic function.

The energy functional related to our problems reads

$$E[\chi] = E[\chi, 0] = \int_\Omega (\frac{1}{2}\chi(\chi - \mathcal{J}[\chi]) + W(\chi))\, \mathrm{d}x.$$

the natural domain of E is $L^2(\Omega)$.

$$E \in C^1(L^2(\Omega); R), \quad E \notin C^2(L^2(\Omega); R)$$

no matter how smooth W is.

$$u_t - \nabla \cdot (\mu \nabla v) = 0$$

$$v = f'(u) + \int_\Omega J(|x-y|)(1-2u(t,y))dy = \log \frac{u}{1-u} + w,$$

$$\mu \nu \cdot \nabla v = 0 \text{ in } (0,T) \times \partial\Omega,$$

$$f(u) = u \ln u + (1-u)\ln(1-u), \;\; \mu = \frac{a}{f''(u)} = u(1-u),$$

$$u_0 \in L^\infty(\Omega), \quad 0 \le u_0(x) \le 1, \quad 0 < \int_\Omega u_0 = u_\alpha < 1.$$

Stationary solutions

$$u^* = \frac{1}{1+\exp(w^* - v^*)}, \quad v^* = const,$$

$$w^* = \int_\Omega K(|x-y|)(1-2u^*(t,y))\mathrm{d}y$$

The energy functional

$$F(z) = \int_\Omega f(z) + u\mathcal{J}(z) + z \cdot K * 1$$

$$\frac{\mathrm{d}}{\mathrm{dt}} F(u(t)) = -\int_\Omega \mu |\nabla v|^2 \mathrm{d}x$$

$$F'(u(t)) = f'(u(t)) - 2\mathcal{J}(u(t)) + l = v(t)$$

$$\int_t^\infty \int_\Omega \mu |\nabla v|^2 \, \mathrm{d}x\mathrm{d}s = F(u(t)) - F_\infty = F(u(t)) - F(u^*).$$

By virtue of the generalized Lojasiewicz theorem (Gajewski, Griepentrog)

$$|F(u(t)) - F(u^*)|^{1-\theta} \leq$$
$$\lambda \inf\{\|v(t) - z\|_{L^2(\Omega)};\ z = const\} = \lambda \|v(t) - \textstyle\fint_\Omega v(t)\|_{L^2(\Omega)}$$

for

$$\|u(t) - u^*\|_{L^2(\Omega)} \leq \delta.$$

$$\frac{4}{a} \int_t^\infty \int_\Omega (\mu |\nabla v|)^2 \mathrm{d}x\mathrm{d}s \leq \int_t^\infty \int_\Omega \mu |\nabla v|^2 \,\mathrm{d}x\mathrm{d}s$$
$$\leq \lambda \|v(t) - \textstyle\fint_\Omega v(t)\|_{L^2(\Omega)}^{\frac{1}{1-\theta}} \leq \|\mu |\nabla v|(t)\|_{L^2(\Omega)}^{\frac{1}{1-\theta}},$$

$$\Rightarrow u_t \in L^1(T, \infty; H^1(\Omega)^*$$

provided that $0 < k \leq \mu \leq 1 - k$ a.a. in $((T, \infty) \times \Omega)$

$$\|\log u_0\|_{L^\infty(\Omega)} < \infty$$

Separation property for general initial condition

$$u_0 \in L^\infty(\Omega),\quad 0 \leq u_0(x) \leq 1,\quad 0 < \textstyle\fint_\Omega u_0 = u_\alpha < 1.$$

$$\|\ln u(t)\|_{L^r\Omega)} \leq B_1 \|\ln u(0)\|_{L^r(\Omega)},\quad t \geq 0,$$

$$\|\ln u(t)\|_{L^r(\Omega)} \leq B_2 r^2,\quad t \geq 1,$$

$$\|\ln u(t)\|_{L^r(\Omega)} \leq B_3,\quad t \geq t_r.$$

Keller-Segel chemotaxis model

$$\partial_t u(t,x) = \operatorname{div}(\nabla u(t,x) - u(t,x)\nabla v(t,x)),$$

$$\partial_t v(t,x) = \alpha\Delta v(t,x) - \beta v(t,x) + \gamma(u(t,x)-1) \text{ for } t>0,\ x\in\Omega,$$

$$\nabla u(t,x)\cdot\nu(x) = \nabla v(t,x)\cdot\nu(x) = 0,\ t>0,\ x\in\partial\Omega,$$

$u = u(t,x)$ -the species density,
$v = v(t,x)$ - a rescaled chemical substance density.

Lyapunov function

$$F[u,v] \equiv \fint_\Omega (u\log(u) + \frac{\alpha}{2\gamma}|\nabla v|^2 + \frac{\beta}{2\gamma}|v|^2 - uv),$$

defined on the space

$$X = \{(u,v) \mid v\in W^{1,2}(\Omega),\ u\in L^2(\Omega),\ \ \fint_\Omega u = 0\}.$$

F *is not* twice continuously differentiable on X. The gradient ∂F is of type "monotone operator + linear compact perturbation" while the usual form is "linear isomorphism + (nonlinear) compact perturbation". The operator associated to the gradient ∂F in the sense of distribution does not map the space X into itself, it does not conserve the (zero) mean of the first component.

$$|F(u,v) - F(U,V)|^{1-\theta} \le m\|\partial F(u,v)\|_{X^*}$$

$$\text{whenever } \|(u,v)-(U,V)\|_X < \epsilon.$$

U, V are solutions of the stationary problem.

References

[1] E. Feireisl, F. Issard-Roch, and H. Petzeltová. A non-smooth version of the Lojasiewicz-Simon theorem with applications to non-local phase-field systems. *J. Differential Equations*, **199**:1–21, 2004.

[2] M. Grasselli, H. Petzeltová, and G. Schimperna. A non-local phase-field system with inertial term. *Quart. Appl. Math.*, **65**:451–469, 2007.

[3] S.-O. Londen, and H.Petzeltová. Convergence of solutions of a nonlocal phase-field system. *Preprint*

[4] E. Feireisl, P. Laurençot, and H. Petzeltová. On convergence to equilibria for the Keller-Segel chemotaxis model. *JDE.*, **236**:551–569, 2007.

[5] E. Feireisl, and H. Petzeltová. Non-standard applications of the Łojasiewicz-Simon theory: stabilization to equilibria of solutions to phase-field models. *Banach Center Publications*, 2008.