Data assimilation problems and Tychonov regularization revisited

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Data Assimilation : Very important problem in

- Weather prediction
- Climatology
- Oceanology
- All environment sciences

Corresponds to an enormous amount of computing time, for example 60% of computing time in meteorology.

For example : weather prediction.

To compute evolution of air masses, pressure, temperature, etc. on an annulus around the planet up to an altitude of 51 km.

Model : enriched every day but rather satisfactory at the moment.

Computers : more and more powerful ... reasonable possibilities.

Missing : knowledge of the state variables to-day or yesterday (initial conditions) on the whole spatial domain !! in order to predict the evolution.

For simplicity of presentation, all the presentation will be given on the heat equation.

Analogous results and proofs for

- Diffusion-convection equations
- Linearized Navier-Stokes equations
- Linearized Boussinesq
- More generally : coupled (linearized) systems of diffusion-convection and Navier-Stokes

All operators considered here are linear.

Up to now : linear method.

Hope of extension to some nonlinear systems...

Mathematical problem : Heat equation

$$\begin{aligned}
&\frac{\partial y}{\partial t} - \Delta y = f \text{ on } \Omega \times (0,T),\\
&y = 0 \text{ on } \partial\Omega \times (0,T),\\
&y(0) = y_0 \text{ in } \Omega.
\end{aligned}$$

y_0 is unknown, but we know measurements of the solution on a sub-domain $\omega \times (0,T_0)$, $T_0 < T$.

$$h = y_{\omega\times(0,T_0)}.$$

For example

$$0 = \text{ yesterday } .\ T_0 = \text{ to-day. } T = 2T_0 = \text{ to-morrow.}$$

A simple translation by $\tilde{y}$ solution (which can be computed) of

$$
\begin{aligned}
&\frac{\partial \tilde{y}}{\partial t} - \Delta \tilde{y} = f \text{ on } \Omega \times (0,T),\\
&\tilde{y} = 0 \text{ on } \partial\Omega \times (0,T),\\
&\tilde{y}(0) = 0 \text{ in } \Omega
\end{aligned}
$$

reduces the problem to the case where $f = 0$ called (HE).

$$
\frac{\partial y}{\partial t} - \Delta y = 0 \text{ on } \Omega \times (0,T), \tag{1}
$$

$$
y = 0 \text{ on } \partial\Omega \times (0,T), \tag{2}
$$

$$
y(0) = y_0 \text{ in } \Omega. \tag{3}
$$

Question :

How to recover the value of the state variable $y(t)$

at a time t, $0 \leq t \leq T_0$ in order to compute y

on the time interval (T_0, T)?

Classical method : Variational data assimilation.

In use at the moment in several meteorological centers (European center Reading, Meteo-France...)

Based on optimal control ideas.

Try to recover $y(0) = y_0$, then run the system from $t = 0$ to $t = T$.

y_0 : control variable. Solve the equation for y_0 given $\to y(y_0)$.

Define

$$J_0(y_0) = \frac{1}{2} \int_0^{T_0} \int_\omega |y(y_0) - h|^2 dx dt$$

To find $\bar{y}_0$ such that

$$J_0(\bar{y}_0) = \min_{y_0} J_0(y_0) \qquad (OC)$$

This problem does not have a solution (ill-posed problem).

If we take a minimizing sequence y_0^n, then we only know that

$$\int_0^{T_0} \int_\omega |y(y_0^n)|^2 dx dt \leq C.$$

This gives no estimate on y_0^n (in any known functional space)!!

Remedy :

Tychonov regularization.

For $\alpha > 0$ (Tychonov regularization parameter) define

$$J_\alpha(y_0) = \frac{1}{2}\int_0^{T_0}\int_\omega |y(y_0) - h|^2 dxdt + \frac{\alpha}{2}||y_0||^2$$

Now find y_α such that

$$J_\alpha(y_\alpha) = \min_{y_0} J_\alpha(y_0) \qquad (TROC)$$

This problem has a unique solution y_α (classical methods)

This problem is known to be unstable when $\alpha \to 0$ ....but seems to work in practice !

In fact :

Global Carleman estimates (difficult) plus energy estimates (standard) provide a weighted estimate on y.

We have the following (magic !!) estimate:

$$|y(T_0)|^2_{L^2(\Omega)} + \int_0^{T_0}\int_\Omega e^{-2s\eta}|\nabla y|^2 dxdt + \int_0^{T_0}\int_\Omega e^{-2s\eta}|y|^2 dxdt \leq C\int_0^{T_0}\int_\omega |y|^2 dxdt \quad (4)$$

where $s > 0$ and η is a weight such that

$$\eta(x,t) > 0 \text{ in } \Omega \times (0, T_0), \ \eta \to +\infty \text{ when } t \to 0,$$

$$\forall \delta > 0, \ \eta(x,t) \leq M_\delta \text{ on } \Omega \times (\delta, T_0).$$

Remarks :

This estimate does not make any reference to the initial value y_0 !!

It gives an estimate in classical Sobolev spaces $L^2(\delta, T_0; H_0^1(\Omega))$ for every $\delta > 0$ and an estimate on $|y(T_0)|_{L^2(\Omega)}$.

It implies the unique continuation property which is classical for this example, but less classical for more complex systems.

This estimate is the main mathematical difficulty to solve when trying to apply the method to your favorite evolution system.

Consider the space

$$\mathcal{V} = \{y,\ y \text{ is a (regular) solution of (HE)},\ y_0 \in L^2(\Omega)\}$$

and define on this space

$$||y||_V^2 = \int_0^{T_0} \int_\omega |y|^2 dx dt.$$

Then $||y||_V$ is a prehilbertian norm on $\mathcal{V}$ because of unique continuation property.

Now define V as the completion of $\mathcal{V}$.

Then V is a Hilbert space for the scalar product associated to the norm $||y||_V$.

Because of Carleman estimate we have

$$\forall \delta > 0,\ V \subset L^2(\delta, T_0; H_0^1(\Omega)).$$

Essentially we have

$$V = \{y \in L^2_{e^{-2s\eta}}(0, T_0; H_0^1(\Omega)),\ y \text{ is a solution of (HE(1)-(2)),}\ \int_0^{T_0}\int_\omega |y|^2 dxdt < +\infty\}.$$

An element of V may not have any value at time $t = 0$!!

But its value at each time $t > 0$, in particular at time $t = T_0$ makes perfect sense in $L^2(\Omega)$ for example.

We still have the Carleman estimate for elements of V

$$|y(T_0)|^2_{L^2(\Omega)} + \int_0^{T_0}\int_\Omega e^{-2s\eta}|\nabla y|^2 dxdt + \int_0^{T_0}\int_\Omega e^{-2s\eta}|y|^2 dxdt \leq C\int_0^{T_0}\int_\omega |y|^2 dxdt \tag{5}$$

For $y \in V$ we define

$$J(y) = \int_0^{T_0}\int_\omega |y-h|^2 dxdt.$$

Notice that for $y \in \mathcal{V}$, J and J_0 have the same value but the argument is different : J_0 depends on the initial value y_0 while J depends on the trajectory $y \in V$.

We now consider the minimization problem (MP):

To find $\widehat{y} \in V$ such that

$$J(\widehat{y}) = \min_{y \in V} J(y). \qquad (MP)$$

We immediately obtain the following result.

Theorem 1 *There exists a unique solution $\widehat{y} \in V$ to the previous minimization problem.*

If we have two datas h_1 and h_2 the corresponding solutions $\widehat{y}_1$ and $\widehat{y}_2$ satisfy

$$||\widehat{y}_1 - \widehat{y}_2||_V^2 \leq \int_0^{T_0} \int_\omega |h_1 - h_2|^2 dxdt.$$

Remark : This result implies in particular the stability (in the V-norm) of the solution with respect to perturbations in the "measurements".

A priori we know that for every $\delta > 0$

$$\widehat{y} \in C([\delta, T_0]; L^2(\Omega)) \cap L^2(\delta, T_0; H_0^1(\Omega)).$$

Without any further hypothesis, we cannot say anything about the convergence of Tychonov regularization. But if we assume some additional regularity on the solution $\widehat{y}$ we can obtain a convergence result.

Theorem 2 *Let us assume that* $\widehat{y} \in C([0,T_0]; L^2(\Omega))$ *with* $\widehat{y}(0) = \widehat{y}_0$. *Then we have*

$$J(\widehat{y}) = J_0(\widehat{y}_0).$$

Moreover, when $\alpha \to 0$, *the solution* y_α *of the Tychonov regularized problem converges to* $\widehat{y}_0$ *strongly in* $L^2(\Omega)$. *If* y^α *is the solution of (HE) corresponding to* y_α *we have*

$$\int_0^{T_0}\int_\omega |\widehat{y} - y^\alpha|^2 dxdt \leq \alpha|\widehat{y}_0|_{L^2(\Omega)}.$$

If in addition $\widehat{y}_0$ is in the range of the controlled adjoint equation we can obtain a better convergence rate.

Theorem 3 *If there exists* $w \in L^2(0,T_0;L^2(\omega))$ *such that the solution* q *of*

$$
\begin{aligned}
&-\frac{\partial q}{\partial t} - \Delta q = w.\chi_\omega \ \textit{in}\ \Omega \times (0,T_0),\\
&q = 0,\ \textit{on}\ \Gamma \times (0,T_0),\\
&q(T_0) = 0,\ \textit{in}\ \Omega,
\end{aligned}
$$

satisfies

$$q(0) = \hat{y}_0,$$

then we have

$$|\hat{y}_0 - y_\alpha|^2_{L^2(\Omega)} \le C\alpha, \quad \int_0^{T_0}\int_\omega |\hat{y} - y^\alpha|^2 dxdt \le C\alpha^2.$$

Recovery of the final state.

We have already seen that, due to the fact that $\widehat{y} \in V$,

$$|\widehat{y}(T_0)|^2_{L^2(\Omega)} \leq C \int_0^{T_0} \int_\omega |\widehat{y}|^2 dxdt.$$

This gives a stability result for the recovery of $\widehat{y}(T_0)$ from "measurements" of $\widehat{y}$ on $\omega \times (0, T_0)$.

On the other hand, from the Euler-Lagrange equation associated to the minimization problem (MP) we have

$$\int_0^{T_0} \int_\omega (h - \widehat{y}) z dxdt = 0, \ \forall z \in V. \qquad (EL)$$

Let us now consider the following exact controllability problem (null controllability) for the adjoint equation.

If $\psi \in L^2(\Omega)$ we know (null controllability results) that there exists $v = v(\psi) \in L^2(0, T_0; L^2(\omega))$ such that the solution φ of

$$
\begin{aligned}
&-\frac{\partial \varphi}{\partial t} - \Delta \varphi = v.\chi_\omega, \text{ in } \Omega \times (0, T_0),\\
&\varphi = 0, \text{ on } \Gamma \times (0, T_0),\\
&\varphi(T_0) = \psi \text{ in } \Omega,
\end{aligned}
$$

satisfies

$$\varphi(0) = 0.$$

Moreover we can take $v(\psi)$ of minimal norm in $L^2(0, T_0; L^2(\omega))$ and we then have

$$|v(\psi)|_{L^2(0,T_0;L^2(\omega))} \leq C|\psi|_{L^2(\Omega)}$$

We also know (by a duality argument) that $v(\psi)$ is obtained as the restriction of z on $\omega \times (0, T_0)$ where z is the minimum (in V) of the functional

$$K(z) = \frac{1}{2} \int_0^{T_0} \int_\omega |z|^2 dx dt + \int_\Omega z(T_0) \psi dx.$$

As $z \in V$ we have from (EL)

$$\int_0^{T_0} \int_\omega h z dx dt = \int_0^{T_0} \int_\omega \widehat{y} z dx dt$$

so that

$$\int_0^{T_0} \int_\omega h v(\psi) dx dt = \int_0^{T_0} \int_\omega \widehat{y} v(\psi) dx dt.$$

Now multiplying the equation for φ by $\widehat{y}$, because $\varphi(0)=0$, we obtain (this can be shown rigorously)

$$\int_\Omega \psi \widehat{y}(T_0)dx = -\int_0^{T_0}\int_\omega v(\psi)\widehat{y}dxdt = -\int_0^{T_0}\int_\omega v(\psi)hdxdt.$$

The control $v(\psi)$ can be "computed" for every $\psi \in L^2(\Omega)$. The measurement h is given. Therefore, at least in theory, this enables us to recover the coefficient

$$\int_\Omega \psi \widehat{y}(T_0)dx$$

for every ψ in a Hilbert basis of $L^2(\Omega)$, and therefore $\widehat{y}(T_0)$, at the price of solving a null controlability problem for each element of a Hilbert basis.....

For $\beta > 0$ we can consider an approximation by an optimal control problem.

Define

$$K_\beta(v) = \frac{1}{2\beta}\int_\Omega |\varphi(0)|^2 dx + \frac{1}{2}\int_0^{T_0}\int_\omega |v|^2 dx dt,$$

Look for $v_\beta \in L^2(0, T_0; L^2(\omega))$ such that

$$K_\beta(v_\beta) = \min_{v \in L^2(0,T_0;L^2(\omega))} K_\beta(v)$$

For every $\beta > 0$ this problem has a unique solution v_β which can be characterized by an optimality system using an adjoint state (classical optimal control).

It is easy to show that when $\beta \to 0$,

$$v_\beta \to v(\psi) \text{ in } L^2(0,T_0;L^2(\omega))$$

$$-\int_0^{T_0}\int_\omega v_\beta h dx dt \to -\int_0^{T_0}\int_\omega v(\psi) h dx dt = \int_\Omega \psi \widehat{y}(T_0) dx$$

Therefore we can compute an approximation of the desired coefficient at the price of solving an optimal control problem (for each ψ).

Rate of convergence : requires a regularity hypothesis on the adjoint state associated to the exact controllability problem.

As

$$q \in V \to \int_\Omega \psi q(T_0)dx$$

is a linear continuous map, from Riesz Theorem there exists $p \in V$ such that

$$\forall q \in V, \ \int_0^{T_0}\int_\omega pq\,dx\,dt = \int_\Omega \psi q(T_0)dx.$$

If we suppose that in addition

$$p \in C([0,T_0];L^2(\Omega))$$

then we obtain

$$|v_\beta - v(\psi)|_{L^2(0,T_0;L^2(\omega))} \leq 2\beta^{\frac{1}{2}}|p(0)|_{L^2(\Omega)},$$

$$|\int_\Omega \hat{y}(T_0)\psi dx + \int_0^{T_0}\int_\omega h.v_\beta dx dt| \leq C\beta^{\frac{1}{2}}|p(0)|_{L^2(\Omega)},$$

Computations with a classical basis done in Garcia-Osses-Puel for a large scale ocean model (reasonable results). New results with a spectral basis. Better results and much cheaper in terms of computing time. (Garcia-Osses-Puel to appear).

It would be important to use a reduced basis such as POD... in order to reduce the number of optimal control problems (or exact controllability problems) to solve. Work of P.Hepperger (Diplomarbeit, T.U. München) which seems very promising in this direction.