Asymptotics in Nonlinear Hyperbolic Thermoelasticity[1]

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Plan:
1. Introduction
2. One-dimensional non-linear Cauchy problem
3. Propagation of singularities in three dimensions
4. Low frequency expansion in exterior domains

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1 Introduction

Thermoelastic systems for the displacement vector u and the the temperature difference θ, where the classical Fourier law

$$q + k\nabla\theta = 0$$

is replaced by Cattaneo's law

$$\tau q_t + q + k\nabla\theta = 0$$

τ: relaxation parameter.

$$\rho u_{tt} - \mathrm{div} S = 0 \tag{1.1}$$

$$\varepsilon_t - \mathrm{tr}\{S \nabla u_t\} + \mathrm{div} q = 0 \tag{1.2}$$

$$(\theta + T_0)\{a(\nabla u, \theta, q)\theta_t - \mathrm{tr}[S_\theta \cdot \nabla u_t]\} + \mathrm{div}\ q = b(\nabla u, \theta, q) q_t \tag{1.3}$$

Cattaneo's law:

$$\tau(\nabla u, \theta) q_t + q + k(\nabla u, \theta)\nabla\theta = 0 \tag{1.4}$$

Topics:

- Decay rates of solutions to the one-dimensional Cauchy problem
- Propagation of singularities depending on the curvature of the initial surface of discontinuities in three space dimensions
- Low frequency asymptotics in three-dimensional exterior domains as first step towards the time asymptotics in exterior domains

2 One-dimensional non-linear Cauchy problem

(Joint work with Y.-G. Wang)

Nonlinear Cauchy problem in one space dimension

$$u_{tt} - a_1(P)u_{xx} + a_2(P)\theta_x + \sigma_1(P)q_x = 0 \tag{2.1}$$

$$\theta_t + a_3(P)u_{tx} + a_4(P)q_x + \sigma_2(P)\theta_x + a_5(P)q = 0 \tag{2.2}$$

$$q_t + a_6(u_x, \theta)\theta_x + a_7(u_x, \theta)q = 0 \tag{2.3}$$

$$u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ \theta(0,x) = \theta_0(x),\ q(0,x) = q_0(x),\ x \in \mathbb{R} \tag{2.4}$$

where

$$P = (u_x, \theta, q)$$

Transforming (2.1) - (2.4) into a first-order system for

$$V := (u_x, u_t, \theta, q)'$$

$$V_t + \underbrace{\begin{pmatrix} 0 & -1 & 0 & 0 \\ -a_1 & 0 & a_2 & \sigma_1 \\ 0 & a_3 & \sigma_2 & a_4 \\ 0 & 0 & a_6 & 0 \end{pmatrix}}_{=:A(V)} \partial_x V + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_5 \\ 0 & 0 & 0 & a_7 \end{pmatrix}}_{=:B(V)} V = 0 \tag{2.5}$$

$$V(t=0) = V_0 := (u_{0,x}, u_1, \theta_0, q_0)' \tag{2.6}$$

This is strictly hyperbolic if $|P|$ is small enough, hence:

Theorem 2.1. *Let $s \geq 2$. Then there is $\delta > 0$ such that for data $V_0 \in W^{s,2}(\mathbb{R})$ with $||V_0||_{s,2} < \delta$ there is a unique local solution V to (2.5), (2.6) in some time interval $[0, T]$ with*

$$V \in C^0([0,T], W^{s,2}(\mathbb{R})) \cap C^1([0,T], W^{s-1,2}(\mathbb{R})),$$

and $T = T(||V_0||_{s,2}) > 0.$

Gobal existence for small data by Tarabek, solution with bounded norm $||V(t)||_{s,2}$. Rewrite the differential equations:

$$u_{tt} - \alpha u_{xx} + \beta\theta_x = h_1 \tag{2.7}$$

$$\theta_t + \gamma q_x + \delta u_{tx} = h_2 \tag{2.8}$$

$$\tau_0 q_t + q + \kappa\theta_x = h_3 \tag{2.9}$$

where

$$\alpha := S_{u_x}(0,0,0), \quad \beta := -S_\theta(0,0,0), \quad \gamma := a_4(0,0,0) \tag{2.10}$$

$$\delta := a_3(0,0,0), \quad \kappa := \tau_0 a_6(0,0,0), \quad \tau_0 := \tau(0,0) \tag{2.11}$$

$$h_1 := (S(u_x,\theta,q) - \alpha\, u_x + \beta\theta)_x \tag{2.12}$$

$$h_2 := (\gamma - a_4)q_x + (\delta - a_3)u_{tx} - \sigma_2\theta_x - a_5 q \tag{2.13}$$

$$h_3 := \tau_0 \left\{ \left(\frac{1}{\tau_0} - \frac{1}{\tau}\right) q + \left(\frac{\kappa}{\tau_0} - \frac{k}{\tau}\right)\theta_x \right\}. \tag{2.14}$$

One knows

$$\sigma_1(0,0,0) = 0, \quad \sigma_2(0,0,0) = 0, \quad a_5(0,0,0) = 0. \tag{2.15}$$

Defining

$$V := (\sqrt{\alpha\kappa\delta}\, u_x, u_t, \theta, q)'$$

we obtain

$$A^0 V_t + A^1 V_x + BV = \widetilde{F}(V, V_x), \quad V(0) = V_0 := (\sqrt{\alpha\kappa\delta}\, u_{0,x}, u_1, \theta_0, q_0)' \quad (2.16)$$

with a nonlinearity that vanishes quadratically near zero. The linearized system, i.e. for $\widetilde{F} = 0$, is solved by

$$V(t) = e^{tR} V_0$$

where

$$R := -(A^0)^{-1} (A^1 \partial_x + B)$$

generates a C_0-semigroup on $D(R) := (W^{1,2}(\mathbb{R}))^4 \subset (L^2(\mathbb{R}))^4$. Then the solution to (2.16) is represented in general by

$$V(t) = e^{tR} V_0 + \int_0^t e^{(t-r)R} F(V, V_x)(r)\, dr \quad (2.17)$$

where

$$F := (A^0)^{-1}\widetilde{F} \tag{2.18}$$

Cattaneo's law is assumed to be the linear one:

$$\tau_0 q_x + q + \kappa\theta_x = 0 \tag{2.19}$$

This assumption implies that in (2.9)

$$h_3 = 0 \tag{2.20}$$

Further assumption:

$$\psi(u_x, \theta, q) \; = \; \psi_0(u_x, \theta) + \psi_1(u_x, \theta)q^2 \tag{2.21}$$

$$e(u_x, \theta, q) \; = \; e_0(u_x, \theta) + e_1(u_x, \theta)q^2 \tag{2.22}$$

Theorem 2.2. *Let* $V_0 \in W^{3,2}(\mathbb{R}) \cap L^1(\mathbb{R})$*, and* $m_0 := ||V_0||_{3,2} + ||V_0||_1$*. For sufficiently small* m_0 *there is* $c > 0$ *such that for* $t \geq 0$*, the solution* V *to (2.16) satisfies*

$$||V(t)||_{1,2} \leq c\,(1+t)^{-\frac{1}{4}} \cdot m_o$$

The fact that F vanishes quadratically in zero is sufficient despite the linear decay behavior like that of a heat equation, because terms like "$V \cdot V_x$" appear. Therefore, the proof rewrites the nonlinearities in divergence form in order to be able to exploit the better decay of derivatives in the later estimates for $||V(t)||_2$.

3 Propagation of singularities in three dimensions

(Joint work with Y.-G. Wang)

$$U_{tt}^{po} - \alpha^2 \Delta U^{po} p + \beta \nabla \theta \;=\; 0, \tag{3.1}$$

$$\theta_t + \gamma \nabla' q^{po} + \delta \nabla' U_t^{po} \;=\; 0, \tag{3.2}$$

$$\tau q_t^{po} + q^{po} + \kappa \nabla \theta \;=\; 0, \tag{3.3}$$

$$U^{po}(0,\cdot) = U_0^{po},\; U_t^{po}(0,\cdot) = U_1^{po},\; \theta(0,\cdot) = \theta^0,\; q^{po}(0,\cdot) = q_0^{po}. \tag{3.4}$$

The data $(\nabla U^0, U^1, \theta^0, q^0)$ are assumed to be smooth away from and possibly having jumps on a given smooth surface

$$\sigma = \{x \in \Omega_0 \subset \mathbb{R}^3 \mid \Phi^0(x) = 0\}$$

described by a C^2-function $\Phi^0 : \Omega_0 \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$, with Ω_0 open, satisfying

$$|\nabla \Phi^0(x)| = 1 \qquad \text{on} \quad \sigma. \tag{3.5}$$

$$\tilde{U} = (\tilde{U}_1, \tilde{U}_2, \tilde{U}_3, \tilde{U}_4)' := (\nabla' U^{po}, U_t^{po}, \theta, q^{po})'.$$

Then

$$\partial_t \tilde{U} + \sum_{j=1}^{3} A_j \partial_j \tilde{U} + A_0 \tilde{U} = 0, \quad \tilde{U}(0, \cdot) = \tilde{U}^0, \tag{3.6}$$

The characteristic polynomial

$$\det(\lambda \mathrm{Id}_{\mathbb{R}^8} - \sum_{j=1}^{3} \xi_j A_j) = \lambda^4 \left(\lambda^4 - \lambda^2 (\alpha^2 + \beta\delta + \frac{\kappa\gamma}{\tau}) |\xi|^2 + \frac{\kappa\gamma\alpha^2}{\tau} |\xi|^4 \right) \tag{3.7}$$

has real roots $\lambda_k, 1 \leq k \leq 8$, and the eigenvalues of

$$B := \sum_{j=1}^{3} \partial_j \Phi^0 A_j \tag{3.8}$$

are

$$\lambda_k = 0, \quad 1 \leq k \leq 4, \tag{3.9}$$

$$\lambda_k = \mu_k = \pm \left\{ \frac{1}{2}(\alpha^2 + \beta\delta + \frac{\kappa\gamma}{\tau}) \pm \frac{1}{2}\sqrt{(\alpha^2 + \beta\delta + \frac{\kappa\gamma}{\tau})^2 - \frac{4\kappa\gamma\alpha^2}{\tau}} \right\}^{\frac{1}{2}}, \quad 5 \leq k \leq 8 \tag{3.10}$$

taking

$$\mu_8 < \mu_6 < 0 < \mu_7 < \mu_5. \tag{3.11}$$

The characteristic surfaces $\Sigma_k = \{(t,x) \mid \Phi_k(t,x) = 0\}, 1 \leq k \leq 8$, evolving from the initial surface $\sigma = \{(0,x) \mid \Phi^0(x) = 0\}$ are determined by

$$\partial_t \Phi_k + \mu_k |\nabla \Phi_k| = 0, \quad \Phi_k(0,\cdot) = \Phi^0 \tag{3.12}$$

hence

$$\Sigma_k = \{(t,x) \mid -\mu_k t + \Phi^0(x) = 0\}. \tag{3.13}$$

The associated right (column) eigenvectors r_k and left (row) eigenvectors l_k of B, can be computed explicitly, as well as expansions of these and the eigenvalues in powers of τ. Let the matrices L and R be given by

$$L := \begin{pmatrix} l_1 \\ \vdots \\ l_8 \end{pmatrix}, \qquad R := (r_1, \ldots, r_8).$$

Then $V := L\tilde{U}$, with $\tilde{U}$ satisfying (3.6), satisfies

$$\partial_t V + \sum_{j=1}^{3} (LA_jR)\partial_j V + \widetilde{A}_0 V = 0, \quad V(t=0) = V^0 := L\tilde{U}^0, \tag{3.14}$$

where

$$\widetilde{A}_0 := LA_0R + \sum_{j=1}^{3} LA_j\partial_j R. \tag{3.15}$$

Let $[H]_{\Sigma_k}$ denote the jump of H along Σ_k. Then $(V_1, V_2, V_3.V_4)$ are continuous at $\cup_{k=5}^{8}\Sigma_k$, and $V_j, j = 5, \ldots, 8$, does not have any jump on $\Sigma_k, k = 1, \ldots, 8$ for $k \neq j$. Moreover, $[V_k]_{\Sigma_0} = 0$ for $1 \leq k \leq 4$. Evolutional equations for $[V_k]_{\Sigma_k}$ for $5 \leq k \leq 8$:

$$\partial_t V_k + \sum_{m=1}^{8}\sum_{j=1}^{3}(LA_jR)_{km}\partial_j V_m = -\sum_{m=1}^{8}(\widetilde{A}_0)_{km}V_m + \widetilde{F_k} \tag{3.16}$$

for $5 \leq k \leq 8$. Since

$$(\lambda_k \mathrm{Id}_{\mathbb{R}^8} - \sum_{j=1}^{3} L\partial_j\Phi^0 A_j R)_{kk} = (\lambda_k \mathrm{Id}_{\mathbb{R}^8} - \widetilde{\Lambda})_{kk} = 0 \tag{3.17}$$

with $\widetilde{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_8)$, we obtain that for each $5 \leq k \leq 8$, $[V_k]_{\Sigma_k}$ satisfies the following transport equation:

$$\Big(\partial_t + \sum_{j=1}^{3} (LA_jR)_{kk}\partial_{x_j} + (\widetilde{A}_0)_{kk}\Big)[V_k]_{\Sigma_k} = [\tilde{F}_k]_{\Sigma_k} \tag{3.18}$$

with initial conditions

$$[V_k]_{\Sigma_k}(t=0) = [V_k^0]_\sigma. \tag{3.19}$$

In order to determine the behavior of $[V_k]_{\Sigma_k}$ from (3.18), (3.19), it is essential to study $(\widetilde{A}_0)_{kk}$, where, by (3.15), $\widetilde{A}_0$ is given by

$$\widetilde{A}_0 = LA_0R + \sum_{j=1}^{3} LA_j\partial_jR. \tag{3.20}$$

$$(\widetilde{A}_0)_{kk} = \begin{cases} \frac{1}{2\tau} \pm \frac{1}{2}\sqrt{\frac{\kappa\gamma}{\tau}}\Delta\Phi^0 + \mathcal{O}(1), & k = 5, 6, \\ \frac{\beta\delta}{2\kappa\gamma} \pm \frac{\alpha}{2}\Delta\Phi^0 + \mathcal{O}(\tau), & k = 7, 8. \end{cases} \tag{3.21}$$

Notice: The mean curvature H of the initial surface σ equals

$$H = \frac{\Delta\Phi^0}{2}$$

and will play an essential role in the behavior of the jumps as $t \to \infty$ or as $\tau \to 0$.

$$\begin{aligned}
[V_k]_{\Sigma_k(t)} &= [V_k^0]_\sigma \; e^{-\int_0^t (\widetilde{A}_0)_{kk}(x(s;0,x^0))\mathrm{ds}} \\
&= \begin{cases} e^{-\frac{t}{2\tau} \mp \frac{1}{2}\sqrt{\frac{\kappa\gamma}{\tau}}\int_0^t (\Delta\Phi^0(x(s;0,x^0)) + O(\sqrt{\tau}))\mathrm{ds}} [V_k^0]_\sigma, & k = 5, 6, \\ e^{-\frac{\beta\delta}{2\kappa\gamma}t \mp \frac{\alpha}{2}\int_0^t (\Delta\Phi^0(x(s;0,x^0)) + \mathcal{O}(\tau))\mathrm{ds}} [V_k^0]_\sigma, & k = 7, 8. \end{cases}
\end{aligned} \tag{3.22}$$

For $\tau \to 0$ the dominating term for $k = 5, 6$ is $e^{-\frac{t}{2\tau}}$, i.e., we have exponential decay of the jumps of V_k on Σ_k as $\tau \to 0$ or $t \to \infty$ for a fixed small $\tau > 0$. If $k = 7, 8$ the dominating term, for $\tau \to 0$, is $\exp(-\int_0^t (\frac{\beta\delta}{2\kappa\gamma} \pm \frac{\alpha}{2}\Delta\Phi^0(x(s;0,x^0)))\mathrm{ds})$, whether the jumps of V_k on Σ_k decay exponentially depends on the size of the mean curvature ($= \Delta\Phi^0/2$).

Example: Let σ be the sphere of radius r:

$$\sigma = \{x \in \mathbb{R}^3 \mid |x| = r\} = \{x \mid \Phi^0(x) \equiv r - |x| = 0\}.$$

Then, we have

$$\Sigma_k = \{(t,x) \mid \mu_k t = r - |x|\} = \{(t,x) \mid |x| = r - \mu_k t\}.$$

Spreading surfaces, as $t \longrightarrow \infty$, are Σ_6, Σ_8, and

$$\Delta\Phi^0(x_0) = \frac{2}{|x_0|} = \frac{2}{r} > 0.$$

Thus, as $t \longrightarrow +\infty$, $[V_6]_{\Sigma_6}$ is decaying exponentially, while $[V_8]_{\Sigma_8}$ decays (grows resp.) exponentially if

$$\frac{\beta\delta}{\alpha\kappa\gamma} > \Delta\Phi^0 = \frac{2}{r} \qquad \left(\frac{\beta\delta}{\alpha\kappa\gamma} < \Delta\Phi^0 = \frac{2}{r} \text{ resp.}\right), \tag{3.23}$$

that is depending on the size of the mean curvature $H = \frac{1}{r}$.

Theorem 3.1. *Suppose that the initial data $\nabla' U^{po}, \partial_t U^{po}, \theta$ and q^{po} may have jumps on $\sigma = \{\Phi^0(x) = 0\}$ with $|\nabla \Phi^0(x)| = 1$, then the propagation of strong singularities of solutions to the linearized problem (3.1)–(3.4) obeys*

(1) The jumps of $\nabla' U^{po}, \partial_t U^{po}, \theta, q^{po}$ on Σ_5 and Σ_6 decay exponentially both when $\tau \longrightarrow 0$ for a fixed $t > 0$ and when $t \longrightarrow +\infty$ for a fixed $\tau > 0$.

(2) The jumps of $\nabla' U^{po}, \partial_t U^{po}, q^{po}$ on Σ_7 (Σ_8 resp.) are propagated, and when $t \longrightarrow +\infty$ they will decay exponentially as soon as $\frac{\beta\delta}{\kappa\gamma} + \alpha \Delta \Phi^0$ ($\frac{\beta\delta}{\kappa\gamma} - \alpha \Delta \Phi^0$ resp.) being positive, more rapidly for smaller heat conductive coefficient $\kappa\gamma$, while the jump of the temperature θ on Σ_7 and Σ_8 vanishes of order $O(\tau)$ when $\tau \longrightarrow 0$, which shows a smoothing effect in the system (3.1)–(3.3) when the thermoelastic model with second sound converges to the hyperbolic-parabolic type of classical thermoelasticity.

4 Low frequency expansion in exterior domains

(Joint work with Y. Naito, Y. Shibata)

Let Ω be an exterior domain in $\mathbb{R}^3$ with $C^{1,1}$ boundary Γ.

$$
\begin{aligned}
&u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla \mathrm{div}\, u + \beta\nabla\theta = 0 \\
&\theta_t + \gamma \mathrm{div}\, q + \delta \mathrm{div}\, u_t = 0 \\
&\tau_0 q_t + q + \kappa\nabla\theta = 0
\end{aligned}
\tag{4.1}
$$

in $\Omega \times (0,\infty)$ subject to the initial conditions

$$
u(x,0) = u_0(x),\;\; u_t(x,0) = u_1(x),\;\; \theta(x,0) = \theta_0(x),\;\; q(x,0) = q_0(x) \quad \text{in } \Omega
$$

and boundary conditions

$$
u = 0, \quad \theta = 0 \quad \text{on } \Gamma \times (0,\infty)
$$

Aim: Low frequency expansion of the corresponding resolvent problem(s) — important to investigate the decay property of solutions to (4.1) as time goes to infinity (essentially: via Laplace transform).
Moreover: Limit as τ_0 tends to zero.

Resolvent problem:

$$\begin{aligned}
&k^2u - \mu\Delta u - (\mu+\lambda)\nabla \operatorname{div} u + \beta\nabla\theta = f && \text{in } \Omega\\
&k\theta + \gamma \operatorname{div} q + \delta k \operatorname{div} u = g && \text{in } \Omega\\
&\tau_0 kq + q + \kappa\nabla\theta = h && \text{in } \Omega\\
&\quad u = 0, \quad \theta = 0 && \text{on } \Gamma
\end{aligned} \tag{4.2}$$

First $\Omega = \mathbb{R}^3$: Use Fourier transform. Eliminate q and consider

$$\begin{aligned} k^2 u - \mu\Delta u - (\mu+\lambda)\nabla \operatorname{div} u + \beta\nabla\theta = f \quad & \text{in } \Omega \\ k\theta - \gamma\kappa(\tau_0 k + 1)^{-1}\Delta\theta + \delta k \operatorname{div} u = g \quad & \text{in } \Omega \\ u = \theta = 0 \quad & \text{on } \Gamma \end{aligned} \tag{4.3}$$

Theorem 4.1. *Let* $1 < q < \infty$ *and* $0 < \tau_0 \leq 1$*. Then, for any small* $\epsilon > 0$ *there exist a constant* $\sigma_0 > 0$ *depending on* ϵ *and an operator* $S_k \in \operatorname{Anal}(U_{\sigma_0,\epsilon}, \mathcal{B}(L_q(\mathbb{R}^3)^3 \times L_q(\mathbb{R}^3), W_q^2(\mathbb{R}^3)^3 \times W_q^2(\mathbb{R}^3)))$ *such that for any* $(f, g) \in L_q(\mathbb{R}^3)^3 \times L_q(\mathbb{R}^3)$, $(u, \theta) = S_k(f, g)$ *solves equation* (4.3)*. Here, for two Banach spaces* X *and* Y, $\mathcal{B}(X, Y)$ *denotes the set of all bounded linear operators from* X *into* Y, $U_{\sigma_0,\epsilon}$ *denotes an open set in* $\mathbb{C}$ *defined by the formula:*

$$U_{\sigma_0,\epsilon} = \{k \in \mathbb{C} \setminus \{0\} \mid |\arg k| \leq (\pi/2) - \epsilon, \ \ |k| \leq \sigma_0\}$$

and $\mathrm{Anal}\,(U_{\sigma_0,\epsilon}, X)$ *denotes the set of all holomorphic functions defined on* $U_{\sigma_0,\epsilon}$ *with their values in* X*.*

Theorem 4.2. *Let* $1 < q < \infty$*,* $0 < \epsilon < \pi/2$*,* $0 < \tau_0 \leq 1$ *and* $R > 0$*. Let* σ_0 *and* S_k *be the same number and solution operator as in Theorem 4.1, respectively. Set*

$$\mathcal{L}_{q,R}(\mathbb{R}^3) = \{(f,g) \in L_q(\mathbb{R}^3)^3 \times L_q(\mathbb{R}^3) \mid (f,g) \text{ vanishes for } |x| > R\}$$
$$\mathcal{W}_{q,\mathrm{loc}}(\mathbb{R}^3) = W^2_{q,\mathrm{loc}}(\mathbb{R}^3)^3 \times W^2_{q,\mathrm{loc}}(\mathbb{R}^3)$$

Then, there exist a σ $(0 < \sigma \leq \sigma_0)$ *and* $G_j(k) \in \mathrm{Anal}\,(U_\sigma, \mathcal{B}(\mathcal{L}_{q,R}(\mathbb{R}^3), \mathcal{W}_{q,\mathrm{loc}}(\mathbb{R}^3)))$ $(j = 0,1)$ *such that when* $(f,g) \in \mathcal{L}_{q,R}(\mathbb{R}^3)$*,* $G_k(f,g) = (k^{1/2}G_0(k)+G_1(k))(f,g)$ *solves equation* (4.3) *for* $k \in U_\sigma$ *and* $G_k(f,g) = S_k(f,g)$ *for* $k \in U_{\sigma,\epsilon}$*.*

Now let Ω be an exterior domain.

Let σ, S_k, $G_0(k)$ and $G_1(k)$ be the same constant and operators as in Theorem 4.2

and set

$$G_k = k^{1/2} G_0(k) + G_1(k) \tag{4.4}$$

Locally we get

Theorem 4.3. *Let* $1 < q < \infty$ *and* $0 < \tau_0 \le 1$. *Let* R *be a large fixed number such that* $\mathbb{R}^3 \setminus \Omega \subset B_R$. *Then, there exists a small number* σ' $(0 < \sigma' \le \sigma)$ *and an operator* $H_k \in \mathcal{B}(\mathcal{L}_{q,R}, \mathcal{W}^2_{q,\mathrm{loc}}(\Omega))$ *for each* $k \in U_{\sigma'} = \{k \in \mathbb{C} \mid |k| \le \sigma'\}$ *such that* $H_k(f, g)$ *satisfies equation* (4.3) *for any* $(f, g) \in \mathcal{L}_{q,R}(\Omega)$ *and* $k \in U_{\sigma'}$ *and* H_k *has the expansion formula:*

$$H_k = k^{1/2} H_0(k) + H_1(k) \quad \textit{for } k \in U_{\sigma'}$$

where H^0_k, $H^1_k \in \mathrm{Anal}(U_{\sigma'}, \mathcal{B}(\mathcal{L}_{q,R}, \mathcal{W}^2_{q,\mathrm{loc}}(\Omega)))$.

Combining this with the result for $\Omega = \mathbb{R}^3$ by cut-off techniques yields

Theorem 4.4. *Let* $1 < q < \infty$, $0 < \epsilon < \pi/2$ *and* $0 < \tau_0 \leq 1$. *Let* $\sigma' > 0$ *be the same constant as in Theorem 4.3. Then, there exists an operator* $T_k \in \mathrm{Anal}\,(U_{\sigma',\epsilon}, \mathcal{B}(L_q(\Omega)^4, W_q^2(\Omega)^4)$ *such that* $T_k(f, g)$ *satisfies equation* (4.3) *for any* $(f, g) \in L_q(\Omega)^4$ *and* $k \in U_{\sigma',\epsilon}$.

Employing the same argument, we can show the theorems corresponding to Theorems 4.3 and 4.4 in the classical thermoelastic case ($\tau_0 = 0$). Moreover, we have

Theorem 4.5. *The solution operators* H_k *constructed in Theorem 4.3 and* T_k *in Theorem 4.4 depend on* $\tau_0 \in (0, 1]$ *continuously. The limit of* H_k *and* T_k *as* $\tau_0 \to 0$ *are the corresponding operators of the classical thermoelastic equations, where the limit is given in the in the operator norm of* $\mathcal{B}(\mathcal{L}_{q,R}(\Omega), \mathcal{W}^2_{q,\mathrm{loc}}(\Omega))$ *when* $k \in U_{\sigma'}$ *and* $\mathcal{B}(L_q(\Omega)^4, W_q^2(\Omega)^4)$ *when* $\mathrm{Re}\,k > 0$ *and* $|k| < \sigma'$, *respectively.*

Thank you for your attention!