Weighted Hardy's Inequality and the Kolmogorov Equation perturbed by an inverse-square potential

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-I- Plan:

1. Introduction.

2. The Cabré-Martel approach.

3. Main result.

1. Introduction:

Problem 1:

Let $0 \leq V \in L^1_{loc}(\mathbf{R}^N)$, $N \geq 3$,

$$\lim_{x \to 0} V(x) = \infty.$$

If V is "too singular" at the origin, can this prevent the existence of positive solution to the heat equation

$$\frac{\partial u}{\partial t} = \Delta u + V u, \quad x \in \mathbf{R}^N, \, t > 0?$$

Theorem 1 *The equation*

$$\frac{\partial u}{\partial t} = \Delta u + \frac{c}{|x|^2}u, \quad x \in \mathbf{R}^N,\, t > 0$$

has many positive solutions
(e.g. one for $u(x,0) = f(x)$ for each $0 \le f \in L^2(\mathbf{R}^N)$)
if $c \le C^*(N)$ and no positive solutions at all if $c > C^*(N)$.

Here

$$C^*(N) := \left(\frac{N-2}{2}\right)^2.$$

Problem 2:

Replacing Δ by the Kolmogorov operator

$$L\varphi := \Delta\varphi + \frac{\nabla\rho}{\rho} \cdot \nabla\varphi.$$

The Baras-Goldstein theorem still remain true?

Example:
Take

$$\rho(x) = e^{-\frac{1}{2}\langle Ax,x\rangle}, \quad x \in \mathbf{R}^N$$

for A a positive hermitian $N \times N$ matrix. One obtains the **Ornstein-Uhlenbeck operator**

$$L_A\varphi := \Delta\varphi - Ax \cdot \nabla\varphi.$$

2. The Cabré-Martel approach:

Assume

(H) $0 < \rho \in H^1_{loc}(\mathbf{R}^N) \cap C(\mathbf{R}^N)$, $0 \le V \in L^1_{loc}(\mathbf{R}^N)$ $(N \ge 3)$, and L generates a C_0-semigroup on $L^2(\mu)$ with $D(L) \subset H^1(\mu)$ and $C_c^\infty(\mathbf{R}^N)$ dense in $H^1(\mu)$,

where $L^2(\mu) := L^2(\mathbf{R}^N, \rho(x)dx)$ and $H^1(\mu) := W^{1,2}(\mathbf{R}^N, \rho(x)dx)$.

Define the bottom of the spectrum of $-(L+V)$ by

$$\lambda_1(L+V) = \inf_{0 \neq \varphi \in H^1(\mu)} \left(\frac{\int_{\mathbf{R}^N} |\nabla \varphi|^2 \, d\mu - \int_{\mathbf{R}^N} V \varphi^2 \, d\mu}{\int_{\mathbf{R}^N} \varphi^2 \, d\mu} \right).$$

Consider the problems

$$(K_V)\begin{cases} \partial_t u(x,t) = Lu(x,t) + V(x)u(x,t), \\ \qquad\qquad\qquad\qquad t > 0, x \in \mathbf{R}^N, \\ u(\cdot,0) = u_0 \in L^2(\mu)_+, \end{cases}$$

$$(K_{V_n})\begin{cases} \partial_t u_n(x,t) = Lu_n(x,t) + V_n(x)u_n(x,t), \\ \qquad\qquad\qquad\qquad t > 0, x \in \mathbf{R}^N, \\ u_n(\cdot,0) = u_0 \in L^2(\mu)_+, \end{cases}$$

where $V_n = \min(V, n)$.

Theorem 2 *(i) If* $\lambda_1(L+V) > -\infty$*, then there exists a positive weak solution* $u \in C([0,\infty), L^2(\mu))$ *of* (K_V) *satisfying*

$$\|u(t)\|_{L^2(\mu)} \leq M e^{\omega t} \|u_0\|_{L^2(\mu)}, \quad t \geq 0 \quad (1)$$

for some constants $M \geq 1$ *and* $\omega \in \mathbf{R}$*.*

(ii) If $\lambda_1(L+V) = -\infty$*, then for any* $0 \leq u_0 \in L^2(\mu)$*,* $u_0 \neq 0$*, there is no positive weak solution of* (K_V) *satisfying (1).*

Idea of the proof:

(ii) Suppose there is a positive weak solution u of (K_V) satisfying (1). Let u_n be the unique positive solution of (K_{V_n}). Then

$$0 \le u_n \le u.$$

Then, there is a positive weak solution $\widetilde{u}(t) = \lim_{n\to\infty} u_n(t)$ of (K_V). Let $\varphi \in C_c^\infty(\mathbf{R}^N)$ with $\int_{\mathbf{R}^N} \varphi^2\, d\mu = 1$. Multiplying (K_{V_n}) by $\frac{\varphi^2}{u_n}$ and integrating we obtain

$$\int_{\mathbf{R}^N} V_n \varphi^2\, d\mu \le \partial_t \left(\int_{\mathbf{R}^N} (\log u_n) \varphi^2\, d\mu \right) + \int_{\mathbf{R}^N} |\nabla \varphi|^2\, d\mu.$$

Integrating with respect to $t > 1$, we get

$$\begin{aligned} (t-1) \int_{\mathbf{R}^N} V_n \varphi^2\, d\mu \ &\le\ \int_{\mathbf{R}^N} \log\left(\frac{u_n(t)}{u_n(1)}\right) \varphi^2\, d\mu \\ &\quad + (t-1) \int_{\mathbf{R}^N} |\nabla \varphi|^2\, d\mu. \end{aligned}$$

Letting $n \to \infty$,

$$\begin{aligned} & \int_{\mathbf{R}^N} V\varphi^2\, d\mu - \int_{\mathbf{R}^N} |\nabla\varphi|^2\, d\mu \\ \leq\ & \frac{1}{t-1}\int_{\mathbf{R}^N} \log(\widetilde{u}(t))\varphi^2\, d\mu \\ & -\frac{1}{t-1}\int_{\mathbf{R}^N} \log(\widetilde{u}(1))\varphi^2\, d\mu. \end{aligned}$$

Using Jensen's and Hölder's inequalities, (1) and letting $t \to +\infty$,

$$\int_{\mathbf{R}^N} V\varphi^2\, d\mu - \int_{\mathbf{R}^N} |\nabla\varphi|^2\, d\mu \leq \omega.$$

3. Main result: (Ornstein-Uhlenbeck operators and weighted Hardy's inequalities)

Put $\mu_A(dx) := e^{-\frac{1}{2}\langle Ax,x\rangle}\,dx$. Then,

$$\int_{\mathbf{R}^N} |\nabla\varphi|^2\,d\mu_A \;-\; C^*(N)\int_{\mathbf{R}^N}\frac{\varphi^2}{|x|^2}\,d\mu_A \;\geq\; -\|A\|\sqrt{C^*(N)}\int_{\mathbf{R}^N}\varphi^2\,d\mu_A.$$

In fact, define $F(x) := \frac{cx}{|x|^2}e^{-\frac{1}{2}\langle Ax,x\rangle}$. Then

$$\begin{aligned}
&\int_{\mathbf{R}^N}\varphi^2\left(c\frac{(N-2)}{|x|^2} - c\frac{\langle Ax,x\rangle}{|x|^2}\right)e^{-\frac{1}{2}\langle Ax,x\rangle}\,dx \\
&= \int_{\mathbf{R}^N}\varphi^2 \mathrm{div} F\,dx \\
&= -2c\int_{\mathbf{R}^N}\varphi\frac{x}{|x|^2}\cdot\nabla\varphi\,d\mu_A \\
&\leq 2c\left(\int_{\mathbf{R}^N}|\nabla\varphi|^2\,d\mu_A\right)^{\frac{1}{2}}\left(\int_{\mathbf{R}^N}\frac{\varphi^2}{|x|^2}\,d\mu_A\right)^{\frac{1}{2}} \\
&\leq \int_{\mathbf{R}^N}|\nabla\varphi|^2\,d\mu_A + c^2\int_{\mathbf{R}^N}\frac{\varphi^2}{|x|^2}\,d\mu_A.
\end{aligned}$$

Hence,

$$\begin{aligned} &\int_{\mathbf{R}^N} \frac{\varphi^2}{|x|^2} \left(c(N-2) - c^2 \right) d\mu_A \\ \leq\ &\int_{\mathbf{R}^N} |\nabla \varphi|^2 \, d\mu_A + c\|A\| \int_{\mathbf{R}^N} \varphi^2 \, d\mu_A. \end{aligned}$$

Theorem 3 *The following hold.*

(i) If $0 \le c \le C^*(N)$*, then there exists a weak solution* $u \in C([0,\infty), L^2(\mu_A))$ *of*

$$\begin{cases} \partial_t u(x,t) = \Delta u(x,t) - Ax \cdot \nabla u(x,t) + \frac{c}{|x|^2} u(x,t), \\ \qquad t > 0,\ x \in \mathbf{R}^N, \\ u(\cdot,0) = u_0 \in L^2(\mu_A), \end{cases}$$

satisfying

$$\|u(t)\|_{L^2(\mu_A)} \le M e^{\omega t} \|u_0\|_{L^2(\mu_A)}. \qquad (2)$$

(ii) If $c > C^*(N)$*, then for any* $0 \le u_0 \in L^2(\mu_A)$*,* $u_0 \neq 0$*, there is no positive weak solution satisfying (2).*

Proof. (ii) Assume $c > C^*(N)$ and take $\varphi(x) := |x|^\gamma$ with $\gamma > 1 - \frac{N}{2}$. Then,

$$\int_{\mathbf{R}^N} \left(|\nabla\varphi|^2 - \frac{c}{|x|^2}\varphi^2 \right) d\mu_A$$
$$= (\gamma^2 - c) \int_{\mathbf{R}^N} |x|^{2(\gamma-1)}\, d\mu_A.$$

Hence, $\lambda_1 := \lambda_1(-\Delta + Ax\cdot\nabla - \frac{c}{|x|^2})$ satisfies

$$\lambda_1 \le (\gamma^2 - c)\frac{\int_{\mathbf{R}^N} |x|^{2(\gamma-1)}\, d\mu_A}{\int_{\mathbf{R}^N} |x|^{2\gamma}\, d\mu_A}.$$

Since

$$\alpha_1 |x|^2 \le |A^{\frac{1}{2}}x|^2 \le \alpha_2 |x|^2,$$

it follows

$$\alpha_2^{-(\frac{N}{2}+\beta)} \int_{\mathbf{R}^N} |x|^{2\beta} e^{-\frac{|x|^2}{2}}\, dx$$
$$\le \int_{\mathbf{R}^N} |x|^{2\beta} e^{-\frac{|A^{\frac{1}{2}}x|^2}{2}}\, dx$$
$$\le \alpha_1^{-(\frac{N}{2}+\beta)} \int_{\mathbf{R}^N} |x|^{2\beta} e^{-\frac{|x|^2}{2}}\, dx.$$

Hence,

$$\frac{\int_{\mathbf{R}^N} |x|^{2(\gamma-1)}\, d\mu_A}{\int_{\mathbf{R}^N} |x|^{2\gamma}\, d\mu_A} \geq \frac{\alpha_2^{-(\frac{N}{2}+\gamma-1)}}{\alpha_1^{-(\frac{N}{2}+\gamma)}(2\gamma+N-2)}.$$

Now, $c > C^*(N) = \left(\frac{N-2}{2}\right)^2$, implies

$$\lambda_1 \leq \lim_{\gamma\to\left(1-\frac{N}{2}\right)^+} \frac{(\gamma^2-c)\alpha_2^{-(\frac{N}{2}+\gamma-1)}}{\alpha_1^{-(\frac{N}{2}+\gamma)}(2\gamma+N-2)} = -\infty.$$