On the Cahn-Hilliard equation with singular potential and dynamic boundary conditions

Giulio Schimperna

Dipartimento di Matematica "F. Casorati",
Università di Pavia, Italy
giusch04@unipv.it

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Introduction and motivation

Well-posedness

Long-time behavior

The Cahn-Hilliard equation

We deal with the following equation of Cahn-Hilliard type:

$$u_t - \Delta\big(- \Delta u + f(u) - h\big) = 0, \qquad \text{(CH)}$$

where

- the unknown u represents an order parameter,
- $w = -\Delta u + f(u) - h$ is the chemical potential, and we assume $\partial_{\boldsymbol{n}} w = 0$,
- f is the derivative of a configuration potential F,
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The boundary condition

- Usually, equation (CH) is coupled with either homogeneous Neumann or periodic boundary condition.
- Recently, the following class of dynamic boundary conditions has been proposed:

$$v_t - \Delta_\Gamma v + f_\Gamma(v) = h_\Gamma - \partial_{\boldsymbol{n}} u, \quad \text{where} \qquad \text{(dynBC)}$$

  - v represents the trace of u on $\Gamma = \partial\Omega$,
  - Δ_Γ is the Laplace-Beltrami operator,
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Motivations for (dynBC)

- It has been recently proposed in a number of physically oriented papers (cf., e.g., [Fischer-Maass-Dieterich]);
- In a variational derivation, it comes from a boundary contribution to the free energy;
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Singular potentials

- The potentials F, F_Γ are assumed to be λ-convex;
- Two situations appear to be physically relevant:
  - Regular potentials: when F (or F_Γ) is a smooth function defined on the whole real line (example: $F(r) \sim (r^2 - 1)^2$);
  - Singular potentials: when F (or F_Γ) has a bounded domain (e.g., $(-1,1)$ or $[-1,1]$) and it is conventionally set $= +\infty$ outside it (example: $F(r) \sim (1+r)\log(1+r) + (1-r)\log(1-r)$).
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A survey on the literature

- The interest on dynamic b.c. for pattern formation systems and phase change models is relatively recent, but rapidly growing;
- Some significant contributions:
  - [Racke-Zheng, 2003], [Wu-Zheng, 2004], [Prüss-Racke-Zheng, 2006]: F regular of controlled growth, $F_\Gamma = 0$, well-posedness, regularity, attractors;
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Objectives of our work

- Consider the widest admissible class of potentials, both on the “bulk” Ω and on the boundary Γ;
- Analyze the growth and/or compatibility compatibility conditions which are required in order to have (at least) existence of a solution;
- Prove a well-posedness theorem via a rigorous approximation argument;
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Dealing with singular potentials

Main trouble:

A control of $f(u)$, $f_\Gamma(v)$ cannot be deduced from a bound on u (the latter holds due to the energy inequality coming from the variational structure of the problem).

But this is true also, e.g., for the Neumann conditions (cf., e.g., [Kenmochi-Niezgodka-Pawłow]);

Is this really a new difficulty?

YES! The coupling between singular potentials and dynamic b.c. creates additional problems.

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...but it is better to do it on the blackboard!

Summarizing – 1

Although the free energy is of the form

$$E(u,v) = \int_\Omega F(u) + \int_\Gamma F_\Gamma(v) + \dots,$$

in order to control $f(u)$ (and $f_\Gamma(v)$) uniformly in time, we need an information on

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We need some condition ensuring that

$$\int_\Gamma f(v)\big(f_\Gamma(v) - h_\Gamma\big) \quad \text{can be controlled;}$$

In particular, if F is singular, we need that

- either F_Γ is singular “in a similar way”
- or that the green integrand has the right sign for values of v close to ± 1 (boundary of dom f).
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Theorem. Assume that:

- f, f_Γ are "compatible" in the sense specified before;
- The initial datum u_0 has finite energy and satisfies $F(v_0) \in L^1(\Gamma)$.

- Then, there exists a unique solution to (CH)+(dynBC)
- and the regularity of the initial datum is maintained for $t \geq 0$.

Remark. We can also consider the case when the graph of f contains vertical segments (this requires some technique of maximal monotone operators).

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Remark. We can also consider the case when the graph of f contains vertical segments (this requires some technique of maximal monotone operators).

Well-posedness

Theorem. Assume that:

- f, f_Γ are "compatible" in the sense specified before;
- The initial datum u_0 has finite energy and satisfies $F(v_0) \in L^1(\Gamma)$.

- Then, there exists a unique solution to (CH)+(dynBC)
- and the regularity of the initial datum is maintained for $t \geq 0$.

Remark. We can also consider the case when the graph of f contains vertical segments (this requires some technique of maximal monotone operators).

The stationary problem

- Notice that, due to the Neumann condition for $w = -\Delta u + f(u) - h$, the spatial average of u is conserved in time. Let us call m its (known) value;
- Then, we obtain the steady-state problem

$$-\Delta u_\infty + f(u_\infty) = w_\infty + h, \qquad |\Omega|^{-1} \int_\Omega u_\infty = m, \quad (\mathrm{CH}_\infty)$$

$$-\Delta_\Gamma v_\infty + f_\Gamma(v_\infty) = h_\Gamma - \partial_{\boldsymbol{n}} u_\infty. \quad (\mathrm{BC}_\infty)$$

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- Note that w_∞ is a constant function.
- We can prove that the ω-limit of our system contains only solutions of (CH_∞)-(BC_∞), where w_∞ is indetermined, but controlled in a way which depends only on the initial value u_0.

Łojasiewicz-Simon method

- Can we, for analytic f and f_Γ prove, via the Łojasiewicz-Simon inequality that the ω-limit is a singleton?
- For singular potentials F, we need that at least the stationary states u_∞ are "separated" from the boundary of dom f. Namely, if dom $f = (-1, 1)$, we need

$$\exists \varepsilon > 0 : \quad -1 + \varepsilon \le u_\infty(x) \le 1 - \varepsilon, \quad \text{a.e. in } \Omega.$$

- We can prove this for f such that (additionally)

$$\lim_{r \to \partial \operatorname{dom} f} |f(r)| = +\infty$$

Main ingredient: the control of w_∞.

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In progress

- We are also studying the problem of determining global and exponential attractors,
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References

G. Gilardi, A. Miranville, G. Schimperna, *On the Cahn-Hilliard equation with irregular potentials and dynamic boundary conditions*, to appear on CPAA.

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Thanks for your attention