Time reversal and data assimilation

Karim RAMDANI and Marius TUCSNAK
Institut Elie Cartan de Nancy

Institut ELIE CARTAN

Statement of the problem

Statement of the problem

Let X and Y be Hilbert spaces, $A : \mathcal{D}(A) \to X$ et $C \in \mathcal{L}(X, Y)$.

$$\dot{w}(t) = Aw(t), \;\; w(0) = x$$

$$y(t) = Cw(t).$$

Assume that A generates a C^0 semigroup, denoted $\mathbb{T}$, in X and that the pair (A, C) is exactly observable in time τ. Then the map $x \mapsto y$ has a bounded left-inverse.

Practical question: How to compute efficiently x from y?

Institut ELIE CARTAN

A naïve solution: inverting the gramian

Solve in the sense of mean squares, i.e., consider the equation

$$Q_\tau x = \int_0^\tau \mathbb{T}_t^* C^* y(s)\,\mathrm{d}s,$$

where $Q_\tau = \int_0^\tau \mathbb{T}_t^* C^* C \mathbb{T}_t \,\mathrm{d}t$ is the *observability gramian.*

Note that $Q_\tau > 0$ since (A, C) observable.

The condition number is generally high and the method is very expensive

Institut ELIE CARTAN

Outline

- A solution in one shot
- An iterative method
- Simulations
- Extensions and comments

A solution in one shot

Basic assumptions

$A = -A^*$, (A, C) exactly observable in time τ and there exists a "time reversal" operator $\mathbf{Я}_\tau \in \mathcal{L}\left(L^2([0,\tau]; X)\right)$ satisfying

$$\mathbf{Я}_\tau^2 = I, \tag{1a}$$

$$\frac{d}{dt}\mathbf{Я}_\tau \mathbb{T}_t z_0 = -\mathbf{Я}_\tau \frac{d}{dt}\mathbb{T}_t z_0, \qquad \forall z_0 \in X \tag{1b}$$

$$A\mathbf{Я}_\tau \mathbb{T}_t z_0 + \mathbf{Я}_\tau A \mathbb{T}_t z_0 = 0, \qquad \forall z_0 \in X \tag{1c}$$

$$C^* C \mathbf{Я}_\tau f = \mathbf{Я}_\tau C^* C f \qquad \forall f \in C([0,\tau], X), \tag{1d}$$

$$\|(\mathbf{Я}_\tau f)(0)\| = \|f(\tau)\| \qquad \forall f \in C([0,\tau], X). \tag{1e}$$

Examples:

$(\mathbf{Я}_\tau v)(s) = \overline{v(\tau - s)}$ for the Schrödinger equation.

$\left(\mathbf{Я}_\tau \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right)(s) = \begin{bmatrix} v_1(\tau - s) \\ -v_2(\tau - s) \end{bmatrix}$ for the wave equation.

Institut ELIE CARTAN

A reversed Luenberger observer

$$\dot{v}(t) = (A - C^*C)v(t) + (\mathbf{Я}_\tau C^* C\mathbb{T}_t x)(t), \qquad t \in (0,\tau)$$
$$v(0) = 0.$$

Denote

$$e(t) = v(t) - \mathbf{Я}_\tau \mathbb{T}_t x.$$

Simple calculations using (1a)-(1e) show that

$$\dot{e}(t) = (A - C^*C)e(t),$$

so that $\|e(\tau)\| \leq Me^{-\omega\tau}\|e(0)\|$.
This means that $v(\tau)$ is a good (and cheap) approximation of x, provided that τ is large.

An iterative method

(abstract version of Phung and Zhang, 2008)

The algorithm

For $\gamma > 0$, we define the sequences (v_n) and (e_n) by:

- $n = 0$:

$$\begin{cases} \dot{v}_0(t) = (A - \gamma C^* C) v_0(t) + \gamma(\mathbf{Я}_\tau C^* C \mathbb{T}_t x)(t), & t \in (0, \tau) \\ v_0(0) = 0, \end{cases}$$

$$e_0 = v_0 - \mathbf{Я}_\tau \mathbb{T}_t x.$$

- $n \geq 1$:

$$\begin{cases} \dot{v}_n(t) = (A - \gamma C^* C) v_n(t) + 2\gamma(\mathbf{Я}_\tau C^* C e_{n-1})(t), \\ v_n(0) = 0, \end{cases}$$

$$e_n(t) = v_n(t) - \mathbf{Я}_\tau e_{n-1}(t).$$

Proposition. *The sequences* (v_n) *and* (e_n) *satisfy :*

1. $\dot{e}_n(t) = (A - \gamma C^* C) e_n(t), \qquad (n \geq 0).$
2. $\|e_0(0)\| = \|x\|, \qquad \|e_n(0)\| = \|e_{n-1}(\tau)\|$ *for all* $n \geq 1$.
3. *For every* $N \geq 1$ *we have :*

$$e_{2N}(t) = \left(\sum_{n=0}^{N} v_{2n}(t) - \sum_{n=1}^{N} (\mathbf{Я}_\tau v_{2n-1})(t) \right) - (\mathbf{Я}_\tau \mathbb{T}_t x)(t). \qquad (1)$$

Lemma. *Let* $\mathbb{S}$ *be the semigroup generated by* $A - \gamma C^* C$ *and assume that* (A, C) *is exactly observable in time* τ. *Then* $\|S_\tau\| < 1$.

Theorem. *Assume that* (A, C) *is exactly observable in time* τ.

$$\left\| x - \sum_{n=0}^{N} (\mathbf{Я}_\tau v_{2n})(0) \right\| \leq \|S_\tau\|^{2N} \|x\| \qquad (N \in \mathbb{N}).$$

Proof. By the Lemma we have

$$\|e_n(\tau)\| \leq \|S_\tau\|^n \, \|e_0(0)\| = \|S_\tau\|^n \, \|x\| \qquad (n \geq 1).$$

On the other hand

$$\|e_{2N}(\tau)\| = \|(\mathbf{Я}_\tau e_{2N})(0)\| = \left\| \sum_{n=0}^{N} (\mathbf{Я}_\tau v_{2n})(0) - x \right\| .$$

Simulations

The wave equation with distributed observation

$$\begin{cases} \ddot{w}(x,t) - \Delta w(x,t) = 0, & x \in \Omega, \quad t \in (0,\tau), \\ w(x,t) = 0, & x \in \partial\Omega, \quad t \in (0,\tau), \\ w(x,0) = w_0(x), & x \in \Omega, \\ \dot{w}(x,0) = w_1(x), & x \in \Omega, \end{cases}$$

with the output

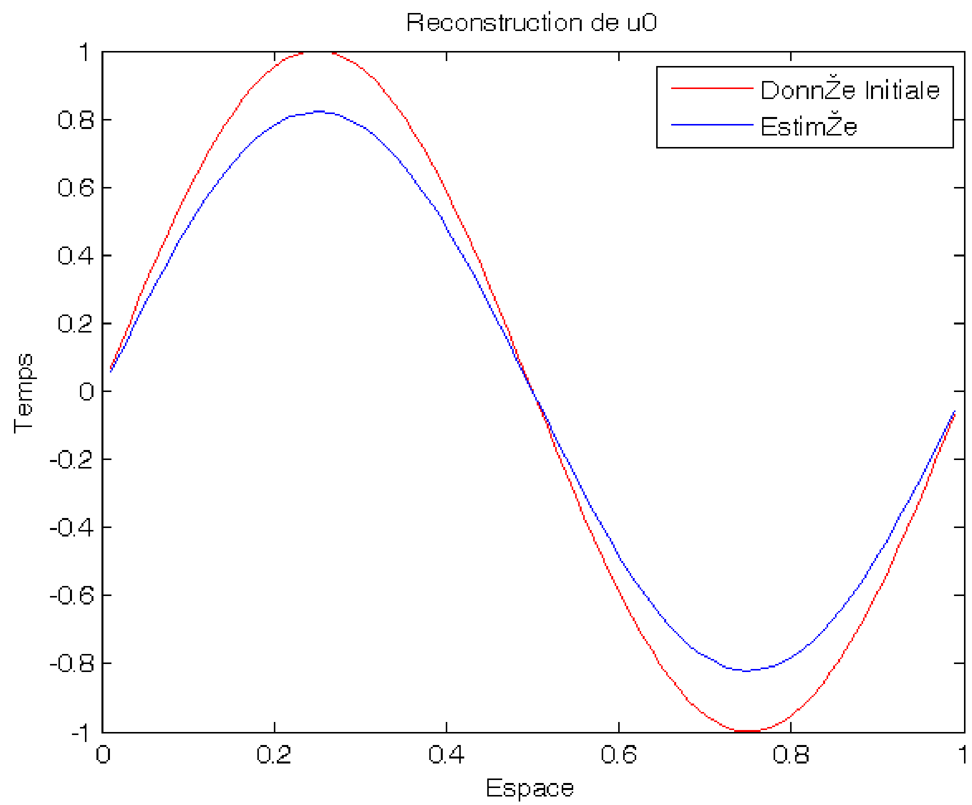
$$y = \dot{w}_{|\mathcal{O}},$$

where $\mathcal{O} \subset \Omega$ satisfies the geometric optics condition.

First simulation in one space dimension

$$\Omega = (0,1), \quad \mathcal{O} = \left(\frac{1}{3}, \frac{2}{3}\right), \quad \tau = 2, \quad \gamma = 1,$$

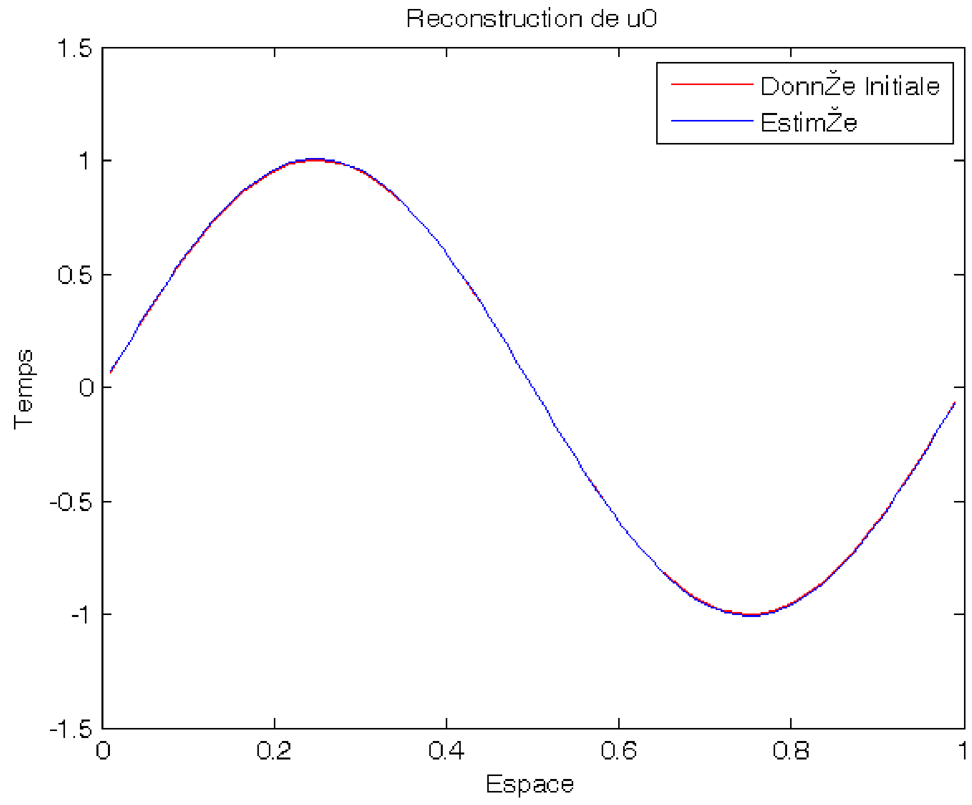
$$w_0(x) = \sin(2\pi x), \quad w_1(x) = 0.$$



Second simulation in one space dimension

$$\Omega = (0,1), \quad \mathcal{O} = \left(\frac{1}{3}, \frac{2}{3}\right), \quad \tau = 1, \quad \gamma = 10,$$

$$w_0(x) = \sin(2\pi x), \quad w_1(x) = 0.$$



Extensions and comments

Extensions and comments

- The boundedness of C is clearly not necessary, but some weaker assumption on the observation operator are necessary
- Exact observability is not necessary for the convergence
- The choice of the damping coefficient plays a crucial role
- The method can be coupled to a solver of Volterra equations to solve inverse source problems (see Alvez, Silvestre, Takahashi and M.T. , 2008)
- It seems possible to generalize the method for operators A with spectrum in a vertical strip
- For a self-adjoint A the problem is strongly ill-posed. Similar algorithms could be interesting to compute the final state.

Institut ELIE CARTAN